

## Research Article

# Composition Formula for Saigo Fractional Integral Operator Associated with V-Function

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In this study, we form integral formulas for Saigo's hypergeometric integral operator involving V-function. Corresponding assertions for the classical Riemann–Liouville (R-L) and Erdélyi–Kober (E-K) fractional integral operator are extrapolated. Also, by putting in the transformations of Beta and Laplace, we can establish their composition formulas. By selecting the appropriate parameter values, the V-function may be reduced to a variety of functions, including the exponential function, Mittag–Leffler, Lommel, Struve, Wright's generalized Bessel function, and Bessel and generalized hypergeometric function.

## 1. Introduction and Preliminaries

Calculus with fractional orders is a branch of mathematics that develops from typical definitions of calculus with integer orders of integral and derivative operators, just how fractional exponents develop from integer exponents. In recent years, fractional calculus has been used in different fields of technology, research, plasma physics, economics, nonlinear control theory, applied maths, and bio-engineering. The V-function plays an important role to develop solutions to critical problem in terms of fractional-order integral and differential equations. The computations of

fractional integrals and fractional derivatives involving transcendental functions of one and several variables are important because of the usefulness of their results, e.g., for evaluating differential and integral equations. The characteristics, execution, and various extensions of a number of fractional calculus operators have studied in depth by researchers (see [1–6]).

We recollect the Saigo fractional integral operator containing Gauss's hypergeometric function  ${}_2F_1(\cdot)$  as a kernel. Let  $l, m, \xi \in \mathbb{C}$  and  $x > 0$ ; then, the generalized fractional integration operators related with Gauss hypergeometric function due to Saigo [7, 8] are defined as follows:

$$(I_{0,x}^{l,m,\xi} f)(x) = \frac{x^{-l-m}}{\Gamma(l)} \int_0^x (x-z)^{l-1} {}_2F_1(l+m, -\xi; l; 1-z/x) f(z) dz, \quad (\Re(l) > 0), \quad (1)$$

$$= \frac{d^k}{dx^k} (I_{0,x}^{l+k,m-k,\xi-k} f)(x), \quad (\Re(l) \leq 0, k = [\Re(-l)] + 1), \quad (2)$$

$$(I_{x,\infty}^{l,m,\xi} f)(x) = \frac{1}{\Gamma(l)} \int_x^\infty (z-x)^{l-1} z^{-l-m} {}_2F_1(l+m, -\xi; l; 1-x/z) f(z) dz, \quad (\Re(l) > 0), \quad (3)$$

$$= (-1)^k \frac{d^k}{dx^k} (I_{x,\infty}^{l-k,m-k,\xi} f)(x), \quad (\Re(l) \leq 0, k = [\Re(-l)] + 1). \tag{4}$$

If we set  $m = 0$  in equations (1) and (3), we get the E-L fractional integral operators identified as

$$(I_{0,x}^{l,\xi} f)(x) = \frac{x^{-l-\xi}}{\Gamma(l)} \int_0^x (x-z)^{l-1} z^\xi f(z) dz, \quad \Re(l) > 0,$$

$$(I_{x,\infty}^{l,\xi} f)(x) = \frac{x^\xi}{\Gamma(l)} \int_x^\infty (z-x)^{\xi-1} z^{-\xi-l} f(z) dz, \quad \Re(l) > 0. \tag{5}$$

The operators  $I_{0,x}^{l,m,\xi}$  and  $I_{x,\infty}^{l,m,\xi}$  include, as their special case,  $m = -l$ , the fractional integral operator of (R-L) and Weyl type as

$$(I_{0,x}^{l,-l,\xi} f)(x) = (I_{0,x}^l f)(x), \quad (I_{x,\infty}^{l,-l,\xi} f)(x) = (I_{x,\infty}^l f)(x). \tag{6}$$

For the present study, power function formulas of the fractional integral operators discussed above are needed as given in the following lemmas.

**Lemma 1.** Let  $x > 0$  and  $l, m, \xi \in \mathbb{C}$  be such that  $\Re(l) > 0$ ; then, there exists the relation

$$(I_{0,x}^{l,m,\xi} z^{\zeta-1})(x) = \frac{\Gamma(\zeta)\Gamma(\zeta + \xi - m)}{\Gamma(\zeta - m)\Gamma(\zeta + \xi + l)} x^{\zeta-m-1} (\Re(\zeta) > \max\{0, \Re(m - \xi)\}), \tag{7}$$

$$(I_{x,\infty}^{l,m,\xi} t^{\zeta-1})(x) = \frac{\Gamma(1 - \zeta + m)\Gamma(1 - \zeta + \xi)}{\Gamma(1 - \zeta)\Gamma(1 - \zeta + l + m + \xi)} x^{\zeta-m-1}, (\Re(\zeta) < 1 + \min\{\Re(m), \Re(\xi)\}). \tag{8}$$

**Lemma 2.** Let  $x > 0$  and  $l, \xi, \zeta \in \mathbb{C}$  be such that  $\Re(l) > 0$ ; then, there exists the relation

$$(I_{0,x}^{l,\xi} t^{\zeta-1})(x) = \frac{\Gamma(\zeta + \xi)}{\Gamma(\zeta + \xi + l)} x^{\zeta-1}, \quad \Re(\zeta) > -\Re(\xi),$$

$$(I_{x,\infty}^{l,\xi} t^{\zeta-1})(x) = \frac{\Gamma(1 - \zeta + \xi)}{\Gamma(1 - \zeta + l + \xi)} x^{\zeta-1}, \quad (\Re(\zeta) < 1 + \Re(\xi)). \tag{9}$$

**Lemma 3.** Let  $x > 0$  and  $l, \zeta \in \mathbb{C}$  be such that  $\Re(l) > 0$ ; then, there exists the relation:

$$(I_{0,x}^l t^{\zeta-1})(x) = \frac{\Gamma(\zeta)}{\Gamma(\zeta + l)} x^{\zeta+l-1}, \quad (\Re(\zeta) > 0),$$

$$(I_{x,\infty}^l t^{\zeta-1})(x) = \frac{\Gamma(1 - \zeta + l)}{\Gamma(1 - \zeta)} x^{\zeta+l-1}, \quad (0 < \Re(\zeta) < 1 - \Re(l)). \tag{10}$$

Kumar [9] recently defined the V-function as follows:

$$V(z) = V_n^{a_\theta, d, b_\nu}(L, \beta, \kappa, \omega, y, k_\theta, A_\nu, b_u, \phi, \nu, \sigma; z)$$

$$= \lambda \sum_{n=0}^\infty \frac{(-L)^n \prod_{\theta=1}^p [(a_\theta)_{n+k_\theta}] (d + \phi n + \nu)^{-\beta} (z/2)^{n\kappa+d\omega+y}}{\prod_{\nu=1}^q [(b_\nu)_{n+A_\nu}] \prod_{u=1}^r (d)_{\phi n\sigma+g_u}}, \tag{11}$$

where

- (1)  $L, \kappa, \omega, y, \nu, \sigma, k_\theta (\theta = 1, \dots, p), A_\nu (\nu = 1, \dots, q)$ , and  $g_u (u = 1, \dots, r)$  are real numbers
- (2)  $p, q$ , and  $r$  are natural numbers
- (3)  $a_\theta, b_\nu \geq 1 (\theta = 1, \dots, p; \nu = 1, \dots, q)$
- (4)  $\phi > 0, \Re(\beta) > 0, \Re(d) > 0, z$  is a complex variable, and  $\lambda$  is an arbitrary constant
- (5) The series on the RHS of (11) converges absolutely if  $q > p$  or  $q = p$  with  $|L(z/2)^k| \leq 1$

For descriptions of the series' convergence constraints on (11) RHS, simply review [10–12]. The V-function defined by (11) is of a general character since it assimilates and applies a variety of valuable functions such as MacRobert's E-function and exponential function [13], generalized Mittag–Leffler function [14–18], Lommel's function, Struve's function, generalized Bessel function and Bessel function [19–24], generalized hypergeometric function [13, 25, 26], and unified Riemann-zeta function [27].

The purpose of this work is to evaluate the compositions of the generalized fractional integration operators (1) and (3) with the (11) V-function. Additionally, equivalent claims for the classical R-L and E-K fractional integral operators are evaluated. The conclusions mentioned in conjunction with the corollaries in theorems are undoubtedly innovative and conceptually valuable. Additionally, we derive the composition formulas for the findings stated in theorems using the Beta and Laplace transforms. Finally, as previously said, establish them as special instances on the primary result in conjunction with various special functions, the hypergeometric function, and so on.

## 2. Fractional Integration of V-Function

In this section, we develop image formulas for the V-function involving the Saigo fractional integral operator's left and right sides. The following theorems give these formulas.

**Theorem 1.** Let  $l, m, \xi, \zeta \in \mathbb{C}, L, \kappa, \omega, y, \nu, \sigma, k_\theta, A_\nu, g_u \in \mathbb{R}, p, q, r \in \mathbb{N}, a_\theta, b_\nu \geq 1, \phi > 0, \Re(\beta) > 0, \Re(d) > 0$ , and  $\lambda > 0$  is an arbitrary constant, such that  $\Re(l) > 0$  and  $\Re(\zeta) > \max[0, \Re(m - l)]$ . Then, the subsequent Saigo hypergeometric fractional integral  $I_{0,x}^{l,m,\xi}$  of V-function holds true:

$$\begin{aligned}
 & \left( I_{0,x}^{l,m,\xi} \left[ z^{\zeta-1} V_n^{a_\theta,d,b_\nu} (L, \beta, \kappa, \omega, y, k_\theta, A_\nu, g_u, \phi, \nu, \sigma; z^\tau) \right] \right) (x) \\
 &= \lambda x^{\zeta-m-1} \sum_{n=0}^{\infty} \frac{(-L)^n \prod_{\theta=1}^p [(a_\theta)_{n+k_\theta}] (d + \phi n + \nu)^{-\beta} (x^\tau/2)^{nk+d\omega+y}}{\prod_{\nu=1}^q [(b_\nu)_{n+A_\nu}] \prod_{u=1}^r (d)_{\phi n\sigma+g_u}} \\
 & \times \frac{\Gamma(\zeta + \tau(n\kappa + d\omega + y))\Gamma(\zeta + \tau(n\kappa + d\omega + y) + \xi - m)}{\Gamma(\zeta + \tau(n\kappa + d\omega + q) - m)\Gamma(\zeta + \tau(n\kappa + d\omega + y) + \xi + l)}.
 \end{aligned} \tag{12}$$

*Proof.* For convenience, using definitions (1) and (11), we denote the L. H. S. of the result (12) by  $\mathfrak{S}_1$ ; by changing the order of summation and integration, we obtain

$$\begin{aligned}
 \mathfrak{S}_1 &= \left( I_{0,x}^{l,m,\xi} \left[ z^{\zeta-1} V_n^{a_\theta,d,b_\nu} (L, \beta, \kappa, \omega, y, k_\theta, A_\nu, g_u, \phi, \nu, \sigma; z^\tau) \right] \right) (x) \\
 &= \left( I_{0,x}^{l,m,\xi} \left[ z^{\zeta-1} \lambda \sum_{n=0}^{\infty} \frac{(-L)^n \prod_{\theta=1}^p [(a_\theta)_{n+k_\theta}] (d + \phi n + \nu)^{-\beta} (z^\tau/2)^{nk+d\omega+y}}{\prod_{\nu=1}^q [(b_\nu)_{n+A_\nu}] \prod_{u=1}^r (d)_{\phi n\sigma+g_u}} \right] \right) (x) \\
 &= \lambda \sum_{n=0}^{\infty} \frac{(-L)^n \prod_{\theta=1}^p [(a_\theta)_{n+k_\theta}] (d + \phi n + \nu)^{-\beta}}{\prod_{\nu=1}^q [(b_\nu)_{n+A_\nu}] \prod_{u=1}^r (d)_{\phi n\sigma+g_u}} (2)^{nk+d\omega+y} \\
 & \times \left( I_{0,x}^{l,m,\xi} \left[ z^{\zeta+\tau(n\kappa+d\omega+y)-1} \right] \right) (x).
 \end{aligned} \tag{13}$$

By applying relation (7) in (13), we get R.H.S. of (12).  $\square$

**Theorem 2.** Let  $l, m, \xi, \zeta \in \mathbb{C}$ ,  $L, \kappa, \omega, y, \nu, \sigma, k_\theta, A_\nu, g_u \in \mathfrak{R}$ ,  $p, q, r \in \mathbb{N}$ ,  $a_\theta, b_\nu \geq 1$ ,  $\phi > 0$ ,  $\Re(\beta) > 0$ ,  $\Re(d) > 0$ , and  $\lambda > 0$  is

an arbitrary constant, such that  $\Re(l) > 0$  and  $\Re(\zeta) < 1 + \min\{\Re(m), \Re(\xi)\}$ . Then, the subsequent Saigo hypergeometric fractional integral  $I_{x,\infty}^{l,m,\xi}$  of V-function holds true:

$$\begin{aligned}
 & \left( I_{x,\infty}^{l,m,\xi} \left[ z^{\zeta-1} V_n^{a_\theta,d,b_\nu} (L, \beta, \kappa, \omega, y, k_\theta, A_\nu, g_u, \phi, \nu, \sigma; z^{-\tau}) \right] \right) (x) \\
 &= x^{\zeta-m-1} \lambda \sum_{n=0}^{\infty} \frac{(-L)^n \prod_{\theta=1}^p [(a_\theta)_{n+k_\theta}] (d + \phi n + \nu)^{-\beta} (x^{-\tau}/2)^{nk+d\omega+y}}{\prod_{\nu=1}^q [(b_\nu)_{n+A_\nu}] \prod_{u=1}^r (d)_{\phi n\sigma+g_u}} \\
 & \times \frac{\Gamma(1 - \zeta + \tau(n\kappa + d\omega + y) + m)\Gamma(1 - \zeta + \tau(n\kappa + d\omega + y) + \xi)}{\Gamma(1 - \zeta + \tau(n\kappa + d\omega + y))\Gamma(1 - \zeta + \tau(n\kappa + d\omega + y) + \xi + l + m)}.
 \end{aligned} \tag{14}$$

*Proof.* To derive (14), we denote (14) by  $\mathfrak{S}_2$  to L.H.S.; using definitions (3) and (11) and by changing the order of summation and integration, we obtain

$$\begin{aligned}
 \mathfrak{S}_2 &= \left( I_{x,\infty}^{l,m,\xi} \left[ z^{\zeta-1} V_n^{a_\theta,d,b_\nu} (L, \beta, \kappa, \omega, y, k_\theta, A_\nu, g_u, \phi, \nu, \sigma; z^{-\tau}) \right] \right) (x) \\
 &= \left( I_{x,\infty}^{l,m,\xi} \left[ z^{\zeta-1} \lambda \sum_{n=0}^{\infty} \frac{(-L)^n \prod_{\theta=1}^p [(a_\theta)_{n+k_\theta}] (d + \phi n + \nu)^{-\beta} (z^{-\tau}/2)^{nk+d\omega+y}}{\prod_{\nu=1}^q [(b_\nu)_{n+A_\nu}] \prod_{u=1}^r (d)_{\phi n\sigma+g_u}} \right] \right) (x) \\
 &= \lambda \sum_{n=0}^{\infty} \frac{(-L)^n \prod_{\theta=1}^p [(a_\theta)_{n+k_\theta}] (d + \phi n + \nu)^{-\beta}}{\prod_{\nu=1}^q [(b_\nu)_{n+A_\nu}] \prod_{u=1}^r (d)_{\phi n\sigma+g_u}} (2)^{nk+d\omega+y} \\
 & \times \left( I_{x,\infty}^{l,m,\xi} \left[ z^{\zeta-\tau(n\kappa+d\omega+y)-1} \right] \right) (x).
 \end{aligned} \tag{15}$$

By applying relation (8) in (15), we get the desired result (14).  $\square$

In relation, by setting  $m = -l$  and  $m = 0$ , respectively, we work out the fractional integral formulas for the classical R-L and E-K fractional integral operators, which are stated by Corollaries 1 to 4.

**Corollary 1.** Let  $l, \zeta \in \mathbb{C}$ ,  $L, \kappa, \omega, y, \nu, \sigma, k_\theta, A_\nu, g_u \in \mathfrak{R}$ ,  $p, q, r \in \mathbb{N}$ ,  $a_\theta, b_\nu \geq 1$ ,  $\phi > 0$ ,  $\Re(\beta) > 0$ ,  $\Re(d) > 0$ , and  $\lambda > 0$  is an arbitrary constant, such that  $\Re(l) > 0$  and  $\Re(\zeta) > 0$ . Thus, the corresponding R-L fractional integral  $I_{0,x}^l$  of V-function holds true:

$$\begin{aligned} & \left( I_{0,x}^l \left[ z^{\zeta-1} V_n^{a_\theta, d, b_\nu} (L, \beta, \kappa, \omega, y, k_\theta, A_\nu, g_u, \phi, \nu, \sigma; z^\tau) \right] \right) (x) \\ &= x^{\zeta+l-1} \lambda \sum_{n=0}^{\infty} \frac{(-L)^n \prod_{\theta=1}^p [(a_\theta)_{n+k_\theta}] (d + \phi n + \nu)^{-\beta} (x^\tau/2)^{n\kappa+d\omega+y}}{\prod_{\nu=1}^q [(b_\nu)_{n+A_\nu}] \prod_{u=1}^r [(d)_{\phi n\sigma+g_u}]} \\ & \quad \times \frac{\Gamma(\zeta + \tau(n\kappa + d\omega + y))}{\Gamma(\zeta + \tau(n\kappa + d\omega + y) + l)}. \end{aligned} \tag{16}$$

**Corollary 2.** Let  $l, \xi, \zeta \in \mathbb{C}$ ,  $L, \kappa, \omega, y, \nu, \sigma, k_\theta, A_\nu, g_u \in \mathfrak{R}$ ,  $p, q, r \in \mathbb{N}$ ,  $a_\theta, b_\nu \geq 1$ ,  $\phi > 0$ ,  $\Re(\beta) > 0$ ,  $\Re(d) > 0$ , and  $\lambda > 0$  is an arbitrary constant, such that  $\Re(l) > 0$  and

$\Re(\zeta) > -\Re(\xi)$ . Therefore, the subsequent E-K fractional integral  $I_{0,x}^{l,\xi}$  of V-function holds true:

$$\begin{aligned} & \left( I_{0,x}^{l,\xi} \left[ z^{\zeta-1} V_n^{a_\theta, d, b_\nu} (L, \beta, \kappa, \omega, y, k_\theta, A_\nu, g_u, \phi, \nu, \sigma; z^\tau) \right] \right) (x) \\ &= x^{\zeta-1} \lambda \sum_{n=0}^{\infty} \frac{(-L)^n \prod_{\theta=1}^p [(a_\theta)_{n+k_\theta}] (d + \phi n + \nu)^{-\beta} (x^\tau/2)^{n\kappa+d\omega+y}}{\prod_{\nu=1}^q [(b_\nu)_{n+A_\nu}] \prod_{u=1}^r [(d)_{\phi n\sigma+g_u}]} \\ & \quad \times \frac{\Gamma(\zeta + \tau(n\kappa + d\omega + y) + \xi)}{\Gamma(\zeta + \tau(n\kappa + d\omega + y) + l)}. \end{aligned} \tag{17}$$

**Corollary 3.** Let  $l, \zeta \in \mathbb{C}$ ,  $L, \kappa, \omega, y, \nu, \sigma, k_\theta, A_\nu, g_u \in \mathfrak{R}$ ,  $p, q, r \in \mathbb{N}$ ,  $a_\theta, b_\nu \geq 1$ ,  $\phi > 0$ ,  $\Re(\beta) > 0$ ,  $\Re(d) > 0$ , and  $\lambda > 0$  is an arbitrary constant, such that  $\Re(l) > 0$  and

$0 < \Re(\zeta) < 1 - \Re(l)$ . Therefore, the subsequent R-L fractional integral  $I_{x,\infty}^l$  of V-function holds true:

$$\begin{aligned} & \left( I_{x,\infty}^l \left[ z^{\zeta-1} V_n^{a_\theta, d, b_\nu} (L, \beta, \kappa, \omega, y, k_\theta, A_\nu, g_u, \phi, \nu, \sigma; z^{-\tau}) \right] \right) (x) \\ &= x^{\zeta+l-1} \lambda \sum_{n=0}^{\infty} \frac{(-L)^n \prod_{\theta=1}^p [(a_\theta)_{n+k_\theta}] (d + \phi n + \nu)^{-\beta} (x^{-\tau}/2)^{n\kappa+d\omega+y}}{\prod_{\nu=1}^q [(b_\nu)_{n+A_\nu}] \prod_{u=1}^r [(d)_{\phi n\sigma+g_u}]} \\ & \quad \times \frac{\Gamma(1 - \zeta + \tau(n\kappa + d\omega + y) - l)}{\Gamma(1 - \zeta + \tau(n\kappa + d\omega + y))}. \end{aligned} \tag{18}$$

**Corollary 4.** Let  $l, \xi, \zeta \in \mathbb{C}$ ,  $L, \kappa, \omega, y, \nu, \sigma, k_\theta, A_\nu, g_u \in \mathfrak{R}$ ,  $p, q, r \in \mathbb{N}$ ,  $a_\theta, b_\nu \geq 1$ ,  $\phi > 0$ ,  $\Re(\beta) > 0$ ,  $\Re(d) > 0$ , and  $\lambda > 0$  is an arbitrary constant, such that  $\Re(l) > 0$  and

$\Re(\zeta) > 1 + \Re(\xi)$ . Therefore, the subsequent E-K fractional integral  $I_{x,\infty}^l$  of V-function holds true:

$$\begin{aligned} & \left( I_{x,\infty}^{l,\xi} \left[ z^{\zeta-1} V_n^{a_\theta, d, b_\nu} (L, \beta, \kappa, \omega, y, k_\theta, A_\nu, g_u, \phi, \nu, \sigma; z^{-\tau}) \right] \right) (x) \\ &= x^{\zeta-1} \lambda \sum_{n=0}^{\infty} \frac{(-L)^n \prod_{\theta=1}^p [(a_\theta)_{n+k_\theta}] (d + \phi n + \nu)^{-\beta} (x^{-\tau}/2)^{n\kappa+d\omega+y}}{\prod_{\nu=1}^q [(b_\nu)_{n+A_\nu}] \prod_{u=1}^r [(d)_{\phi n\sigma+g_u}]} \\ & \quad \times \frac{\Gamma(1 - \zeta + \tau(n\kappa + d\omega + y) + \xi)}{\Gamma(1 - \zeta + \tau(n\kappa + d\omega + y) + \xi + l)}. \end{aligned} \tag{19}$$

### 3. Beta Transform of the Composition Formulas (12) and (14)

In this section, we develop some theorem that includes the results obtained in the previous section concerning the integral transformation.

In this regard, we recall the definition of Beta function [28] as follows:

$$B(f(t); g, h) = \int_0^1 t^{g-1} (1-t)^{h-1} f(t) dt, (\Re(g) > 0, \Re(h) > 0). \tag{20}$$

**Theorem 3.** Let  $l, m, \xi, \zeta \in \mathbb{C}$ ,  $L, \kappa, \omega, y, \nu, \sigma, k_\theta, A_\nu, g_u \in \Re$ ,  $p, q, r \in \mathbb{N}$ ,  $a_\theta, b_\nu \geq 1$ ,  $\Re(g) > 0$ ,  $\Re(h) > 0$ ,  $\phi > 0$ ,  $\Re(\beta) > 0$ ,  $\Re(d) > 0$ , and  $\lambda > 0$  is an arbitrary constant, such that  $\Re(l) > 0$  and  $\Re(\zeta) > \max[0, \Re(m-l)]$ . Therefore, the preceding fractional integral holds true:

$$\begin{aligned} & B\left(\left(I_{0,x}^{l,m,\xi} \left\{ z^{\zeta-1} V_n^{a_\theta,d,b_\nu} (L, \beta, \kappa, \omega, y, k_\theta, A_\nu, g_u, \phi, \nu, \sigma; tz^\tau) \right\}\right)(x); g, h\right) \\ &= x^{\zeta-m-1} \lambda \sum_{n=0}^{\infty} \frac{(-L)^n \prod_{\theta=1}^p [(a_\theta)_{n+k_\theta}] (d + \phi n + \nu)^{-\beta} (x^\tau t/2)^{n\kappa+d\omega+y}}{\prod_{\nu=1}^q [(b_\nu)_{n+A_\nu}] \prod_{u=1}^r [(d)_{\phi n\sigma+g_u}]} \\ &\times \frac{\Gamma(\zeta + \tau\wp)\Gamma(\zeta + \tau\wp + \xi - m)\Gamma(b + \wp)\Gamma(a)}{\Gamma(\zeta + \tau\wp - m)\Gamma(\zeta + \tau\wp + \xi + l)\Gamma(b + \wp + h)}, \end{aligned} \tag{21}$$

where  $\wp = n\kappa + d\omega + y$ .

*Proof.* On using (20), the L.H.S. of assertion (21) leads to

$$B\left(\left(I_{0,x}^{l,m,\xi} \left\{ z^{\zeta-1} \lambda \sum_{n=0}^{\infty} \frac{(-L)^n \prod_{\theta=1}^p [(a_\theta)_{n+k_\theta}] (d + \phi n + \nu)^{-\beta} A}{\prod_{\nu=1}^q [(b_\nu)_{n+A_\nu}] \prod_{u=1}^r [(d)_{\phi n\sigma+g_u}]} \right\}\right)(x); g, h\right), \tag{22}$$

where

$$\begin{aligned} A &= (tz^\tau/2)^{n\kappa+d\omega+y} \\ &= \int_0^1 y^{g-1} (1-y)^{h-1} \\ &\times \left( I_{0,x}^{l,m,\xi} \left[ z^{\zeta-1} \lambda \sum_{n=0}^{\infty} \frac{(-L)^n \prod_{\theta=1}^p [(a_\theta)_{n+k_\theta}] (d + \phi n + \nu)^{-\beta} A}{\prod_{\nu=1}^q [(b_\nu)_{n+A_\nu}] \prod_{u=1}^r [(d)_{\phi n\sigma+g_u}]} \right] \right)(x) dy. \end{aligned} \tag{23}$$

By interchanging the order of integration of the above term, it is reduced to

$$\begin{aligned} &= \lambda \sum_{n=0}^{\infty} \frac{(-L)^n \prod_{\theta=1}^p [(a_\theta)_{n+k_\theta}] (d + \phi n + \nu)^{-\beta}}{\prod_{\nu=1}^q [(b_\nu)_{n+A_\nu}] \prod_{u=1}^r [(d)_{\phi n\sigma+g_u}] (2)^\wp} \\ &\times \left( I_{0,x}^{l,m,\xi} \left[ z^{\zeta+\tau\wp-1} \right] \right)(x) \int_0^1 t^{g+\tau\wp-1} (1-t)^{h-1} dt. \end{aligned} \tag{24}$$

By using the definition of Beta transform and (7), we obtain the R.H.S. of (21).  $\square$

**Theorem 4.** Let  $l, m, \xi, \zeta \in \mathbb{C}$ ,  $L, \kappa, \omega, y, \nu, \sigma, k_\theta, A_\nu, g_u \in \Re$ ,  $p, q, r \in \mathbb{N}$ ,  $a_\theta, b_\nu \geq 1$ ,  $\phi > 0$ ,  $\Re(\beta) > 0$ ,  $\Re(d) > 0$ ,  $\Re(g) > 0$ ,  $\Re(h) > 0$ , and  $\lambda > 0$  is an arbitrary constant, such that  $\Re(l) > 0$  and  $\Re(\zeta) < 1 + \min\{\Re(m), \Re(\xi)\}$ . Therefore, the preceding fractional integral holds true:

$$\begin{aligned}
& B\left(I_{x,\infty}^{l,m,\xi}\left[z^{\zeta-1}V_n^{a_\theta,d,b_\nu}(L,\beta,\kappa,\omega,y,k_\theta,A_\nu,g_u,\phi,\nu,\sigma;yz^{-\tau})\right]\right)(x) \\
&= x^{\zeta-m-1}\lambda\sum_{n=0}^{\infty}\frac{(-L)^n\prod_{\theta=1}^p[(a_\theta)_{n+k_\theta}](d+\phi n+\nu)^{-\beta}(x^{-\tau}/2)^\varrho\Gamma(a)}{\prod_{\nu=1}^q[(b_\nu)_{n+A_\nu}]\prod_{u=1}^r[(d)_{\phi n\sigma+g_u}]\Gamma(b+\varrho+h)} \\
&\quad \times\frac{\Gamma(1-\zeta+\tau\varrho+m)\Gamma(1-\zeta+\tau\varrho+\xi)\Gamma(b+\varrho)}{\Gamma(1-\zeta+\tau\varrho)\Gamma(1-\zeta+\tau\varrho+\xi+l+m)}.
\end{aligned} \tag{25}$$

*Proof.* The proof of Theorem 4 is similar manner of Theorem 3.  $\square$

#### 4. Laplace Transform of the Composition Formulas (12) and (14)

In this section, we develop two theorems, including the Laplace transform, involving the result obtained in Section 2 concerning the integral transform.

The Laplace transform [28] of  $f(z)$  is defined as

$$F(s) = L(f(z); s) = \int_0^\infty e^{-st} f(t) dt, \quad \Re(s) > 0. \tag{26}$$

**Theorem 5.** Let  $l, m, \xi, \zeta \in \mathbb{C}$ ,  $L, \kappa, \omega, y, \nu, \sigma, k_\theta, A_\nu, g_u \in \mathfrak{R}$ ,  $p, q, r \in \mathbb{N}$ ,  $a_\theta, b_\nu \geq 1$ ,  $\Re(g) > 0$ ,  $\Re(s) > 0$ ,  $\phi > 0$ ,  $\Re(\beta) > 0$ ,  $\Re(d) > 0$ , and  $\lambda > 0$  is an arbitrary constant, such that  $\Re(l) > 0$  and  $\Re(\zeta) > \max[0, \Re(m-l)]$ . Then, the subsequent fractional integral holds true:

$$\begin{aligned}
& L\left(y^{g-1}\left(I_{0,x}^{l,m,\xi}\left\{z^{\zeta-1}V_n^{a_\theta,d,b_\nu}(L,\beta,\kappa,\omega,y,k_\theta,A_\nu,g_u,\phi,\nu,\sigma;yz^\tau)\right\}\right)\right)(x) \\
&= \frac{x^{\zeta-m-1}}{s^g}\lambda\sum_{n=0}^{\infty}\frac{(-L)^n\prod_{\theta=1}^p[(a_\theta)_{n+k_\theta}](d+\phi n+\nu)^{-\beta}((x/s)^\tau/2)^\varrho}{\prod_{\nu=1}^q[(b_\nu)_{n+A_\nu}]\prod_{u=1}^r[(d)_{\phi n\sigma+g_u}]} \\
&\quad \times\frac{\Gamma(\zeta+\tau\varrho)\Gamma(\zeta+\tau\varrho+\xi-m)\Gamma(b+\tau\varrho)}{\Gamma(\zeta+\tau\varrho-m)\Gamma(\zeta+\tau\varrho+\xi+l)}.
\end{aligned} \tag{27}$$

*Proof.* On using (26), the left-hand side of assertion (27), leads to

$$\begin{aligned}
& L\left(\left(y^{g-1}I_{0,x}^{l,m,\xi}\left\{z^{\zeta-1}\lambda\sum_{n=0}^{\infty}\frac{(-L)^n\prod_{\theta=1}^p[(a_\theta)_{n+k_\theta}](d+\phi n+\nu)^{-\beta}(yz^\tau/2)^\varrho}{\prod_{\nu=1}^q[(b_\nu)_{n+A_\nu}]\prod_{u=1}^r[(d)_{\phi n\sigma+g_u}]}\right\}\right)(x)\right) \\
&= \int_0^\infty \lambda\sum_{n=0}^{\infty}\frac{(-L)^n\prod_{\theta=1}^p[(a_\theta)_{n+k_\theta}](d+\phi n+\nu)^{-\beta}}{\prod_{\nu=1}^q[(b_\nu)_{n+A_\nu}]\prod_{u=1}^r[(d)_{\phi n\sigma+g_u}]}(2)^\varrho \\
&\quad \times\left(I_{0,x}^{l,m,\xi}\left[z^{\zeta+\tau(\varrho)-1}\right]\right)(x)y^{g+\tau(\varrho)-1}e^{-sy}dy.
\end{aligned} \tag{28}$$

By interchanging the order of integration and summation, the right-hand side of (28) reduces to

$$\begin{aligned}
& \lambda\sum_{n=0}^{\infty}\frac{(-L)^n\prod_{\theta=1}^p[(a_\theta)_{n+k_\theta}](d+\phi n+\nu)^{-\beta}}{\prod_{\nu=1}^q[(b_\nu)_{n+A_\nu}]\prod_{u=1}^r[(d)_{\phi n\sigma+g_u}]}(2)^\varrho \\
&\quad \times\left(I_{0,x}^{l,m,\xi}\left[z^{\zeta+\tau(\varrho)-1}\right]\right)(x)\int_0^\infty y^{g+\tau(\varrho)-1}e^{-sy}dy.
\end{aligned} \tag{29}$$

Calculating the integral and using (7), we get the R.H.S. of (27).  $\square$

**Theorem 6.** Let  $l, m, \xi, \zeta \in \mathbb{C}$ ,  $L, \kappa, \omega, y, \nu, \sigma, k_\theta, A_\nu, g_u \in \mathfrak{R}$ ,  $p, q, r \in \mathbb{N}$ ,  $a_\theta, b_\nu \geq 1$ ,  $\phi > 0$ ,  $\Re(\beta) > 0$ ,  $\Re(d) > 0$ ,  $\Re(g) > 0$ ,  $\Re(s) > 0$ , and  $\lambda > 0$  is an arbitrary constant, such that  $\Re(l) > 0$  and  $\Re(\zeta) < 1 + \min\{\Re(m), \Re(\xi)\}$ . Then, the subsequent fractional integral holds true:

$$\begin{aligned}
 &L\left(y^{g-1}\left(I_{x,\infty}^{l,m,\xi}\left[z^{\zeta-1}V_n^{a_\theta,d,b_\nu}(L,\beta,\kappa,\omega,y,k_\theta,A_\nu,g_u,\phi,\nu,\sigma;yz^{-\tau})\right]\right)(x)\right) \\
 &= \frac{x^{\zeta-m-1}}{s^g} \lambda \sum_{n=0}^{\infty} \frac{(-L)^n \prod_{\theta=1}^p [(a_\theta)_{n+k_\theta}] (d+\phi n+\nu)^{-\beta} (sx^{-\tau}/2)^{\wp}}{\prod_{\nu=1}^q [(b_\nu)_{n+A_\nu}] \prod_{u=1}^r [(d)_{\phi n\sigma+g_u}]} \\
 &\quad \times \frac{\Gamma(1-\zeta+\tau\wp+m)\Gamma(1-\zeta+\tau\wp+\xi)\Gamma(b+\wp)}{\Gamma(1-\zeta+\tau\wp)\Gamma(1-\zeta+\tau+\wp+\xi+l+m)}.
 \end{aligned} \tag{30}$$

*Proof.* The proof of Theorem 6 is similar manner of Theorem 5.  $\square$

### 5. Special Cases

In this section, we develop some new result as special cases of V-Function by selecting the appropriate parameter values in (12) and (14); we have some important results regarding the other special functions such as exponential function, Mittag-Leffler function, Lommel’s function, Struve’s function, Wright’s generalized Bessel function, Bessel function, and generalized hypergeometric function, given in the following corollaries:

(1) If we set  $d = 1, u = 1, L = -2, p = P, q = Q, \kappa = 1, \beta = 1, y = 0, \omega = 0, k_\theta = 0, A_\nu = 0, g_1 = -1, \phi = 1, \nu = -1, \sigma = 1,$  and  $\lambda = 1$  in (12) and (14), the V-function reduces to generalized hypergeometric function [13], using the result from equations (12) and (13), page 34, [29] as

$$(\mathfrak{F})_{\delta n} = \delta^{n\delta} \prod_{j=1}^{\delta} \left[ \frac{\mathfrak{F} + j - 1}{\delta} \right]_n. \tag{31}$$

under the condition stated in Theorems 1 and 2, respectively, as

$$\begin{aligned}
 &\left(I_{0,x}^{l,m,\xi}\left[z^{\zeta-1}{}_pF_Q(a_p; b_Q; z^\tau)\right]\right)(x) = x^{\zeta-m-1} \frac{\Gamma(\zeta)\Gamma(\zeta+\xi-m)}{\Gamma(\zeta-m)\Gamma(\zeta+\xi+l)} \\
 &\quad \times {}_{P+2\tau}F_{Q+2\tau} \left[ \begin{matrix} a_1, a_2, \dots, a_p, \frac{\zeta}{\tau}, \frac{\zeta+1}{\tau}, \dots, \frac{\zeta+\tau-1}{\tau}, \\ b_1, b_2, \dots, b_Q, \frac{\zeta-m}{\tau}, \frac{\zeta-m-1}{\tau}, \dots, \frac{\zeta-m+\tau-1}{\tau}, \\ \frac{\zeta+\xi-m}{\tau}, \frac{\zeta+\xi-m+1}{\tau}, \dots, \frac{\zeta+\xi-m+\tau-1}{\tau} \\ \frac{\zeta+\xi+l}{\tau}, \frac{\zeta+\xi+l-1}{\tau}, \dots, \frac{\zeta+\xi+l+\tau-1}{\tau} \end{matrix} \middle| x^\tau \right],
 \end{aligned} \tag{32}$$

$$\begin{aligned}
 &\left(I_{x,\infty}^{l,m,\xi}\left[z^{\zeta-1}{}_pF_Q(a_p; b_Q; z^{-\tau})\right]\right)(x) = x^{\zeta-m-1} \frac{\Gamma(1-\zeta+m)\Gamma(1-\zeta+\xi)}{\Gamma(1-\zeta)\Gamma(1-\zeta+\xi+l+m)} \\
 &\quad \times {}_{P+2\tau}F_{Q+2\tau} \left[ \begin{matrix} a_1, a_2, \dots, a_p, \frac{1-\zeta+m}{\tau}, \frac{2-\zeta+m}{\tau}, \dots, \frac{\tau-\zeta+m}{\tau}, \\ b_1, b_2, \dots, b_Q, \frac{1-\zeta}{\tau}, \frac{2-\zeta}{\tau}, \dots, \frac{\tau-\zeta}{\tau}, \\ \frac{1-\zeta+\xi}{\tau}, \frac{2-\zeta+\xi}{\tau}, \dots, \frac{\xi-\zeta+\tau}{\tau} \\ \frac{1-\zeta+\xi+l+m}{\tau}, \frac{2-\zeta+\xi+l+m}{\tau}, \dots, \frac{\xi-\zeta+l+m+\tau}{\tau} \end{matrix} \middle| x^{-\tau} \right].
 \end{aligned} \tag{33}$$

(2) If we take  $u = 1, \tau = 1, \theta = 1, \nu = 2, a_1 = 1, b_1 = 1, b_2 = 1, L = 1, \beta = 1, \kappa = 2, \omega = 1, y = 0, k_1 = 0, a_1 = 0, a_2 = 0, b_1 = 0, \phi = 1, \nu = 0, \sigma = 1,$  and  $\lambda = 1/\Gamma(d)$  in (12) and (14), the V-function reduces to Bessel

function [20], applying the result from equations (12) and (33), page 38, [29] as

$$(\theta)_{2n} = 2^{2n} (\theta/2)_n \left\{ \frac{(\theta+1)}{2} \right\}_n, \tag{34}$$

under the condition stated in Theorems 1 and 2, respectively, as below:

$$\begin{aligned} \left( I_{0,x}^{l,m,\xi} \left[ z^{\zeta-1} J_d(z) \right] \right) (x) &= \frac{\Gamma(\zeta+d)\Gamma(\zeta+d+\xi-m)x^{\zeta+d-m-1}}{\Gamma(\zeta+d-m)\Gamma(\zeta+d+\xi+l)\Gamma(d+1)2^d} \\ &\times {}_4F_5 \left[ \begin{matrix} \frac{\zeta+d}{2}, \frac{\zeta+d+1}{2}, \frac{\zeta+d+\xi-m}{2}, \frac{\zeta+d+\xi-m+1}{2} \\ d+1, \frac{\zeta+d-m}{2}, \frac{\zeta+d-m+1}{2}, \frac{\zeta+d+\xi+l}{2}, \frac{\zeta+d+\xi+l+1}{2} \end{matrix} \middle| -\frac{x^2}{4} \right], \end{aligned} \tag{35}$$

$$\begin{aligned} \left( I_{x,\infty}^{l,m,\xi} \left[ z^{\zeta-1} J_d\left(\frac{1}{z}\right) \right] \right) (x) &= \frac{\Gamma(m-\zeta+d+1)\Gamma(\xi-\zeta+d+1)x^{\zeta-d-m-1}}{\Gamma(1-\zeta+d)\Gamma(l+m+\xi-\zeta+d+1)\Gamma(d+1)2^d} \\ &\times {}_4F_5 \left[ \begin{matrix} \frac{1-\zeta+\xi+d}{2}, \frac{\xi-\zeta+d+2}{2}, \frac{1-\zeta+d+m}{2}, \frac{m-\zeta+d+2}{2} \\ d+1, \frac{1-\zeta+d}{2}, \frac{2-\zeta+d}{2}, \frac{l+m+\xi-\zeta+d+1}{2}, \frac{l+m+\xi-\zeta+d+2}{2} \end{matrix} \middle| -\frac{1}{4x^2} \right]. \end{aligned} \tag{36}$$

(3) If we take  $\theta = 1, \nu = 2, u = 1, a_1 = 1, b_1 = 1, b_2 = 1, L = 2, \beta = 1, \kappa = 1, \omega = 0, \gamma = 0, k_1 = 0, A_1 = 0, A_2 = 0, g_1 = 0, \nu = 0, \sigma = 1$ , and  $\lambda = 1/\Gamma(d)$  in (12) and (14), the V-function reduces to Wright's

generalized Bessel function [20], using result (31) under the condition stated in Theorems 1 and 2, respectively:

$$\begin{aligned} \left( I_{0,x}^{l,m,\xi} \left[ z^{\zeta-1} J_d^\phi(z^\tau) \right] \right) (x) &= x^{\zeta-m-1} \frac{\Gamma(\zeta)\Gamma(\zeta+\xi-m)}{\Gamma(d+1)\Gamma(\zeta-m)\Gamma(\zeta+\xi+l)} \\ &\times {}_{2\tau}F_{\phi+2\tau} \left[ \begin{matrix} \frac{\zeta}{\tau}, \frac{\zeta+1}{\tau}, \dots, \frac{\zeta+\tau-1}{\tau} \\ d+1, \dots, \frac{d+\phi}{\phi}, \frac{\zeta-m}{\tau}, \frac{\zeta-m-1}{\tau}, \dots, \frac{\zeta-m+\tau-1}{\tau} \end{matrix} \right] \\ &\times \left[ \begin{matrix} \frac{\zeta+\xi-m}{\tau}, \frac{\zeta+\xi-m+1}{\tau}, \dots, \frac{\zeta+\xi-m+\tau-1}{\tau} \\ \frac{\zeta+\xi+l}{\tau}, \frac{\zeta+\xi+l-1}{\tau}, \dots, \frac{\zeta+\xi+l+\tau-1}{\tau} \end{matrix} \middle| \frac{x^\tau}{\phi^\phi} \right], \end{aligned} \tag{37}$$

$$\begin{aligned} \left( I_{x,\infty}^{l,m,\xi} \left[ z^{\zeta-1} J_d^\phi(z^{-\tau}) \right] \right) (x) &= x^{\zeta-m-1} \frac{\Gamma(1-\zeta+m)\Gamma(1-\zeta+\xi)}{\Gamma(d+1)\Gamma(1-\zeta)\Gamma(1-\zeta+\xi+l+m)} \\ &\times {}_{2\tau}F_{\phi+2\tau} \left[ \begin{matrix} \frac{1-\zeta+m}{\tau}, \frac{2-\zeta+m}{\tau}, \dots, \frac{\tau-\zeta+m}{\tau} \\ d+1, \dots, \frac{d+\phi}{\phi}, \frac{1-\zeta}{\tau}, \frac{2-\zeta}{\tau}, \dots, \frac{\tau-\zeta}{\tau} \end{matrix} \right] \\ &\times \left[ \begin{matrix} \frac{1-\zeta+\xi}{\tau}, \frac{2-\zeta+\xi}{\tau}, \dots, \frac{\tau-\zeta+\xi}{\tau} \\ \frac{1-\zeta+\xi+l+m}{\tau}, \frac{2-\zeta+\xi+l+m}{\tau}, \dots, \frac{\tau-\zeta+\xi+l+m}{\tau} \end{matrix} \middle| \frac{x^{-\tau}}{\phi^\phi} \right]. \end{aligned} \tag{38}$$



(4) If we take  $u = 1, \tau = 1, \theta = 1, \nu = 2, a_1 = 1, b_1 = 3/2, b_2 = 1, L = 1, \beta = 1, \kappa = 2, \omega = 1, y = 1, k_1 = 0, A_1 = 0, A_2 = 0, g_1 = 1/2, \phi = 1, \nu = 1/2, \sigma = 1,$  and  $\lambda = 1/\{\Gamma(d)\Gamma(3/2)\}$  in (12) and (14), the V-function

reduces to Struve's function [20], using the result from (34) under the condition stated in Theorems 1 and 2, respectively, as below:

$$\begin{aligned} (I_{0,x}^{l,m,\xi} [z^{\zeta-1} H_d(z)])(x) &= \frac{\Gamma(\zeta + d + 1)\Gamma(\zeta + d + \xi - m + 1)x^{\zeta+d-m}}{\Gamma(\zeta + d - m + 1)\Gamma(\zeta + d + \xi + l + 1)\Gamma(d + 3/2)\Gamma(3/2)2^{d+1}} \\ &\times {}_5F_6 \left[ \begin{matrix} 1, \frac{\zeta + d + 1}{2}, \frac{\zeta + d + 2}{2}, \frac{\zeta + d + \xi - m + 1}{2}, \frac{\zeta + d + \xi - m + 2}{2} \\ \frac{3}{2}, d + \frac{3}{2}, \frac{\zeta + d - m + 1}{2}, \frac{\zeta + d - m + 2}{2}, \frac{\zeta + d + \xi + l + 1}{2}, \frac{\zeta + d + \xi + l + 2}{2} \end{matrix} \middle| -\frac{x^2}{4} \right], \end{aligned} \tag{39}$$

$$\begin{aligned} (I_{x,\infty}^{l,m,\xi} [z^{\zeta-1} H_d(1/z)])(x) &= \frac{\Gamma(m - \zeta + d + 2)\Gamma(\xi - \zeta + d + 2)x^{\zeta-d-m-2}}{\Gamma(2 - \zeta + d)\Gamma(l + m + \xi - \zeta + d + 2)\Gamma(d + 1)2^{d+1}} \\ &\times {}_5F_6 \left[ \begin{matrix} 1, \frac{2 - \zeta + \xi + d}{2}, \frac{\xi - \zeta + d + 3}{2}, \frac{2 - \zeta + d + m}{2}, \frac{m - \zeta + d + 3}{2} \\ \frac{3}{2}, d + \frac{3}{2}, \frac{2 - \zeta + d}{2}, \frac{3 - \zeta + d}{2}, \frac{l + m + \xi - \zeta + d + 2}{2}, \frac{l + m + \xi - \zeta + d + 3}{2} \end{matrix} \middle| -\frac{1}{4x^2} \right]. \end{aligned} \tag{40}$$

(5) If we take  $u = 1, \tau = 1, \theta = 1, \nu = 2, a_1 = 1, k_1 = 0, b_1 = (\delta' + \nu' + 3)/2, b_2 = (\delta' - \nu' + 3)/2, L = 1, \beta = 1, \kappa = 2, d = 1, \omega = \delta', y = 1, A_1 = 0, A_2 = 0, g_1 = -1, \phi = 1, \nu = -1, \sigma = 1,$  and  $\lambda = 2^{\delta'+1}/\{(\delta' + \nu' + 1)(\delta' - \nu' + 1)\}$  in (12) and (14), the V-function

reduces to Lommel's function [20], using the result from (34) under the condition stated in Theorems 1 and 2, respectively, as below:

$$\begin{aligned} (I_{0,x}^{l,m,\xi} [z^{\zeta-1} S_{\delta',\nu'}(z)])(x) &= \frac{\Gamma(\zeta + \delta' + 1)\Gamma(\zeta + \delta' + \xi - m + 1)x^{\zeta+\delta'-m}}{(\delta' + \nu' + 1)(\delta' - \nu' + 1)\Gamma(\zeta + \delta' - m + 1)\Gamma(\zeta + \delta' + \xi + l + 1)} \\ &\times {}_5F_6 \left[ \begin{matrix} 1, \frac{\zeta + \delta' + 1}{2}, \frac{\zeta + \delta' + 2}{2}, \frac{\zeta + \delta' + \xi - m + 1}{2}, \frac{\zeta + \delta' + \xi - m + 2}{2} \\ \frac{\delta' + \nu' + 3}{2}, \frac{\delta' - \nu' + 3}{2}, \frac{\zeta + \delta' - m + 1}{2}, \frac{\zeta + \delta' - m + 2}{2}, \frac{\zeta + \delta' + \xi + l + 1}{2}, \frac{\zeta + \delta' + \xi + l + 2}{2} \end{matrix} \middle| -\frac{x^2}{4} \right], \end{aligned} \tag{41}$$

$$\begin{aligned} (I_{x,\infty}^{l,m,\xi} [z^{\zeta-1} S_{\delta',\nu'}(1/z)])(x) &= \frac{\Gamma(m - \zeta + \delta' + 2)\Gamma(\xi - \zeta + \delta' + 2)x^{\zeta-\delta'-m-2}}{\Gamma(2 - \zeta + \delta')\Gamma(l + m + \xi - \zeta + \delta' + 2)} \\ &\times {}_5F_6 \left[ \begin{matrix} 1, \frac{2 - \zeta + \xi + \delta'}{2}, \frac{\xi - \zeta + \delta' + 3}{2}, \frac{2 - \zeta + \delta' + m}{2}, \frac{m - \zeta + \delta' + 3}{2} \\ \frac{\delta' + \nu' + 3}{2}, \frac{\delta' - \nu' + 3}{2}, \frac{2 - \zeta + \delta'}{2}, \frac{3 - \zeta + \delta'}{2}, \frac{l + m + \xi - \zeta + \delta' + 2}{2}, \frac{l + m + \xi - \zeta + \delta' + 3}{2} \end{matrix} \middle| -\frac{1}{4x^2} \right]. \end{aligned} \tag{42}$$

(6) If we take  $u = 1, \tau = 1, \theta = 1, \nu = 1, a_1 = 1, b_1 = 1, L = -2, \beta = 1, \kappa = 1, \omega = 0, y = 0, k_1 = 0, a_1 = 0, b_1 = -1, \nu = -1, \sigma = 1,$  and  $\lambda = 1/\Gamma(d)$  in (12) and (14), the

V-function reduces to generalized Mittag-Leffler function [14, 16] using result (31), under the condition stated in Theorems 1 and 2, respectively, as below:

$$\begin{aligned} \left( I_{0,x}^{l,m,\xi} \left[ z^{\zeta-1} E_{\phi,d}(z) \right] \right) (x) &= \frac{\Gamma(\zeta)\Gamma(\zeta+\xi-m)x^{\zeta-m-1}}{\Gamma(d)\Gamma(\zeta-m)\Gamma(\zeta+\xi+l)} \\ &\times {}_3F_{\phi+2} \left[ \begin{matrix} 1, \zeta, \zeta+\xi-m \\ \frac{d}{\phi}, \dots, \frac{d+\phi-1}{\phi}, \zeta-m, \zeta+\xi+l \end{matrix} \middle| \frac{x}{\phi^\phi} \right], \end{aligned} \quad (43)$$

$$\begin{aligned} \left( I_{x,\infty}^{l,m,\xi} \left[ z^{\zeta-1} E_{\phi,d}(1/z) \right] \right) (x) &= \frac{\Gamma(1-\zeta+m)\Gamma(1-\zeta+\xi)x^{\zeta-m-1}}{\Gamma(d)\Gamma(1-\zeta)\Gamma(1-\zeta+\xi+l+m)} \\ &\times {}_3F_{\phi+2} \left[ \begin{matrix} 1, 1-\zeta+m, 1-\zeta+\xi \\ \frac{d}{\phi}, \dots, \frac{d+\phi-1}{\phi}, 1-\zeta, 1-\zeta+\xi+l+m \end{matrix} \middle| \frac{1}{x\phi^\phi} \right]. \end{aligned} \quad (44)$$

(7) If we take  $u = 1, \tau = 1, \theta = 1, \nu = 1, d = 1, a_1 = 1, b_1 = 1, L = 2, \kappa = 1, d = 0, \omega = 0, y = 0, k_1 = 0, a_1 = 0, b_1 = -1, \phi = 1, \beta = 1, \nu = -1, \sigma = 1$ , and  $\lambda = 1$  in (12) and (14), the V-function reduces to exponential function under the condition stated in Theorems 1 and 2, respectively, as below:

$$\begin{aligned} \left( I_{0,x}^{l,m,\xi} \left[ z^{\zeta-1} e^{-z} \right] \right) (x) &= \frac{\Gamma(\zeta)\Gamma(\zeta+\xi-m)x^{\zeta-m-1}}{\Gamma(\zeta-m)\Gamma(\zeta+\xi+l)} \\ &\times {}_2F_2 \left[ \begin{matrix} \zeta, \zeta+\xi-m \\ \zeta-m, \zeta+\xi+l \end{matrix} \middle| -x \right], \end{aligned} \quad (45)$$

$$\begin{aligned} \left( I_{x,\infty}^{l,m,\xi} \left[ z^{\zeta-1} e^{-1/z} \right] \right) (x) &= \frac{\Gamma(1-\zeta+m)\Gamma(1-\zeta+\xi)x^{\zeta-m-1}}{\Gamma(1-\zeta)\Gamma(1-\zeta+\xi+l+m)} \\ &\times {}_2F_2 \left[ \begin{matrix} 1-\zeta+m, 1-\zeta+\xi \\ 1-\zeta, 1-\zeta+\xi+l+m \end{matrix} \middle| -\frac{1}{x} \right]. \end{aligned} \quad (46)$$

## 6. Concluding Remarks

In our present investigation, fractional integral operators and the V-function have advantage that it generalizes the R-L, Weyl, and Erdélyi–Kober fractional integral operators. Thus, we conclude the study by highlighting that many other important image formulas can be obtained as the special cases of our leading results (Theorems 1 and 2) involving common fractional integral operators as stated above. Various special cases with similar outcomes of the report may be evaluated by taking acceptable values of the parameters concerned. We refer to [30, 31] for a variety of other special cases and give the results to interested readers. Our study ends with the remark that the stated outcome is important and can result in the yield of the number of other special function involving various types of Generalized hypergeometric function, Bessel function, Wright's generalized Bessel function, Lommel's function, Struve's function, Mittag-Leffler, and exponential function.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

There are no conflicts of interest regarding the publication of this article.

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