Hindawi
Journal of Mathematics
Volume 2022, Article ID 2171089, 10 pages
https://doi.org/10.1155/2022/2171089

Research Article

On Existence of Fixed Points for Multivalued Generalized w_b-Contractive Mappings and Applications

Abdul Latif, Reem Fahad Al Subaie, and Monairah Omar Alansari

[1] *Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[2] *Department of Mathematics, College of Science, Imam Abdulrahman Bin Faisal University, P.O. Box 1982, Dammam 31441, Saudi Arabia*

Correspondence should be addressed to Abdul Latif; alatif@kau.edu.sa

Received 2 January 2022; Accepted 5 February 2022; Published 15 March 2022

Academic Editor: Sun Young Cho

In this study, we present some new results on the existence of fixed points for multivalued generalized w_b-contractive mappings in the frame work of metric type spaces. Consequently, presented results unify and generalize several known metric fixed-point results. In support of our main results, examples are provided to show that the results are genuine generalization of the known corresponding results of metric fixed-point theory.

1. Introduction

The concept of a metric space plays a vital role in the development of metric fixed-point theory and nonlinear functional analysis and also in various scientific branches. In the literature, this notion of metric space has been extended in several directions by reducing or modifying the metric axioms. Czerwik [1, 2] introduced and studied the concept of b-metric space (metric type space), where the triangle inequality replaced with the weaker condition. In fact, the basic idea of b-metric was given by Bakhtin [3]. It has been observed that the class of metric type spaces is effectively larger than the class of metric spaces [1]. In literature, a number of metric fixed-point results for single-valued and multivalued mappings have been shown; see, for example, [4–11] and references therein.

Using a concept of the Hausdorff–Pompieu metric, Nadler [12] introduced a notion of multivalued contraction mappings and proved splendid result in metric fixed-point theory for such mappings, known as Nadler contraction principle. Due to its importance, matric fixed-point theory of multivalued contractions has been further developed in various directions by a number of authors. A real generalization of the Nadler contraction principle is obtained by Mizoguchi and Takahashi [13]. Without using the idea of the Hausdorff–Pompieu metric, a number of authors obtained interesting fixed-point results and improved various results of metric fixed-point theory including the results of Nadler and Mizoguchi–Takahashi and others. See, for example, [14–17] and references therein.

In [18] Kada et al. introduced a concept of generalized distance, namely, w-distance on metric spaces, and improved some classical results by replacing the involved metric by a generalized distance. Based on this set up, a number of authors studied fixed-point results of mappings with respect to w-distance. Suzuki and Takahashi [19] introduced notions of single-valued and multivalued weakly contractive mappings and studied the existence of fixed points for such mappings. Consequently, they generalized the Banach contraction principle and Nadler contraction principle. For further fixed-point results with applications, see, for example, [20–24] and references therein. In [25], Hussain et al. defined w-distance on metric-type spaces called wt-distance (here, we call it w_b-distance), and they proved fixed-point and common fixed-point results for single-valued mappings with respect to w_b-distance. A number of articles with applications on this topic can also be found in [26–29] and references therein.

2. Preliminaries

Now, we recall some notations, concepts, and facts which are useful for our results.

Let (S, d) be a metric space. Let 2^S denote a collection of nonempty subsets of S, $Cl(S)$ denote a collection of nonempty closed subsets of S, $CB(S)$ denote a collection of nonempty closed bounded subsets of S, and $K(S)$ denote a collection of all nonempty compact subsets of S. An element $u \in S$ is called a *fixed point* of a multivalued mapping $J: S \longrightarrow 2^S$ if $u \in J(u)$. We denote $Fix(J) = \{u \in S: u \in J(u)\}$. A sequence $\{u_n\}$ in S is called an *orbit* of J at $u_0 \in S$ if $u_n \in J(u_{n-1})$, for all $n \geq 1$. A map $f: S \longrightarrow \mathbb{R}$ is called *J-orbitally lower semicontinuous* at $z \in S$ if, for any orbit $\{u_n\}$ of J at $u_0 \in S$ with $u_n \longrightarrow z$ implies $f(z) \leq \liminf_{n\longrightarrow\infty} f(u_n)$. For a constant $c \in (0,1)$, we say a function $\xi_c: [0,\infty) \longrightarrow [0,c)$ is a strong MT-function if $\limsup_{r\longrightarrow s^+} \xi_c(r) < c$, for all $s \in [0,\infty)$. In case $c = 1$, the function ξ_1 is denoted by ξ, known as MT-function. It has been observed that a function ξ is MT-function if and only if, for any strictly decreasing sequence $\{u_n\}$ in $[0,\infty)$, we have $0 \leq \sup_n \xi(u_n) < 1$. For more characterizations of MT-function, see [30].

Using the concept of Hausdorff–Pompieu metric, Nadler [12] proved a multivalued version of the well-known Banach contraction principle.

Theorem 1 (see [12]). *Let (S, d) be a complete metric space and let $J: S \longrightarrow CB(S)$ be a multivalued contraction mapping (that is, for a fixed constant $h \in (0,1)$ and for each $u, v \in S$, $H(J(u), J(v)) \leq h\, d(u, v)$, where H is the Hausdorff–Pompieu metric on $CB(S)$). Then, $Fix(J) \neq \emptyset$.*

This result known as Nadler contraction principle, which has been generalized in various directions. The first real generalization of Theorem 1 is obtained by Mizoguchi and Takahashi [13].

Theorem 2 (see [13]). *Let (S, d) be a complete metric space and let $J: S \longrightarrow CB(S)$ be a multivalued mapping. Assume that there exists MT-function ξ such that, for each $u, v \in S$,*

$$H(J(u), J(v)) \leq \xi(d(u, v))d(u, v). \quad (1)$$

Then, $Fix(J) \neq \emptyset$.

On the contrary, without using the concept of the Hausdorff–Pompieu metric, Feng and Liu [16] generalized Theorem 1 as follows.

Theorem 3 (see [16]). *Let (S, d) be a complete metric space and let $J: S \longrightarrow Cl(S)$ be a multivalued mapping. Suppose that a real-valued function g on S, $g(u) = d(u, J(u))$, is lower semicontinuous. Then, $Fix(J) \neq \emptyset$ provided there exist constants $c, h \in (0,1), h < c$, such that, for any $u \in S$, there is $v \in J(u)$ satisfying*

$$\begin{aligned} c d(u, v) &\leq g(u), \\ g(v) &\leq h d(u, v). \end{aligned} \quad (2)$$

Later, Klim and Wardowski [17] generalized Theorem 3 as follows.

Theorem 4 (see [17]). *Let (S, d) be a complete metric space and let $J: S \longrightarrow Cl(S)$ be a multivalued mapping such that a real-valued function g on S, $g(u) = d(u, J(u))$, is lower semicontinuous. Then, $Fix(J) \neq \emptyset$ provided that there exists a strong MT-function ξ_c such that, for any $u \in S$, there is $v \in J(u)$ satisfying*

$$\begin{aligned} c d(u, v) &\leq g(u), \\ g(v) &\leq \xi_c(d(u, v))d(u, v). \end{aligned} \quad (3)$$

Using MT-functions, Klim and Wardowski [17] also proved fixed-point result for compact valued mappings of metric spaces as follows.

Theorem 5 (see [17]). *Let (S, d) be a complete metric space and let $J: S \longrightarrow K(S)$ be a multivalued mapping such that a real-valued function g on S, $g(u) = d(u, J(u))$, is lower semicontinuous. Then, $Fix(J) \neq \emptyset$ provided that there exists MT-function ξ such that, for any $u \in S$, there is $v \in J(u)$ satisfying*

$$\begin{aligned} d(u, v) &= g(u), \\ g(v) &\leq \xi(d(u, v))d(u, v). \end{aligned} \quad (4)$$

It is worth mentioning that Theorem 4 generalizes Theorem 1 and Theorem 3 but does not generalize Theorem 2 because the strong-MT-function ξ_c in Theorem 4 is stronger than the MT-function ξ used in Theorem 2 as $c < 1$.

However, some more general interesting fixed-point results in this direction obtained by Ćirić [14, 15] unify and generalize the corresponding abovementioned results.

In [18], Kada et al. introduced the concept of w-distance as follows.

Let (S, d) be a metric space. A function $p: S \times S \longrightarrow [0,\infty)$ is called w-distance on S if it satisfies the following, for each $u, v, t \in S$:

(1) $p(u, t) \leq p(u, v) + p(v, t)$
(2) A function $p(u, \cdot): S \longrightarrow [0,\infty)$ is lower semicontinuous
(3) For any $\epsilon > 0$, there exists $\delta > 0$ such that $p(t, u) \leq \delta$ and $p(t, v) \leq \delta$ imply $d(u, v) \leq \epsilon$

Using the concept of w-distances, they improved a number of known important results of metric fixed-point theory. Note that, in general, for $u, v \in S$, $p(u, v) \neq p(v, u)$ and not either of the implications $p(u, v) = 0 \Leftrightarrow u = v$ necessarily hold. Clearly, the metric d is a w-distance on S. Let $(W, \|.\|)$ be a normed space. Then, the functions $p_1, p_2: W \times W \longrightarrow [0,\infty)$ defined by $p_1(u, v) = \|v\|$ and $p_2(u, v) = \|u\| + \|v\|$, for all $u, v \in W$, are w-distances [18]. For more examples and properties of the w-distance, see [18, 19, 24]. Using the concept of w-distance, Suzuki and Takahashi [19] introduced single-valued and multivalued weakly contractive mappings and then improved the Banach contraction principle and Nadler contraction principle. For further general fixed-point results in this direction, see [20, 21, 23, 24] and references therein.

Czerwik [1, 2] introduced a concept of b-metric space as follows.

Let S be a nonempty set. Let $\Delta: S \times S \longrightarrow [0, \infty)$ be a function which satisfies the following, for all $u, v, t \in S$:

(1) $\Delta(u, v) = 0$ if and only if $u = v$

(2) $\Delta(u, v) = \Delta(v, u)$

(3) $\Delta(u, v) \leq b[\Delta(u, t) + \Delta(t, v)]$, for some $b \geq 1$

Then, Δ is called a b-metric on S and (S, Δ) is known as b-metric space (also known a metric-type space [8, 25]). In the sequel, we also call it metric-type space. Obviously, for $b = 1$, we obtain a standard metric on S. In fact, the class of metric-type spaces is effectively larger than the class of metric spaces. Indeed, let $S = \mathbb{R}$ be endowed with a mapping $\Delta: S \times S \longrightarrow \mathbb{R}^+$ defined by $\Delta(u, v) = (u - v)^2$, for each $u, v \in S$. Then, (S, Δ) is a metric-type space with $b = 2$, but it is not a metric space [3]. For more examples of metric-type spaces, see [1, 31]. It is worth to point out that, unlike the case of standard metric, the b-metric Δ is not necessarily continuous due to the modified triangle inequality. In general, Δ is not continuous in each variable [5]. However, a metric-type space can be endowed with a topology induced by its convergence [5] and almost all the concepts and results which are valid for metric spaces can be extended to the framework of metric type spaces. In fact, for metric-type spaces, the notions of convergence, Cauchy sequence, and completeness and continuity can be defined similarly as in metric spaces, see [4, 8, 25]. Let us recall few such useful notions and facts in the framework of metric-type spaces.

Let (S, Δ) be a metric-type space and let $\{u_n\}$ be a sequence in S. Then,

(1) $\{u_n\}$ converges in S if there exists $u \in S$ such that $\lim_{n \longrightarrow \infty} \Delta(u_n, u) = 0$

(2) $\{u_n\}$ is a Cauchy sequence in S if $\lim_{n,m \longrightarrow \infty} \Delta(u_n, u_m) = 0$

(3) (S, Δ) is complete if every Cauchy sequence in S is convergent in S

(4) A real-valued function f on S is b-lower semicontinuous at a point $u \in S$ if $f(u) \leq \liminf_{n \longrightarrow \infty} b f(u_n)$ whenever $u_n \longrightarrow u$

Recently, fixed-point theory for metric-type spaces studied and developed by many authors, for example, see [5, 7, 28, 32] and references therein.

Motivated by the work of Kada et al. [18] and Hussain et al. [25] introduced wt-distance (here, we call it w_b-distance) in the setting of metric-type space as follows.

Let (S, Δ) be a metric-type space with constant $b \geq 1$. Then, a function $p_b: S \times S \longrightarrow [0, \infty)$ is called a w_b-distance on S if, for any $u, v, t \in S$, the following hold:

(1) $p_b(u, t) \leq b[p_b(u, v) + p_b(v, t)]$

(2) $p_b(u, \cdot): S \longrightarrow [0, \infty)$ is b-lower semicontinuous (i.e., if, for any sequence $\{v_n\}$ in S, $v_n \longrightarrow v \in S$, then $p_b(u, v) \leq \liminf_{n \longrightarrow \infty} b p_b(u, v_n)$)

(3) For any $\epsilon > 0$, there exists $\delta > 0$ such that $p_b(t, u) \leq \delta$ and $p_b(t, v) \leq \delta$ imply $\Delta(u, v) \leq \epsilon$

Remark 1. Note that, for $b = 1$, each w_b-distance is a w-distance. In general, w_b-distance is not symmetric, see [25]. In fact, the class of w_b-distances is much larger than the class of w-distance, see [28]. Every b-metric is w_b-distance [25], but the converse is not true, see [28].

Example 1 (see [25]). Let $S = \mathbb{R}$ and $\Delta(u, v) = (u - v)^2$; then,

(1) The function $p_b: S \times S \longrightarrow [0, \infty)$ defined by $p_b(u, v) = |u|^2 + |v|^2$ for every $u, v \in S$ is a w_b-distance on S

(2) The function $p_b: S \times S \longrightarrow [0, \infty)$ defined by $p_b(u, v) = |v|^2$ for every $u, v \in S$ is a w_b-distance on S

The following result is an analogue of Lemma 1 of [18], stated and used in [25, 26].

Lemma 1 (see [25]). *Let (S, Δ) be a metric-type space with constant $b \geq 1$ and let p_b be a w_b-distance on S. Let $\{u_n\}$ and $\{v_n\}$ be sequences in S; let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to zero. Then, the following hold, for each $u, v, t \in S$.*

(i) If $p_b(u_n, v) \leq \alpha_n$ and $p_b(u_n, t) \leq \beta_n$, for any $n \in \mathbb{N}$, then $v = t$. In particular, if $p_b(u, v) = 0$ and $p_b(u, t) = 0$, then $v = t$.

(ii) If $p_b(u_n, v_n) \leq \alpha_n$ and $p_b(u_n, t) \leq \beta_n$, for any $n \in \mathbb{N}$, then $\Delta(v_n, t) \longrightarrow 0$.

(iii) If $p_b(u_n, u_m) \leq \alpha_n$, for any $n, m \in \mathbb{N}$ with $m > n$, then $\{u_n\}$ is a Cauchy sequence.

(iv) If $p_b(v, u_n) \leq \alpha_n$, for any $n \in \mathbb{N}$, then $\{u_n\}$ is a Cauchy sequence.

Lemma 2 (see [33]). *Let A be a closed subset of a metric-type space (S, Δ) and p_b be a w_b-distance on S. Suppose that there exists $z \in S$ such that $p_b(z, z) = 0$. Then, $p_b(z, A) = 0$ $\Leftrightarrow z \in A$, where $p_b(z, A) = \inf\{p_b(z, w): \ w \in A\}$.*

Recently, some interesting results appeared in metric fixed-point theory with respect to w_b-distance on metric-type spaces; for example, see [25–28] and references therein.

Let $A \in (0, +\infty]$. Consider a real-valued function ψ on $[0, A)$ satisfying the following conditions:

(1) $\psi(0) = 0$ and $\psi(r) > 0$, for each $r \in (0, A)$.

(2) ψ is nondecreasing on $[0, A)$.

(3) ψ is subadditive, that is,

$$\psi(r_1 + r_2) \leq \psi(r_1) + \psi(r_2), \quad \text{for all } r_1, r_2 \in (0, A). \tag{5}$$

Examples and properties of such functions can be found in [34].

We define

$$\Omega[0, A) = \{\psi: \ \psi \text{ satisfies } (1) - (3)\}. \tag{6}$$

The real-valued function ψ plays an important role in metric fixed-point theory; for example, see [22, 34, 35] and references therein. Among others, Latif and Abdou [22] proved some interesting fixed-point results for multivalued mapping with respect to w-distance. For example, the following results unify and extend a number of known metric fixed-point results.

Theorem 6 (see [22]). *Let (S, d) be a complete metric space with a w-distance p. Let $J: S \longrightarrow Cl(S)$ be a multivalued mapping such that a real-valued function g on S, $g(u) = p(u, J(u))$, is lower semicontinuous. Assume that there exist $c \in (0,1)$ and $\eta \in \Omega[0, A)$ such that, for any $u \in S$, there is $v \in J(u)$ satisfying*

$$\begin{aligned} &c\eta(p(u,v)) \leq \eta(g(u)),\\ &\eta(g(v)) \leq \xi_c(p(u,v))\eta(p(u,v)), \end{aligned} \tag{7}$$

where ξ_c is a strong MT-function. Then, there exists $w_0 \in S$ such that $g(w_0) = 0$. Moreover, if $p(w_0, w_0) = 0$, then $w_0 \in J(w_0)$.

Theorem 7 (see [22]). *Let (S, d) be a complete metric space with a w-distance p. Let $J: S \longrightarrow Cl(S)$ be a multivalued mapping such that a real-valued function g on S, $g(u) = p(u, J(u))$, is lower semicontinuous. Assume that there exists $\eta \in \Omega[0, A)$ such that, for any $u \in S$, there is $v \in J(u)$ satisfying*

$$\begin{aligned} &\eta(p(u,v)) = \eta(g(u)),\\ &\eta(g(v)) \leq \xi(p(u,v))\eta(p(u,v)), \end{aligned} \tag{8}$$

where ξ is MT-function. Then, there exists $w_0 \in S$ such that $g(w_0) = 0$. Moreover, if $p(w_0, w_0) = 0$, then $w_0 \in J(w_0)$.

The aim of this paper is to present some more general results on the existence of fixed points related to multivalued generalized w_b-contractive mappings defined on metric-type spaces. In particular, such mappings involve the function $\psi \circ p_b$, where $\psi \in \Omega[0, A)$ and the function p_b is a w_b-distance on a metric-type space. Consequently, our results unify and generalize the corresponding several known metric fixed-point results.

3. Results

Now, we present our first main result on the existence of fixed points for multivalued mapping with respect to w_b-distance, which improve and generalize a number of known fixed-point results including Theorem 6.

Throughout this section, (S, Δ) is a complete metric-type space and p_b is a w_b-distance on S.

Theorem 8. *Let $J: S \longrightarrow Cl(S)$ be a multivalued mapping. Assume that there exist a strong MT-function ξ_c and a function $\psi \in \Omega[0, A)$ such that, for any $u \in S$, there is $v \in J(u)$ satisfying*

$$\begin{aligned} &c\psi(p_b(u,v)) \leq \psi(q(u)),\\ &\psi(q(v)) \leq \xi_c(p_b(u,v))\psi(p_b(u,v)), \end{aligned} \tag{9}$$

where q is a real-valued function on S defined by $q(u) = p_b(u, J(u))$. Then,

(1) For any $u_0 \in S$, there exist an orbit $\{u_n\}$ of J at u_0 and $z_0 \in S$ such that $\lim_{n\longrightarrow\infty} u_n = z_0$.

(2) $p_b(z_0, J(z_0)) = 0$ if and only if the function q is J-orbitally b-lower semicontinuous at z_0. Moreover, if $p_b(z_0, z_0) = 0$, then $z_0 \in Fix(J)$.

Proof. Let u_0 be an arbitrary but fixed element of S. Then, there exists $u_1 \in J(u_0)$ such that

$$c\psi(p_b(u_0,u_1)) \leq \psi(q(u_0)), \tag{10}$$

$$\psi(q(u_1)) \leq \xi_c(p_b(u_0,u_1))\psi(p_b(u_0,u_1)), \tag{11}$$

$$\xi_c(p_b(u_0,u_1)) < c. \tag{12}$$

Thus, we have

$$\psi(q(u_0)) - \psi(q(u_1)) \geq [c - \xi_c(p_b(u_0,u_1))]\psi(p_b(u_0,u_1)) > 0. \tag{13}$$

Similarly, for $u_1 \in S$, there exists $u_2 \in J(u_1)$ such that

$$\begin{aligned} &c\psi(p_b(u_1,u_2)) \leq \psi(q(u_1)),\\ &\psi(q(u_2)) \leq \xi_c(p_b(u_1,u_2))\psi(p_b(u_1,u_2)),\\ &\xi_c(p_b(u_1,u_2)) < c, \end{aligned} \tag{14}$$

$$\begin{aligned} \psi(q(u_1)) - \psi(q(u_2)) \geq{}& [c - \xi_c(p_b(u_1,u_2))]\\ &\psi(p_b(u_1,u_2)) > 0. \end{aligned} \tag{15}$$

Continuing this process, we obtain an orbit $\{u_n\}$ of J at $u_0 \in S$ such that $u_{n+1} \in J(u_n)$ satisfying

$$c\psi(p_b(u_n,u_{n+1})) \leq \psi(q(u_n)), \tag{16}$$

$$\psi(q(u_{n+1})) \leq \xi_c(p_b(u_n,u_{n+1}))\psi(p_b(u_n,u_{n+1})), \tag{17}$$

$$\xi_c(p_b(u_n,u_{n+1})) < c, \tag{18}$$

$$\begin{aligned} \psi(q(u_n)) - \psi(q(u_{n+1})) \geq{}& [c - \xi_c(p_b(u_n,u_{n+1}))]\\ &\cdot \psi(p_b(u_n,u_{n+1})) > 0, \end{aligned} \tag{19}$$

which imply that

$$\psi(q(u_{n+1})) < \psi(q(u_n)), \quad n \in \mathbb{N} \cup \{0\}. \tag{20}$$

While from (11), (12), and (14), it follows that

$$\begin{aligned} \psi(p_b(u_1,u_2)) &\leq \frac{1}{c}\psi(q(u_1)) \leq \frac{1}{c}\xi_c(p_b(u_0,u_1))\psi(p_b(u_0,u_1))\\ &< \psi(p_b(u_0,u_1)). \end{aligned} \tag{21}$$

Thus, for each $n \in \mathbb{N}$, we obtain

$$\psi(p_b(u_n, u_{n+1})) < \psi(p_b(u_{n-1}, u_n)). \tag{22}$$

By (20) and (22), we note that the sequences $\{\psi(q(u_n))\}$ and $\{\psi(p_b(u_n, u_{n+1}))\}$ are decreasing. Now, since ψ is nondecreasing, it follows that $\{q(u_n)\}$ and $\{p_b(u_n, u_{n+1})\}$ are decreasing sequences and are bounded from below, thus convergent. Now, by the definition of the function ξ_c, there exists $\delta \in [0, c)$ such that

$$\limsup_{n\longrightarrow\infty} \xi_c(p_b(u_n, u_{n+1})) = \delta. \tag{23}$$

Thus, for any $c_0 \in (\delta, c)$ with $c_0 c^{-1} \in (0, b^{-1})$, there exists $n_0 \in \mathbb{N}$ such that

$$\xi_c(p_b(u_n, u_{n+1})) < c_0, \quad \text{for all } n > n_0, \tag{24}$$

and thus, for all $n > n_0$, we have

$$\xi_c(p_b(u_n, u_{n+1})) \times \cdots \times \xi_c\big(p_b\big(u_{n_0+1}, u_{n_0+2}\big)\big) < c_0^{n-n_0}. \tag{25}$$

Note that, for all $n > n_0$, we have

$$\begin{aligned}
\psi(q(u_{n+1})) &\leq \xi_c(p_b(u_n, u_{n+1}))\psi(p_b(u_n, u_{n+1})) \\
&\leq \frac{1}{c}\xi_c(p_b(u_n, u_{n+1}))\psi(q(u_n)) \\
&\leq \frac{1}{c^2}\xi_c(p_b(u_n, u_{n+1}))\xi_c(p_b(u_{n-1}, u_n))\psi(q(u_{n-1})) \\
&\vdots \\
&\leq \frac{1}{c^n}\left[\xi_c(p_b(u_n, u_{n+1})) \times \cdots \times \xi_c(p_b(u_1, u_2))\right]\psi(q(u_1)) \\
&= \frac{\xi_c(p_b(u_n, u_{n+1})) \times \cdots \times \xi_c\big(p_b\big(u_{n_0+1}, u_{n_0+2}\big)\big)}{c^{n-n_0}} \times \frac{\xi_c\big(p_b\big(u_{n_0}, u_{n_0+1}\big)\big) \times \cdots \times \xi_c(p_b(u_1, u_2))\psi(q(u_1))}{c^{n_0}},
\end{aligned} \tag{26}$$

and thus,

$$\psi(q(u_{n+1})) < \left(\frac{c_0}{c}\right)^{n-n_0} \frac{\xi_c\big(p_b\big(u_{n_0}, u_{n_0+1}\big)\big) \times \cdots \times \xi_c(p_b(u_1, u_2))\psi(q(u_1))}{c^{n_0}}. \tag{27}$$

Put $\lambda = c_0 c^{-1}$ and since $\lambda < 1$, we have $\lim_{n\longrightarrow\infty}\lambda^{n-n_0} = 0$, and hence, the decreasing sequence $\{\psi(q(u_n))\}$ converges to 0. Thus, we have

$$\lim_{n\longrightarrow\infty} q(u_n) = 0. \tag{28}$$

Now, we show that $\{u_n\}$ is a Cauchy sequence. From (16), (17), and (24), we note that, for all $n > n_0$,

$$\begin{aligned}
\psi(p_b(u_n, u_{n+1})) &\leq \frac{1}{c}\psi(q(u_n)) \\
&\leq \frac{1}{c}\xi_c(p_b(u_{n-1}, u_n))\psi(p_b(u_{n-1}, u_n)) \\
&< \lambda\psi(p_b(u_{n-1}, u_n)) \\
&< \lambda^2\psi(p_b(u_{n-2}, u_{n-1})) \\
&\vdots \\
&< \lambda^n\psi(p_b(u_0, u_1)).
\end{aligned} \tag{29}$$

Thus, we have

$$\psi(p_b(u_n, u_{n+1})) \leq \lambda^n\psi(p_b(u_0, u_1)), \quad n \in \mathbb{N} \cup \{0\}. \tag{30}$$

Now, for any $n, m \in \mathbb{N}, m > n$,

$$\begin{aligned}\psi(p_b(u_n,u_m)) &\le b(\psi(p_b(u_n,u_{n+1})) + \psi(p_b(u_{n+1},u_m)))\\ &\le b\psi(p_b(u_n,u_{n+1})) + b(b(\psi(p_b(u_{n+1},u_{n+2})) + \psi(p_b(u_{n+2},u_m))))\\ &= b\psi(p_b(u_n,u_{n+1})) + b^2\psi(p_b(u_{n+1},u_{n+2})) + b^2\psi(p_b(u_{n+2},u_m))\\ &\le b\psi(p_b(u_n,u_{n+1})) + b^2\psi(p_b(u_{n+1},u_{n+2})) + b^2(b(\psi(p_b(u_{n+2},u_{n+3})) + \psi(p_b(u_{n+3},u_m))))\\ &= b\psi(p_b(u_n,u_{n+1})) + b^2\psi(p_b(u_{n+1},u_{n+2})) + b^3(\psi(p_b(u_{n+2},u_{n+3})) + \psi(p_b(u_{n+3},u_m)))\\ &\vdots\\ &\le b\psi(p_b(u_n,u_{n+1})) + b^2\psi(p_b(u_{n+1},u_{n+2})) + \ldots + b^{m-n-1}(\psi(p_b(u_{m-2},u_{m-1})) + \psi(p_b(u_{m-1},u_m)))\\ &\le b\lambda^n\psi(p_b(u_0,u_1)) + b^2\lambda^{n+1}\psi(p_b(u_0,u_1)) + \ldots + b^{m-n-1}\lambda^{m-2}\psi(p_b(u_0,u_1)) + b^{m-n-1}\lambda^{m-1}\psi(p_b(u_0,u_1))\\ &= b\lambda^n\left(1 + b\lambda + (b\lambda)^2 + \cdots + (b\lambda)^{m-n-2} + b^{m-n-2}\lambda^{m-n-1}\right)\psi(p_b(u_0,u_1)).\end{aligned} \tag{31}$$

Since $\lambda < b^{-1}$, thus, for $m, n \in \mathbb{N}$ with $m > n > n_0$, we obtain

$$\psi(p_b(u_n,u_m)) \le \frac{b\lambda^n}{1 - b\lambda}\psi(p_b(u_0,u_1)). \tag{32}$$

Thus, since $(b\lambda^n/(1 - b\lambda)) \longrightarrow 0$ as $n \longrightarrow +\infty$, we have $\lim_{n,m\longrightarrow+\infty}\psi(p_b(u_n,u_m)) = 0$, and thus,

$$\lim_{n,m\longrightarrow+\infty} p_b(u_n,u_m) = 0. \tag{33}$$

By Lemma 1 (iii), $\{u_n\}$ is Cauchy sequence in S. Since S is complete, $\{u_n\}$ converges to some point $z_0 \in S$. Note that the sequence $\{u_n\}$ is an orbit of J at $u_0 \in S$ with $u_n \longrightarrow z_0$. Now, suppose that the function q is J-orbitally b-lower semi-continuous at z_0; then, using (28), we have

$$0 \le q(z_0) \le \liminf_{n\longrightarrow\infty} bq(u_n) = 0, \tag{34}$$

and hence, $q(z_0) = p_b(z_0, J(z_0)) = 0$. Conversely, if $p_b(z_0, J(z_0)) = q(z_0) = 0$, then, clearly, the function q is J-orbitally b-lower semicontinuity at z_0 because $q(z_0) = 0 \le \liminf_{n\longrightarrow\infty} bq(u_n)$. Furthermore, if $p_b(z_0, z_0) = 0$, then, since $J(z_0)$ is closed, it follows from Lemma 2 that $z_0 \in J(z_0)$.

If we consider in Theorem 8, a constant mapping $\xi_c(s) = \tau$ and $s \in (0,\infty)$, where $\tau \in (0, c)$; then, we have the following result. □

Corollary 1. *Let $J: S \longrightarrow Cl(S)$ be a multivalued mapping satisfying that, for any constants $c \in (0,1)$ and for each $u \in S$, there is $v \in L_c^u$ such that*

$$\psi(q(v)) \le \tau\psi(p_b(u,v)), \tag{35}$$

where $L_c^u = \{v \in J(u): c\psi(p_b(u,v)) \le \psi(q(u))\}$ and a real-valued function q on S defined by $q(u) = p_b(u, J(u))$ is b-lower semicontinuous. Then, there exists $z_0 \in S$ such that $q(z_0) = 0$. Furthermore, if $p_b(z_0, z_0) = 0$, then $z_0 \in Fix(J)$.

Remark 2.

(1) Theorem 8 generalizes the fixed-point result (Theorem 2.2 of [36]). Indeed, if we take $\psi(s) = s$, for all $s \in [0, A)$ and $b = 1$ (i.e., w_b-distance as a w-distance) in Theorem 8, then we get Theorem 2.2 of [36]. Consequently, Theorem 8 also extends Theorem 4, which contains Theorem 3.

(2) Theorem 8 generalizes the fixed-point result (Theorem 2.1 of [22]) for w_b-distance in the frame work of metric-type spaces.

(3) Theorem 8 contains Theorem 2.1 of [35] as a special case.

(4) Corollary 1 contains the fixed-point results (Corollary 2.2 of [22] and Theorem 3.3 of [37]).

Now, without using b-lower semicontinuity of the function q, we present a fixed-point result for multivalued mappings which extends the fixed-point result of Theorem 2.4 of [36] and reduces to Theorem 2.4 of [22].

Theorem 9. *Suppose that all the hypotheses of Theorem 8 (except the b-lower semicontinuity of the function q) hold. Assume that*

$$\inf\{\psi(p_b(u,z)) + \psi(q(u)): u \in S\} > 0, \tag{36}$$

for every $z \in S$ with $z \notin J(z)$. Then, $Fix(J) \ne \emptyset$.

Proof. Following the proof of Theorem 8, there exists an orbit $\{u_n\}$ of J at $u_0 \in S$, which turns as a Cauchy sequence in a complete space S. Then, there exists $z_0 \in S$ such that the sequence $\{u_n\}$ converges to z_0. Thus, by the b-lower semicontinuity of the function $p_b(u_n, \cdot)$, and from the proof of Theorem 8, it follows that, for all $n > n_0$,

$$\psi(p_b(u_n,z_0)) \le \liminf_{m\longrightarrow\infty} b\psi(p_b(u_n,u_m)) \le \frac{b^2\lambda^n}{1 - b\lambda}\psi(p_b(u_0,u_1)), \tag{37}$$

where $\lambda = c_0/c < 1$. Also, note that

$$q(u_n) = p_b(u_n, J(u_n)) \leq p_b(u_n, u_{n+1}), \tag{38}$$

for all n, and since the function ψ is nondecreasing, we have

$$\psi(q(u_n)) \leq \psi(p_b(u_n, u_{n+1})), \tag{39}$$

and thus,

$$\psi(q(u_n)) < \lambda^n \psi(p_b(u_0, u_1)). \tag{40}$$

Assume that $z_0 \notin J(z_0)$. Then, we have

$$\begin{aligned} 0 &< \inf\{\psi(p_b(u, z_0)) + \psi(q(u)) \colon u \in S\} \\ &\leq \inf\{\psi(p_b(u_n, z_0)) + \psi(q(u_n)) \colon n > n_0\} \\ &< \inf\left\{\frac{b^2\lambda^n}{1 - b\lambda}\psi(p_b(u_0, u_1)) + \lambda^n \psi(p_b(u_0, u_1)) \colon n > n_0\right\} \\ &= \left\{\frac{b^2 - b\lambda + 1}{1 - b\lambda}\psi(p_b(u_0, u_1))\right\}\inf\{\lambda^n \colon n > n_0\} = 0, \end{aligned} \tag{41}$$

which is impossible, and hence, $z_0 \in Fix(J)$.

Using MT-functions (instead of strong MT-functions), we present a fixed-point result for multivalued w_b-contraction mappings which extends Theorem 2.5 of [22] and thus contains a number of known metric fixed-point results. □

Theorem 10. *Let $J \colon S \longrightarrow Cl(S)$ be a multivalued mapping. Assume that there exist an MT-function ξ and a function $\psi \in \Omega[0, A)$ such that, for any $u \in S$, there is $v \in J(u)$ satisfying*

$$\begin{aligned} &\psi(p_b(u, v)) = \psi(q(u)), \\ &\psi(q(v)) \leq \xi(p_b(u, v))\psi(p_b(u, v)), \end{aligned} \tag{42}$$

where q is a real-valued function on S defined by $q(u) = p_b(u, J(u))$. Then,

(1) For any $u_0 \in S$, there exist an orbit $\{u_n\}$ of J at u_0 and $z_0 \in S$ such that $lim_{n \longrightarrow \infty} u_n = z_0$.

(2) $p_b(z_0, J(z_0)) = 0$ if and only if the function q is J-orbitally b-lower semicontinuous at z_0. Moreover, if $p_b(z_0, z_0) = 0$, then $z_0 \in Fix(J)$.

Proof. Let u_0 be an arbitrary but fixed element of S. Then, there is $u_1 \in J(u_0)$ such that

$$\begin{aligned} &\psi(p_b(u_0, u_1)) = \psi(q(u_0)), \\ &\psi(q(u_1)) \leq \xi(p_b(u_0, u_1))\psi(p_b(u_0, u_1)), \\ &\xi(p_b(u_0, u_1)) < 1. \end{aligned} \tag{43}$$

Following the proof of Theorem 8, there exists a Cauchy sequence $\{u_n\}$ in the complete space S such that $u_{n+1} \in J(u_n)$ (that is, $\{u_n\}$ is an orbit of J at u_0) satisfying

$$\begin{aligned} &\psi(p_b(u_n, u_{n+1})) = \psi(q(u_n)), \\ &\psi(q(u_{n+1})) \leq \xi(p_b(u_n, u_{n+1}))\psi(p_b(u_n, u_{n+1})), \\ &\xi(p_b(u_n, u_{n+1})) < 1. \end{aligned} \tag{44}$$

Consequently, there exists $z_0 \in S$ such that the sequence $\{u_n\}$ converges to z_0. Now, the rest of the proof follows as of Theorem 8. □

Remark 3.

(1) For $b = 1$, Theorem 10 reduces to Theorem 2.5 of [22].

(2) If we take $b = 1$ and $\psi(s) = s$, for each $s \in [0, A)$ in Theorem 10, the we obtain Theorem 2.5 of [38].

(3) It turns out that Theorem 10 also generalizes Theorem 7 of [14] and Theorem 2.4 of [35].

Using the same technique as in the proof of Theorem 9, we get the following fixed-point result (in the absence of the b-lower semicontinuity of the function q), which contains fixed-point result (Theorem 2.7 of [22]) as a special case.

Theorem 11. *Suppose that all the hypotheses of Theorem 10 except the b-lower semicontinuity of the function q hold. Assume that*

$$\inf\{\psi(p_b(u, z)) + \psi(q(u)) \colon u \in S\} > 0, \tag{45}$$

for every $z \in S$ with $z \notin J(z)$. Then, $Fix(J) \neq \varnothing$.

4. Example

Now, we present examples which show that our main results, namely, Theorems 8 and 10 are genuine generalizations of Theorem 2.1 of [22] and Theorem 2.5 of [22], respectively.

Example 2. Let $S = [0, 1]$. Define $\Delta(u, v) = (u - v)^2$, for all $u, v \in S$. Then, S is a metric-type space with $b = 2$. Define a w_b-distance function on S by $p_b(u, v) = v^2$, for all $u, v \in S$. Let $J \colon S \longrightarrow Cl(S)$ be defined by

$$J(u) = \begin{cases} \left\{\frac{1}{2}u^2\right\}; & u \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right], \\ \left\{0, \frac{17}{96}, \frac{1}{4}\right\}; & u = \frac{15}{32}. \end{cases} \tag{46}$$

Let $A \in [1, \infty)$ and let $c = 1/2$. Define a function $\psi \colon [0, A) \longrightarrow \mathbb{R}$ by $\psi(s) = s^{1/2}$. Clearly, $\psi \in \Omega[0, A)$. Define $\xi_c \colon [0, \infty) \longrightarrow [0, c)$ as follows:

$$\xi_c(s) = \begin{cases} \frac{3}{4}s^{1/2}; & s \in \left[0, \frac{1}{2}\right), \\ \frac{3}{8}; & s \in \left[\frac{1}{2}, \infty\right). \end{cases} \tag{47}$$

Clearly, ξ_c is a strong MT-function. Also, note that

$$q(u) = p_b(u, J(u)) = \begin{cases} \frac{1}{4}u^4; & u \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right], \\ 0; & u = \frac{15}{32}. \end{cases} \tag{48}$$

Now, for each $u \in [0, 15/32)t \cup n(q15/32, 1]$, we have $J(u) = \{(1/2)u^2\}$. Take $v = (1/2)u^2 \in J(u)$; then, we have

$$p_b(u, v) = q(u) = \frac{1}{4}u^4. \tag{49}$$

Thus, for $u \in [0, 1], u \neq 15/32$, we have

$$c\psi(p_b(u, v)) \leq \psi(q(u)), \tag{50}$$

$$\begin{aligned}\psi(q(v)) &= \psi\left(p_b\left(\frac{1}{2}u^2, \frac{1}{2}\left(\frac{1}{2}u^2\right)^2\right)\right) = \psi\left(\frac{1}{64}u^8\right) = \frac{1}{8}u^4 \\ &\leq \frac{3}{16}u^4 = \xi_c(p_b(u, v))\psi(p_b(u, v)).\end{aligned} \tag{51}$$

Now, let $u = 15/32$; then, we have $J(u) = \{0, 17/96, 1/4\}$. Clearly, there exists $v = 0 \in J(u)$ such that

$$c\psi(p_b(u, v)) = 0 = \psi(q(u)), \tag{52}$$

$$\psi(q(v)) = \psi(p_b(0, 0)) = \xi_c(p_b(u, v))\psi(p_b(u, v)). \tag{53}$$

Note that, for each $u \in [0, 1]$, all the conditions of Theorem 8 are satisfied, and hence, it follows that $Fix(J) \neq \varnothing$. Note that $Fix(J) = \{0\}$.

Clearly, p_b is not a metric d, even not a w-distance p on S, and thus, J does not satisfy the hypotheses of Theorem 2.1 of [22]. Note that the mapping J also does not satisfy the hypotheses of Theorem 2.5 of [22].

Example 3. Let $S = \{0\} \cup \{(1/n)\colon n \in \mathbb{N}\}$. Denote $\Lambda = \{0\} \cup \{(1/2n)\colon n \in \mathbb{N}\}$. Clearly, $\Lambda \subseteq S$. Let $\Delta\colon S \times S \longrightarrow [0, \infty)$ be defined by

$$\Delta(u, v) = \begin{cases} 0; & \text{if } u = v, \\ 2; & \text{if } u \neq v \in \{0, 1\}, \\ |u - v|; & \text{if } u \neq v \in \Lambda, \\ 4; & \text{otherwise.} \end{cases} \tag{54}$$

Then, S is a metric-type space with $b = 8/3$ (see [28]). Define a w_b-distance $p_b\colon S \times S \longrightarrow [0, \infty)$ by

$$p_b(u, v) = \begin{cases} 0; & \text{if } u = v, \\ 2; & \text{if } u \neq v \in \{0, 1\}, \\ \max\{3(u - v), 2(v - u)\}; & \text{if } u \neq v \in \Lambda, \\ 4; & \text{otherwise.} \end{cases} \tag{55}$$

Let $J\colon S \longrightarrow Cl(S)$ be defined by

$$J(u) = \begin{cases} \left\{\frac{1}{11}u\right\}; & \text{if } u \in \Lambda, \\ \left\{0, \frac{1}{3}\right\}; & \text{otherwise.} \end{cases} \tag{56}$$

Let $A \in [1, \infty)$. Define a function $\psi\colon [0, A) \longrightarrow \mathbb{R}$ by $\psi(s) = s^{1/2}$. Clearly, $\psi \in \Omega[0, A)$. Define $\xi\colon [0, \infty) \longrightarrow [0, 1)$ as follows:

$$\xi(s) = \begin{cases} \frac{1}{4}s; & \text{if } s \in \Lambda, \\ \frac{1}{2}; & \text{otherwise.} \end{cases} \tag{57}$$

Clearly, ξ is MT-function. We need to examine the following cases:

Case I: suppose $u \in \Lambda \setminus \{0\}$; we have $J(u) = \{(1/11)u\}$ and so

$$\begin{aligned}q(u) &= p_b(u, J(u)) = \max\left\{3\left(u - \frac{1}{11}u\right), 2\left(\frac{1}{11}u - u\right)\right\} \\ &= \frac{30}{11}u.\end{aligned} \tag{58}$$

Take $v = (1/11)u \in J(u)$; then, clearly, $v \in \Lambda$, and we have

$$p_b(u, v) = q(u) = \frac{30}{11}u. \tag{59}$$

Thus, for $u \in \Lambda \setminus \{0\}$, we have

$$\psi(p_b(u, v)) = \psi(q(u)), \tag{60}$$

$$\begin{aligned}\psi(q(v)) &= \psi\left(p_b\left(\frac{1}{11}u, \frac{1}{(11)^2}u\right)\right) \\ &= \psi\left(\frac{30}{(11)^2}u\right) = \frac{\sqrt{30}}{11}u^{1/2} \\ &\leq \frac{1}{2}\sqrt{\frac{30}{11}}u^{1/2} = \xi(p_b(u, v))\psi(p_b(u, v)).\end{aligned} \tag{61}$$

Case II: suppose $u = 0$; then, we have $J(u) = \{(1/11)u\} = \{0\}$. Clearly, there exists $v = 0 \in J(u)$ such that

$$\psi(p_b(u, v)) = 0 = \psi(q(u)), \tag{62}$$

$$\psi(q(v)) = \psi(p_b(0, 0)) = 0 = \xi(p_b(u, v))\psi(p_b(u, v)). \tag{63}$$

Case III: suppose $u = 1$; then, we have $J(u) = \{0, 1/3\}$. Clearly, there exists $v = 0 \in J(u)$ such that

$$\psi(p_b(u,v)) = \sqrt{2} = \psi(q(u)), \tag{64}$$

$$\begin{aligned}\psi(q(v)) &= \psi(p_b(0,0)) = 0 \le \frac{1}{\sqrt{2}} \\ &= \xi(p_b(u,v))\psi(p_b(u,v)).\end{aligned} \tag{65}$$

Case IV: suppose $u = (1/3)$; then, we have $J(u) = \{0, 1/3\}$. Clearly, there exists $v = (1/3) \in J(u)$ such that

$$\psi(p_b(u,v)) = 0 = \psi(q(u)), \tag{66}$$

$$\psi(q(v)) = \psi\left(p_b\left(\frac{1}{3},\frac{1}{3}\right)\right) = 0 = \xi(p_b(u,v))\psi(p_b(u,v)). \tag{67}$$

Case V: suppose $u \in S\setminus(\Lambda \cup \{1, (1/3)\})$; then, we have $J(u) = \{0, (1/3)\}$. Clearly, there exists $v = (1/3) \in J(u)$ such that

$$\psi(p_b(u,v)) = 2 = \psi(q(u)), \tag{68}$$

$$\psi(q(v)) = \psi\left(p_b\left(\frac{1}{3},\frac{1}{3}\right)\right) = 0 \le 1 = \xi(p_b(u,v))\psi(p_b(u,v)). \tag{69}$$

Note that, for each $u \in S$, all the conditions of Theorem 10 are satisfied, and hence, it follows that $Fix(J) \neq \emptyset$. Note that $Fix(J) = \{0\}$.

Not that the w_b-distance p_b is not a metric d, even not a w-distance p on S, and thus, J do not satisfy the hypotheses of Theorem 2.5 of [22].

5. Conclusion

Among others, Feng-Liu [16], Klim and Wardowski [17], and Ćirić [14] studied the existence of fixed points for multivalued contractive-type mappings without using the Hausdorff–Pompieu metric, and consequently, they generalized some classical known fixed-point results including Theorems 1 and 2. In this study, we established some general fixed-point results for multivalued generalized contractive mappings on metric-type spaces (instead of normal metric spaces) with respect to w_b-distances. Presented results generalize and improve a number of known fixed-point results, including the corresponding fixed-point results which are stated in Section 2. In support of our main fixed-point theorems, examples are also provided. Note that the family of metric-type spaces is effectively larger than one of metric spaces, and hence, our theorems are more general, different from the classical results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] S. Czerwik, "Nonlinear set-valued contraction mappings in b-metric spaces," *Atti del Seminario Matematico e Fisico dell'Universita di Modena*, vol. 46, pp. 263–276, 1998.

[2] S. Czerwick, K. Dlutek, and S. L. Singh, "Round-off stability of iteration procedures for set-valued operators in b-metric spaces," *Journal of Natural Physics and Science*, vol. 11, pp. 87–94, 2001.

[3] I. A. Bakhtin, "The contraction mapping principle in almost metric spaces," *Funct. Anal. Gos. Ped. Inst. Unianowsk*, vol. 30, pp. 26–37, 1989.

[4] R. P. Agarwal, E. Karapinar, D. O'Regan, and A. F. Roldán-López-de-Hierro, *Fixed Point Theory in Metric Type Spaces*, Springer- International Publishing, Switzerland, 2015.

[5] T. V. An, L. Q. Tuyen, and N. V. Dung, "Stone-type theorem on b-metric spaces and applications," *Topology and Its Applications*, vol. 185-186, pp. 50–64, 2015.

[6] P. Hu and F. Gu, "Some fixed point theorems of λ-contractive mappings in Menger PSM-spaces," *Journal of Nonlinear Functional Analysis*, vol. 2020, p. 33, 2020.

[7] M. Iqbal, A. Batool, O. Ege, and M. de la Sen, "Fixed point of generalized weak contraction in b-metric spaces," *Journal of Function Spaces*, vol. 2021, Article ID 2042162, 8 pages, 2021.

[8] M. A. Khamsi and N. Hussain, "KKM mappings in metric type spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 9, pp. 3123–3129, 2010.

[9] R. Miculescu and A. Mihail, "New fixed point theorems for set-valued contractions in b-metric spaces," *Journal of Fixed Point Theory and Applications*, vol. 19, no. 3, pp. 2153–2163, 2017.

[10] S. Reich and A. J. Zaslavski, "Fixed points and convergence results for a class of contractive mappings," *Journal of Nonlinear and Variational Analysis*, vol. 5, pp. 665–671, 2021.

[11] N. Shahzad and H. Zegeye, "Convergence theorems for a fixed point of η-demimetric mappings in Banach spaces," *Applied Set-Valued Analysis and Optimization*, vol. 3, pp. 193–202, 2021.

[12] S. Nadler, "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, no. 2, pp. 475–488, 1969.

[13] N. Mizoguchi and W. Takahashi, "Fixed point theorems for multivalued mappings on complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 141, no. 1, pp. 177–188, 1989.

[14] L. B. Ćirić, "Fixed point theorems for multi-valued contractions in complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 348, no. 1, pp. 499–507, 2008.

[15] L. B. Ćirić, "Multivalued nonlinear contraction mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 2716–2723, 2009.

[16] Y. Feng and S. Liu, "Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings," *Journal of Mathematical Analysis and Applications*, vol. 317, no. 1, pp. 103–112, 2006.

[17] D. Klim and D. Wardowski, "Fixed point theorems for set-valued contractions in complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 1, pp. 132–139, 2007.

[18] O. Kada, T. Suzuki, and W. Takahashi, "Nonconvex minimization theorems and fixed point theorems in complete metric spaces," *Mathematica Japonica*, vol. 44, no. 2, pp. 381–391, 1996.

[19] T. Suzuki and W. Takahashi, "Fixed point theorems and characterizations of metric completeness," *Topological Methods in Nonlinear Analysis*, vol. 8, no. 2, pp. 371–382, 1996.

[20] A. M. Alkhammash, A. A. N. Abdou, and A. Latif, "On existence of fixed points for nonlinear maps in metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 19, no. 1, pp. 89–95, 2018.

[21] S. Kaneko, W. Takahashi, C.-F. Wen, and J.-C. Yao, "Existence theorems for single-valued and multivalued mappings with w-distances in metric spaces," *Fixed Point Theory and Applications*, vol. 2016, no. 1, pp. 1–15, 2016.

[22] A. Latif and A. A. N. Abdou, "Some new weakly contractive type multimaps and fixed point results in metric spaces," *Journal of Fixed Point Theory and Applications*, vol. 2009, p. 12, Article ID 412898, 2009.

[23] A. Latif, L. Guran, and M.-F. Bota, "Ulam-Hyers stability problems and fixed points for contractive type operators on KST-spaces," *Linear and Nonlinear Analysis*, vol. 5, no. 3, pp. 379–390, 2019.

[24] W. Takahashi, N.-C. Wong, and J.-C. Yao, "Fixed point theorems for general contractive mappings with w-distances in metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 14, pp. 637–648, 2013.

[25] N. Hussain, R. Saadati, and R. P. Agrawal, "On the topology and wt-distance on metric type spaces," *Fixed Point Theory and Applications*, vol. 88, no. 1, p. 14, 2014.

[26] M. Demma, R. Saadati, and P. Vetro, "Multi-valued operators with respect wt-distance on metric type spaces," *Bulletin of the Iranian Mathematical Society*, vol. 42, no. 6, pp. 1571–1582, 2016.

[27] K. Fallahi, D. Savic, and G. Soleimani Rad, "The existence theorem for contractive mappings on wt-distance in b-metric spaces endowed with a graph and its application," *Sahand Communications in Mathematical Analysis*, vol. 13, no. 1, pp. 1–15, 2019.

[28] S. K. Ghosh and C. Nahak, "An extension of Lakzian-Rhoades results in the structure of ordered b-metric spaces via w-distance with an application," *Applied Mathematics and Computation*, vol. 378, p. 18, 2020.

[29] A. Latif, R. F. Al Subaie, and M. O. Alansari, "Fixed points of generalized multi-valued contractive mappings in metric type spaces," *Journal of Nonlinear and Variational Analysis*, vol. 6, no. 1, pp. 123–138, 2022.

[30] W. S. Du, "New existence results of best proximity points and fixed points for MT (λ)-functions with applications to differential equations,," *Linear and Nonlinear Analysis*, vol. 2, no. 2, pp. 199–213, 2016.

[31] M. Bota, A. Molnar, and C. Varga, "On Ekeland's variational principle in b-metric spaces," *Fixed Point Theory*, vol. 12, no. 2, pp. 21–28, 2011.

[32] N. V. Dung and V. T. L. Hang, "On relaxations of contraction constants and Caristi's theorem in b-metric spaces," *Journal of Fixed Point Theory and Applications*, vol. 18, no. 2, pp. 267–284, 2016.

[33] A. Latif, R. F. Al Subaie, and M. O. Alansari, "Metric fixed points for contractive type mappings and applications," *Journal of Nonlinear and Convex Analysis*, vol. 23, no. 3, pp. 501–511, 2022, In press.

[34] D. W. Boyd and J. S. W. Wong, "On nonlinear contractions," *Proceedings of the American Mathematical Society*, vol. 20, no. 2, p. 458, 1969.

[35] H. K. Pathak and N. Shahzad, "Fixed point results for set-valued contractions by altering distances in complete metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 7, pp. 2634–2641, 2009.

[36] A. Latif and A. A. N. Abdou, "Fixed points of generalized contractive maps," *Journal of Fixed Point Theory and Applications*, vol. 2009, p. 9, Article ID 487161, 2009.

[37] A. Latif and W. A. Albar, "Fixed point results in complete metric spaces," *Demonstratio Mathematica*, vol. 41, no. 1, pp. 145–150, 2008.

[38] A. Latif and A. A. N. Abdou, "Fixed points results for generalized contractive multimaps in metric spaces," *Journal of Fixed Point Theory and Applications*, vol. 2009, p. 16, Article ID 432130, 2009.