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Research Article

The Strong Stability of Optimal Nonlinear Dynamical System in Batch Fermentation

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For the bio-dissimilation of glycerol to 1,3-propanediol by *Klebsiella pneumoniae*, the nonlinear dynamical system of the complex metabolism in microbial batch fermentation is studied in this study. Since the analytical solution and equilibrium point cannot be found for the nonlinear dynamical system of batch fermentation, the system stability cannot be analyzed using general methods. Therefore, in this study, the stability of the system is analyzed from another angle. We present the corresponding linear variational system for the solution to the nonlinear dynamical system of complex metabolism. In addition, the boundedness of fundamental matrix solutions for the linear variational system is obtained. With this in mind, strong stability with respect to the perturbation of the initial state vector is proved for the nonlinear dynamical system of the complex metabolism.

1. Introduction

The chemical substance 1,3-propanediol (simply denoted as 1,3-PD) is an important raw material for many products. It is produced by using two methods: chemical synthesis and microbial fermentation. As for the chemical synthesis, the raw material is petroleum, and the catalyst is precious metal, and serious environmental pollution and high costs are generated in the production process. As for microbial fermentation, waste glycerol from biodiesel production is used as raw material. Microbial fermentation has the advantages of environmental friendliness, simple operation, and fewer by-products. Hence, 1,3-PD production through the microbial fermentation of glycerol has been widely studied due to its great research and practice value [1–4].

During 1,3-PD production through microbial fermentation, the experimental scheme is firstly designed. 1,3-PD is produced by means of microbial fermentation of glycerol, and the extracellular concentration variations of substances in the fermentation are directly tested. Obviously, a substantial amount of human and material resources and funds should be invested in the experiment, and the experimental data measured might be different even under the same conditions due to the long duration of the experiment and the susceptibility to the environment. Most importantly, the intracellular concentration cannot be measured through this experimental method. Therefore, it is challenging to complete considerable experimental schemes limited by time, funds, and human resources.

If a numerical experiment of 1,3-PD production through microbial fermentation is conducted on a computer, the data which cannot be collected in the lab will be obtained, such as the intracellular concentration. However, the numerical experiment should be made on the basis of a mathematical model that reflects the real experimental fermentation. It exhibits the main advantages of completing multiple experimental schemes and saving costs of time, human resources, and funds, as well as the disadvantages of requiring reliability tests, namely, identifying the mathematical model.

Batch fermentation is a technique to produce 1,3-PD from glycerol (also named substrate). In 1995, Zeng and Deckwer [5] firstly carried out quantitative research on microbial batch fermentation without providing the specific form of the model. In 2000, Xiu et al. [6] proposed the

dynamic model during the microbial transformation based on the Monod-type material balance equation and studied the multistability, but the influence of superfluous substrates was not considered in this model. Yuan et al. [7] considered an optimal minimal variation control and proposed a parallel algorithm based on the genetic algorithm and the gradients of the constraint functions with respect to decision variables. The robust biobjective optimal control of 1, 3-PD microbial batch production process is researched in [8]. Wang et al. [9] formulated the batch process as an optimal control problem subject to continuous state constraints and stochastic disturbances and proposed a modified particle swarm algorithm.

In the fermentation process, factors such as time delay need to be taken into account. In addition, studies on the stability of the system are necessary. Modelling and optimal state-delay control in microbial batch process is studied in [10]. An optimal state-delay control strategy is proposed in nonlinear dynamical systems [11]. The optimal control of glycerol producing 1,3-PD in the batch process with uncertain time-delay and kinetic parameters is studied and a novel gradient-based solution method is proposed in [8]. Robust optimal control for a batch nonlinear enzyme-catalytic switched time-delayed process with noisy output measurements is given in [12]. Chio and Koo [13, 14] studied the stability of the linear dynamical system and nonlinear dynamical system in accordance with the u_∞-similarity. Zheng and Gao [15, 16] studied multiple Lyapunov functions of the stable positive linear system and the switched linear system.

In this study, as for the stability of the nonlinear dynamical system without analytical solution and equilibrium point, properties and relevant theory of the fundamental matrix solution of the linear variational system were utilized to prove the strong stability of the optimal nonlinear dynamical system of complex metabolism of a type of batch fermentation.

The remainder of the study is organized as follows. Section 2 presents the optimal nonlinear dynamical system of complex metabolism. Section 3 provides the linear variational system and fundamental matrix solution. Section 4 analyzes the strong stability of the optimal nonlinear dynamical system of complex metabolism. Finally, Section 5 concludes the study.

2. Optimal Nonlinear Dynamical System of Complex Metabolism

With transmembrane transport of glycerol (also named substrate) and 1,3-PD considered, three means of transport are assumed, including active transport and passive diffusion, as well as active transport and passive diffusion, denoted as A, P, and AP, respectively. GTW stands for the transport pathway of glycerol and PDTW represents the pathway of 1,3-PD. $D \coloneqq \{(\zeta, \tau) \in [0,1]^2\}$ is defined as the set of possible metabolic pathways, where ζ and τ are parameters of pathways. As ζ and τ are continuous variables, there will be numerous metabolic pathways. The transport pathway of glycerol corresponding to ζ is as follows:

$\zeta = 0$, when GTW is P (passive)

$\zeta = 1$, when GTW is A (active)

$\zeta \in (0,1)$, when GTW is AP (active and passive)

The transport pathway of 1,3-PD corresponding to τ, namely, PDTW, is as follows:

$\tau = 0$, when PDTW is P (passive)

$\tau = 1$, when PDTW is A (active)

$\tau \in (0,1)$, when PDTW is AP (active and passive)

Let $x(t) = [x_1(t), x_2(t), \ldots, x_8(t)]^\top \in \mathbb{R}^8$, $t \in [0, t_f] \subset R_+$, be the state vector whose components are the concentrations of biomass, extracellular glycerol, extracellular 1,3-PD, acetate, ethanol, intracellular glycerol, 3-HPA, and intracellular 1,3-PD at time $t \in [0, t_f]$, respectively, and $x_0 \in \mathbb{R}^8$ is an initial state. The complex metabolism system corresponding to (ζ, τ) is denoted as $\mathrm{CMS}(\zeta, \tau)$, which is represented as nonlinear dynamical system as follows:

$$\begin{cases} \dot{x}(t) = h\left(x, u_{\zeta,\tau}, \zeta, \tau\right), & t \in \left[0, t_f\right], \\ x(0) = x_0, \end{cases} \tag{1}$$

where $u_{\zeta,\tau} = [u^1_{\zeta,\tau}, u^2_{\zeta,\tau}, \ldots, u^{12}_{\zeta,\tau}] \in \mathbb{R}^{12}$ is the system parameter of $\mathrm{CMS}(\zeta, \tau)$, and the components of $h(x, u_{\zeta,\tau}, \zeta, \tau) = [h_1(x, u_{\zeta,\tau}, \zeta, \tau), h_2(x, u_{\zeta,\tau}, \zeta, \tau), \ldots, h_8(x, u_{\zeta,\tau}, \zeta, \tau)]^\top$ are shown in [17]. The admissible set of system parameters is defined as $U_{a\,d} \coloneqq [u_l; u^u] = \prod_{j=1}^{12} [u_{lj}, u^u_j] \subset \mathbb{R}^{12}$, and the admissible domain of state variables is defined as $W_0 \coloneqq [x_l; x^u] = \prod_{j=1}^{8} [x_{lj}, x^u_j] \subset \mathbb{R}^8_+$. The values of u_l, u^u, x_l, and x^u can be obtained from [17].

For the metabolic system, there are properties as follows:

Property 1. $\zeta, \tau \in [0,1]$ is given as follows:

(I) $\forall\, x \in W_0$, $h(x, u_{\zeta,\tau}, \zeta, \tau)$ about $u_{\zeta,\tau}$ on $U_{a\,d}$ is continuous.

(II) $\forall\, u_{\zeta,\tau} \in U_{a\,d}$, $h(x, u_{\zeta,\tau}, \zeta, \tau)$ about x on W_0 is Lipschitz continuous.

(III) $\forall\, u_{\zeta,\tau} \in U_{a\,d}$, $h(x, u_{\zeta,\tau}, \zeta, \tau)$ satisfies the linear growth conditions. That is, there exist constants $\alpha, \beta > 0$, making $\|h(x, u_{\zeta,\tau}, \zeta, \tau)\| \le \alpha\|x\| + \beta$. It is defined that the norm of vector function $x = x(t)$: $[0, t_f] \longrightarrow \mathbb{R}^n$ is $\|x(t)\| = \|x\| \coloneqq \max_{s\in[0,t_f]} \|x(s)\|$, where $\|\cdot\|$ represents the norm of the vector on R^n, that is, $\forall\, x \in \mathbb{R}^n$, $\|x\| \coloneqq \sum_{i=1}^n |x_i|$; and the norm of matrix function $A(t)$: $[0, t_f] \longrightarrow \mathbb{R}^{n\times n}$ is $\|A(t)\| = \|A\| \coloneqq \max_{s\in[0,t_f]} \|A(s)\|$, where $\|\cdot\|$ represents the norm of the matrix on $R^{n\times n}$, that is, $\forall\, A = [a_{ij}]_{n\times n} \in \mathbb{R}^{n\times n}$ and $\|A\| \coloneqq \max_{j\in I_n} \sum_{i=1}^n |a_{ij}|$.

Property 2. $\zeta, \tau \in [0,1]$ is given, for $\forall\, x_0 \in W_0$ and $\forall\, u_{\zeta,\tau} \in U_{a\,d}$, and there is a unique solution in the system $\mathrm{CMS}(\zeta, \tau)$, denoted by $x(t; x_0, u_{\zeta,\tau}, \zeta, \tau)$. The solution $x(t; x_0, u_{\zeta,\tau}, \zeta, \tau)$ about parameter $u_{\zeta,\tau} \in U_{a\,d}$ and initial state $x_0 \in W_0$ is continuous.

Suppose that $I_N \coloneqq \{1, 2, \ldots, N\}$ presents the experimental sequence set, and it is preset that $\zeta, \tau \in [0, 1]$, $\forall l \in I_N$, and $\forall x_0^l \in W_0$. The solution set of $\mathrm{CMS}(\zeta, \tau)$ is denoted as

$$S_0(\zeta, \tau) \coloneqq \left\{x\left(t; x_0^l, u_{\zeta,\tau}, \zeta, \tau\right) | x\left(t; x_0^l, u_{\zeta,\tau}, \zeta, \tau\right) \in W_0, x_0^l \in W_0, u_{\zeta,\tau} \in U_{a\,d}, l \in I_N\right\}. \tag{2}$$

The feasible set of the parameter vectors u is defined as

$$U_{S_0}(\zeta, \tau) \coloneqq \left\{u_{\zeta,\tau} \in U_{a\,d} | x\left(t; x_0^l, u_{\zeta,\tau}, \zeta, \tau\right) \in S_0(\zeta, \tau), \forall t \in \left[0, t_f\right], l \in I_N\right\}. \tag{3}$$

According to H_3 and Property 3 [18], for $\forall \zeta, \tau \in [0, 1]$, $S_0(\zeta, \tau)$ and $U_{s_0}(\zeta, \tau)$ are nonempty sets, and for given $\zeta, \tau \in [0, 1]$, $S_0(\zeta, \tau)$ and $U_{s_0}(\zeta, \tau)$ are compact.

Due to the lack of data on the concentration of intracellular substances, biological robustness is introduced to judge the reliability of the numerical solution of System (1). The smaller the value of biological robustness, the more robust the system is [19]. The identification model (abbreviated as IM) of $\mathrm{CMS}(\zeta, \tau)$ about the pathway variable and the system variable is given as follows:

$$\begin{aligned} &\text{IM: } \min J\left(u_{\zeta,\tau}, \zeta, \tau\right), \\ &s.\text{t.} \begin{array}{l} u_{\zeta,\tau} \in U_w(\zeta, \tau), \\ \zeta, \tau \in [0, 1], \end{array} \end{aligned} \tag{4}$$

where $J(u_{\zeta,\tau}, \zeta, \tau)$ is the biological robustness for $u_{\zeta,\tau}$ of the solution to system $\mathrm{CMS}(\zeta, \tau)$.

$$J\left(u_{\zeta,\tau}, \zeta, \tau\right): = \frac{1}{N \cdot t_f} \sum_{l=1}^{N} \int_U \left(\phi\left(u_{\zeta,\tau}{}'\right) \cdot \frac{\int_0^{t_f} \left\| x\left(t; x_0^l, u_{\zeta,\tau}{}', \zeta, \tau\right) - x\left(t; x_0^l, u_{\zeta,\tau}, \zeta, \tau\right) \right\| / \left\| x\left(t; x_0^l, u_{\zeta,\tau}, \zeta, \tau\right) \right\| \mathrm{d}t}{\left\| u_{\zeta,\tau}{}' - u_{\zeta,\tau} \right\|} \right) d\left(u_{\zeta,\tau}{}'\right), \tag{5}$$

where $U \subset U_w(\zeta, \tau)$ is the perturbation space of $u_{\zeta,\tau}$ and $\phi(u_{\zeta,\tau}{}')$ is the probability density function of the uniform distribution in $u_{\zeta,\tau}{}'$ on U. $U_w(\zeta, \tau) \coloneqq \left\{u_{\zeta,\tau} | u_{\zeta,\tau} \in U_{s_0}(\zeta, \tau), \mathrm{RE}(u_{\zeta,\tau}, \zeta, \tau) \le \varepsilon\right\}$ is the set of $u_{\zeta,\tau}$, and $\varepsilon > 0$ is the allowable error limit of $\mathrm{RE}(u, \zeta, \tau)$. Assume that there exits $[\zeta, \tau]$ such that $U_w(\zeta, \tau)$ is nonempty. Since $U_{s_0}(\zeta, \tau)$ is compact, the set $U_w(\zeta, \tau)$ is compact for some $\zeta, \tau \in [0, 1]$. The relative error (abbreviated as RE) between calculated concentration $x_j(t; x_0^l, u_{\zeta,\tau}, \zeta, \tau)$, $j \in I_3$ and measured concentration $y_j^l(t)$, $j \in I_3$ of extracellular components is defined as follows:

$$\mathrm{RE}\left(u_{\zeta,\tau}, \zeta, \tau\right) \coloneqq \frac{1}{3 \cdot N \cdot t_f} \sum_{l=1}^{N} \sum_{j=1}^{3} \int_0^{t_f} \frac{\left[x_j\left(t; x_0^l, u_{\zeta,\tau}, \zeta, \tau\right) - y_j^l(t)\right]^2}{\left|y_j^l(t)\right|^2} \mathrm{d}t. \tag{6}$$

The existence of the optimal solution of IM is given in Theorem 1 of [18]; that is, given $\zeta, \tau \in [0, 1]$, there exists $u_{\zeta,\tau}^* \in U_w(\zeta, \tau)$ such that $J(u_{\zeta,\tau}^*, \zeta, \tau) \le J(u_{\zeta,\tau}, \zeta, \tau)$.

By constructing the parallel particle swarm optimization algorithm, the optimal solution of the problem IM is calculated as $\zeta^* = 0.9$ and $\tau^* = 0.9$, and the value of parameter u_{ζ^*,τ^*}^* is given as follows [18]:

$$u_{0.9,0.9}^* = [34.0297, 107.204, 150.417, 4.71128, 115.59, 57.2065, 1.98077, 17.7808, 10.6496, 5263.6, 44.8808, 68.8345]. \tag{7}$$

So, the optimal nonlinear dynamical system of complex metabolism of batch fermentation is obtained as

$$\begin{cases} \dot{x}(t) = h\left(x, u_{\zeta,\tau}^*, \zeta^*, \tau^*\right), & t \in \left[0, t_f\right], \\ x(0) = x_0, \end{cases} \tag{8}$$

where

$$\begin{cases} h_1\left(x, u^*_{\zeta,\tau}, \zeta^*, \tau^*\right) = \mu x_1, \\ h_2\left(x, u^*_{\zeta,\tau}, \zeta^*, \tau^*\right) = -q_2\left(\zeta^*, \tau^*\right)x_1, \\ h_3\left(x, u^*_{\zeta,\tau}, \zeta^*, \tau^*\right) = q_3\left(\zeta^*, \tau^*\right)x_1, \\ h_4\left(x, u^*_{\zeta,\tau}, \zeta^*, \tau^*\right) = q_4\left(\zeta^*, \tau^*\right)x_1, \\ h_5\left(x, u^*_{\zeta,\tau}, \zeta^*, \tau^*\right) = q_5\left(\zeta^*, \tau^*\right)x_1, \\ h_6\left(x, u^*_{\zeta,\tau}, \zeta^*, \tau^*\right) = \dfrac{1}{u^{9*}_{\zeta,\tau}}\left[\zeta^*\dfrac{46.1633x_2}{x_2 + 2.7467} + \left(1-\zeta^*\right)\mu^{10*}_{\zeta,\tau}\left(x_2 - x_6\right)\cdot N_{R_+}\left(x_2 - x_6\right) - q_{20}\right] - \mu x_6, \\ h_7\left(x, u^*_{\zeta,\tau}, \zeta^*, \tau^*\right) = u^{11*}_{\zeta,\tau}\dfrac{x_6}{0.53\left(1 + x_7/185.242\right) + x_6} - \dfrac{45.9992x_7}{0.14 + x_7\left(1 + x_7/1.3341\right)} - \mu x_7, \\ h_8\left(x, u^*_{\zeta,\tau}, \zeta^*, \tau^*\right) = \dfrac{45.9992x_7}{0.14 + x_7\left(1 + x_7/1.3341\right)} - \tau^*\dfrac{6.9401x_8}{x_8 + 26.6321} - \left(1-\tau^*\right)\cdot u^{12*}_{\zeta,\tau}\left(x_8 - x_3\right)N_{R_+}\left(x_8 - x_3\right) - \mu x_8. \end{cases} \tag{9}$$

The consumption rate and the specific production rate in the microbial transformation are

$$\begin{cases} q_{20} = 3.1453 + \dfrac{\mu}{0.0066} + \dfrac{36.8436x_2}{x_2 + 16.2526}, \\ \mu = \dfrac{0.67x_2}{x_2 + 0.28}\displaystyle\prod_{j=2}^{5}\left(1 - \frac{x_j}{x^*_j}\right), \\ q_2\left(\zeta^*, \tau^*\right) = \zeta^* u^{1*}\dfrac{x_2}{x_2 + 3.87} + \left(1-\zeta^*\right)u^{2*}\left(x_2 - x_6\right)N_{R_+}\left(x_2 - x_6\right), \\ q_3\left(\zeta^*, \tau^*\right) = \tau^* u^{3*}\dfrac{x_8}{x_8 + u^{4*}} + \left(1-\tau^*\right)u^{5*}\left(x_8 - x_3\right)N_{R_+}\left(x_8 - x_3\right), \\ q_4 = -1.0522 + \mu u^{6*}, \\ q_5 = u^{7*} + \mu u^{8*}, \end{cases} \tag{10}$$

where $N_{R_+}(\theta)$ is the indicator function; if $\theta > 0$, then $N_{R_+}(\theta) = 1$; otherwise, $N_{R_+}(\theta) = 0$.

3. Linear Variational System and Fundamental Matrix Solution

To simplify notation, we denote $u^*_{\zeta,\tau}$ as u^*. In accordance with Property 1, the function $h(x, u^*, \zeta^*, \tau^*) \in \mathbb{R}^8$ exhibited its continuous partial derivative with respect to $x_0 \in W_0$ in System (1), so the linear variational system corresponding to System (1) could be built as

$$\dot{y} = \frac{\partial h(x, u^*, \zeta^*, \tau^*)}{\partial x} \cdot y, t \in I_0, \tag{11}$$

where $x = x(t; x_0, u^*, \zeta^*, \tau^*)$ is the solution to System (1) with $x(0; x_0, u^*, \zeta^*, \tau^*) = x_0$ as an initial state. It is assumed that

$$x(t) = z(t) + x(t; x_0, u^*, \zeta^*, \tau^*), \tag{12}$$

was also the solution to System (1) with the following x(0) as the initial state,

$$x(0) = z(0) + x(0; x_0, u^*, \zeta^*, \tau^*) = x_0. \tag{13}$$

Next, the derivative of t on both sides of (12) is calculated as

$$\begin{aligned} \dot{x}(t) &= \dot{z}(t) + \frac{\partial x(t; x_0, u^*, \zeta^*, \tau^*)}{\partial t} \\ &= \dot{z}(t) + h(x(t; x_0, u^*, \zeta^*, \tau^*), u^*, \zeta^*, \tau^*). \end{aligned} \tag{14}$$

Since (12) is the solution to System (1), we have

$$\dot{z}(t) + \frac{\partial x(t; x_0, u^*, \zeta^*, \tau^*)}{\partial t} = h(z(t) + x(t; x_0, u^*, \zeta^*, \tau^*), u^*, \zeta^*, \tau^*). \tag{15}$$

From (14) and (15), it could be calculated that

$$\begin{aligned} \dot{z}(t) &= h(z(t) + x(t; x_0, u^*, \zeta^*, \tau^*), u^*, \zeta^*, \tau^*) - h(x(t; x_0, u^*, \zeta^*, \tau^*), u^*, \zeta^*, \tau^*) \\ &= \frac{\partial h(x(t; x_0, u^*, \zeta^*, \tau^*), u^*, \zeta^*, \tau^*)}{\partial x} \cdot z(t) + o(\|z(t)\|). \end{aligned} \tag{16}$$

When $\|z(t)\|$ is sufficiently small, we have

$$\dot{z}(t) = \frac{\partial h(x(t; x_0, u^*, \zeta^*, \tau^*), u^*, \zeta^*, \tau^*)}{\partial x} \cdot z(t). \tag{17}$$

Therefore, the variational system (11) is obtained by ignoring $o(\|z(t)\|)$ in the differential equation of $z(t)$.

According to Theorem 3.3 in [20], if $x = x(t; x_0, u^*, \zeta^*, \tau^*)$ is the solution to System (1) with $x(0; x_0, u^*, \zeta^*, \tau^*) = x_0$ as an initial state matrix,

$$\frac{\partial x(t; x_0, u^*, \zeta^*, \tau^*)}{\partial x_0} \in \mathbb{R}^{8\times 8}, \tag{18}$$

which is the fundamental matrix solution to the linear variational system (11) with the unit matrix as

$$\frac{\partial x(0; x_0, u^*, \zeta^*, \tau^*)}{\partial x_0} = I \quad \left(I \in \mathbb{R}^{8\times 8} \text{ unit matrix}\right), \tag{19}$$

as an initial state.

According to Theorem 2.6.4 in [20], we have those as follows.

Lemma 1. *Suppose that $x(t; x_0, u^*, \zeta^*, \tau^*)$ and $x(t; y_0, u^*, \zeta^*, \tau^*)$ are the solutions to System (1) with $x(0; x_0, u^*, \zeta^*, \tau^*) = x_0 \in W_0$ and $x(0; y_0, u^*, \zeta^*, \tau^*) = y_0 \in W_0$ as an initial state, then*

$$\begin{aligned} &x(t; y_0, u^*, \zeta^*, \tau^*) - x(t; x_0, u^*, \zeta^*, \tau^*) \\ &= \int_0^1 \Phi(t, x_0 + s(y_0 - x_0))\mathrm{d}s \cdot (y_0 - x_0), \ t \in I_0, \end{aligned} \tag{20}$$

where Φ is the fundamental matrix solution to the linear variational system (11), corresponding to the solution $x(t; x_0 + s(y_0 - x_0), u^*, \zeta^*, \tau^*)$ of System (1) with $x_0 + s(y_0 - x_0)$ as an initial state. The term s is used to portray the perturbation process of the initial values.

4. Strong Stability of the Optimal Nonlinear Dynamical System of Complex Metabolism

According to Properties 1 and 2, there is a unique solution $x(t; x_0, u^*, \zeta^*, \tau^*) \in W_0$ to $\forall x_0 \in W_0$ in the optimal nonlinear dynamical system (1) of complex metabolism. However, the analytical solution cannot be calculated at present, and only the numerical solution can be obtained. From the numerical solution, there is no stationary point in System (1), so the strong stability is defined as follows.

Definition 1. Suppose that $x(t) = x(t; x_0, u^*, \zeta^*, \tau^*)$ is the solution to System (1) with $x(0) = x_0 \in W_0$ as an initial state, if $\forall \varepsilon > 0$, then there exists $\delta = \delta(\varepsilon) > 0$ making the solution $y(t) = y(t; y_0, u^*, \zeta^*, \tau^*)$ of system (1) satisfy

$$y_0 \in W_0, \ |x_0 - y_0| < \delta, \ |x(t) - y(t)| < \varepsilon, \quad \forall t \in I_0. \tag{21}$$

Then, the solution $x(t; x_0, u^*, \zeta^*, \tau^*)$ in System (1) has strong stability for the initial state perturbation.

To prove the strong stability of the solution to System (1), the boundedness of the fundamental matrix solution of the linear variational system (11) corresponding to solution $x(t; x_0, u^*, \zeta^*, \tau^*)$ to System (1) must be firstly discussed.

Theorem 1. *If $x(t) = x(t; x_0, u^*, \zeta^*, \tau^*)$ is the solution to System (1) with $x(0) = x_0 \in W_0$ as an initial state and $\Phi(t, 0, I)$, $t \in I_0$, is the fundamental matrix solution of the linear variational system (11) corresponding to solution $x(t; x_0, u^*, \zeta^*, \tau^*)$ to System (1), then $\Phi(t, 0, I)$ is bounded on $I_0 \subset \mathbb{R}_+$.*

Proof. $\Phi(t, 0, I) \coloneqq [x^1(t, 0, e^1), x^2(t, 0, e^2) \ldots, x^8(t, 0, e^8)] \in \mathbb{R}^{8\times 8}$ is the fundamental matrix solution of the linear variational system (11), namely,

$$\begin{cases} \dot{x}^i(t) = \dfrac{\partial h(x(t, x_0), u^*, \zeta^*, \tau^*)}{\partial x} \cdot x^i(t), \\ x^i(0) = e^i,\ i \in I_8,\ t \in I_0, \end{cases} \tag{22}$$

where $e^i \in \mathbb{R}^8$ is the ith column of the unit matrix $I \in \mathbb{R}^{8\times 8}$.

As $h(x, u^*, \zeta^*, \tau^*)$ about $x \in W_0$ shows continuous partial derivative and $W_0 \subset \mathbb{R}^8$ is also a nonempty bounded closed set, so the given $x_0 \in W_0$ could make $\partial h(x, u^*, \zeta^*, \tau^*)/\partial x$ bounded on $W_0 \subset \mathbb{R}^8$, that is, $\exists M_1 > 0$, such that

$$\left|\frac{\partial h(x, u^*, \zeta^*, \tau^*)}{\partial x}\right| \le M_1, \quad \forall t \in I_0. \tag{23}$$

Let

$$u_i(t) = \arg\min\left\{\|u_i(t)\| \mid u_i(t) \in C^1(I_0, \mathbb{R}_+), u_i(t) \ge \max_{j\in I_8}\left|x_j^i(t)\right|,\ t \in I_0\right\}, \tag{24}$$

$$V_i(t, u_i(t)) \coloneqq 8 \cdot M_1 \cdot u_i(t),\ i \in I_8. \tag{25}$$

Obviously, $V_i(t, u_i(t))$ is continuous on $I_0 \times \mathbb{R}_+$. Hence, the system is

$$\begin{cases} \dot{u}_i(t) = V_i(t, u_i(t)), \\ u_i(0) = 1,\ t \in I_0,\ i \in I_8, \end{cases} \tag{26}$$

which has a unique solution $u_i(t)$, $t \in I_0$, and it is also obtained that

$$u_i(t) \ge 1,\ t \in I_0. \tag{27}$$

Therefore, the right-hand side of the state equation in system (22) is

$$\left|\frac{\partial h(x(t, x_0), u^*, \zeta^*, \tau^*)}{\partial x} \cdot x^i(t)\right| \le \left|\frac{\partial h(x(t, x_0), u^*, \zeta^*, \tau^*)}{\partial x}\right| \cdot \left|x^i(t)\right|, \tag{28}$$

$$\le M_1 \cdot \left|x^i(t)\right|, \tag{29}$$

$$\le M_1 \cdot 8 \cdot \max_{j\in I_8}\left|x_j^i(t)\right|, \tag{30}$$

$$\le 8 \cdot M_1 \cdot u_i(t), \quad \forall t \in I_0, \tag{31}$$

$$\le V_i(t, u_i(t)), \quad \forall t \in I_0, \tag{32}$$

where (29), (30), and (32) are obtained according to (23), (24), and (25), respectively. It is acquired from the equation above that

$$\left|\frac{\partial h(x(t, x_0), u^*, \zeta^*, \tau^*)}{\partial x} \cdot x^i(t)\right| \le V_i(t, u_i(t)), \quad \forall t \in I_0,\ i \in I_8. \tag{33}$$

By Theorem 6.1 and Inference 6.3 in [20], the solution $u_i(t)$ to System (26) satisfies

$$\left|x^i(t, 0, e^i)\right| \le u_i(t) \le \|u_i(t)\| = \max_{t\in I_0}\{u_i(t)\},\ i \in I_8. \tag{34}$$

Because of $u_i(t) \in C^1((I_0, \mathbb{R}_+)$, let m_i denote the supremum of $u_i(t)$; then,

$$\|u_i(t)\| \le m_i < \infty, \quad \forall t \in I_0, \tag{35}$$

and it is obtained from (34) that

$$\left|x^i(t, 0, e^i)\right| \le m_i,\ t \in I_0,\ i \in I_8. \tag{36}$$

According to the definition of norm, it is acquired that

$$\left|x^i(t, 0, e^i)\right| = \sum_{j\in I_8}\left|x_j^i(t, 0, e^i)\right| \le m_i, \quad \forall t \in I_0. \tag{37}$$

Let $M \coloneqq \sum_{j\in I_8} m_i$; then,

$$|\Phi(t, 0, I)| = \sum_{i\in I_8}\sum_{j\in I_8}\left|x_j^i(t, 0, e^i)\right| \le \sum_{i\in I_8} m_i \le M,\ t \in I_0. \tag{38}$$

So, it is verified that the fundamental matrix solution $\Phi(t, 0, I)$ of the linear variational system (11) is bounded on $I_0 \subset \mathbb{R}_+$.

Then, the strong stability of the optimal nonlinear dynamical system (1) of complex metabolism is considered. □

Theorem 2. *If $x(t) = (t; x_0, u^*, \zeta^*, \tau^*)$ is the solution to System (1) with $x(0) = x_0 \in W_0$ as an initial state, the solution $x(t; x_0, u^*, \zeta^*, \tau^*)$ is of strong stability about the initial state perturbation.*

Proof. Suppose that System (1) with $y(0) = y_0 \in W_0$ is an initial state. As for $\forall \varepsilon > 0$, the initial state $y_0 \in W_0$ satisfies

$$|x_0 - y_0| \le \frac{\varepsilon}{M} = \delta(\varepsilon), \tag{39}$$

where $M > 0$, and according to Lemma 1, M is constant in Theorem 1.

$$y(t; y_0, u^*, \zeta^*, \tau^*) - x(t; x_0, u^*, \zeta^*, \tau^*) = \int_0^1 \Phi(t, 0, x_0 + S(y_0 - x_0))\mathrm{d}S \cdot (y_0 - x_0). \tag{40}$$

Hence,

$$\begin{aligned} |y(t; y_0, u^*, \zeta^*, \tau^*) - x(t; x_0, u^*, \zeta^*, \tau^*)| &\le \int_0^1 |\Phi(t, 0, x_0 + S(y_0 - x_0))|\mathrm{d}S \cdot |y_0 - x_0| \\ &\le M \cdot |y_0 - x_0| \\ &\le M \cdot \frac{\varepsilon}{M} = \varepsilon, \quad \forall t \in I_0. \end{aligned} \tag{41}$$

In accordance to Definition 1, the solution $x(t; x_0, u^*, \zeta^*, \tau^*)$ to System (1) has strong stability about the initial state perturbation. □

5. Conclusions

The optimal nonlinear dynamical system of the complex metabolism of batch fermentation and its properties are taken into consideration in this study. It is important to analyze the stability of the system. However, the stability of the dynamical system of the complex metabolism cannot be analyzed by the general methods because the analytical solution and equilibrium point of the system cannot be obtained. To address this problem, we propose the linear variational system and the fundamental matrix solution corresponding to the system. Based on the properties of the system and the proposed content, the strong stability of the system solution about the initial state perturbation is proved.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no known conflicts of interest or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] C. Y. Liu and Z. H. Gong, "Modelling and optimal control of a time-delayed switched system in fed-batch process," *Journal of the Franklin Institute*, vol. 351, no. 2, pp. 840–856, 2014.

[2] C. Y. Liu, Z. H. Gong, K. L. Teo, J. Sun, and L. Caccetta, "Robust multi-objective optimal switching control arising in 1, 3-propanediol microbial fed-batch process," *Nonlinear Analysis: Hybrid Systems*, vol. 25, pp. 1–20, 2017.

[3] G. X. Xu, Y. Liu, and Q. W. Gao, "Multi-objective optimization of a continuous bio-dissimilation process of glycerol to 1, 3-propanediol," *Journal of Biotechnology*, vol. 219, pp. 59–71, 2016.

[4] H. Bei, L. Wang, Y. Ma, J. Sun, and L. Zhang, "A linear optimal feedback control for producing 1, 3-propanediol via microbial fermentation," *Discrete & Continuous Dynamical Systems - S*, vol. 13, no. 6, pp. 1623–1635, 2020.

[5] A. P. Zeng and W. D. Deckwer, "A Kinetic model for substrate and energy consumption of microbial growth under substrate-sufficient conditions," *Biotechnology Progress*, vol. 11, no. 1, pp. 71–79, 1995.

[6] Z. L. Xiu, A. P. Zeng, and L. J. An, "Mathematical modeling of kinetics and research on multiplicity of glycerol bioconversion to 1, 3-propanediol," *Journal of Dalian University of Technology*, vol. 40, no. 4, pp. 428–433, 2000.

[7] J. Yuan, L. Wang, J. Xie, and K. L. Teo, "Optimal minimal variation control with quality constraint for fed-batch fermentation processes involving multiple feeds," *Journal of the Franklin Institute*, vol. 357, no. 11, pp. 6571–6594, 2020.

[8] C. Liu, Z. Gong, H. W. Joseph Lee, and K. L. Teo, "Robust bi-objective optimal control of 1, 3-propanediol microbial batch

production process," *Journal of Process Control*, vol. 78, pp. 170–182, 2019.

[9] L. Wang, J. Yuan, C. Wu, and X. Wang, "Practical algorithm for stochastic optimal control problem about microbial fermentation in batch culture," *Optimization Letters*, vol. 13, no. 3, pp. 527–541, 2019.

[10] C. Liu, Z. Gong, K. L. Teo, and S. Wang, "Modelling and optimal state-delay control in microbial batch process," *Applied Mathematical Modelling*, vol. 89, pp. 792–801, 2021.

[11] C. Liu, R. Loxton, K. L. Teo, and S. Wang, "Optimal state-delay control in nonlinear dynamic systems," *Automatica*, vol. 135, Article ID 109981, 2022.

[12] J. Yuan, L. Wang, J. Zhai et al., "Robust optimal control for a batch nonlinear enzyme-catalytic switched time-delayed process with noisy output measurements," *Nonlinear Analysis: Hybrid Systems*, vol. 41, Article ID 101059, 2021.

[13] S. K. Choi and N. Koo, "On the stability of linear dynamic systems on time scales," *Journal of Difference Equations and Applications*, vol. 15, no. 2, pp. 167–183, 2009.

[14] S. K. Choi, Y. H. Cui, and N. Koo, "Variationally stable dynamic systems on time scales," *Advances in Difference Equations*, vol. 2012, no. 1, p. 129, 2012.

[15] Z. Cheng and Y. Gao, "Two classes of common lyapunov functions for switched systems," *Control and Decision*, vol. 28, no. 4, pp. 623–631, 2013.

[16] Z. Chen and Y. Gao, "On common linear copositive lyapunov functions for pairs of stable positive linear systems," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 4, pp. 467–474, 2009.

[17] Y. Q. Sun, W. T. Qi, H. Teng, Z. L. Xiu, and A. P. Zeng, "Mathematical modeling of glycerol fermentation by *Klebsiella pneumoniae*: concerning enzyme-catalytic reductive pathway and transport of glycerol and 1, 3-propanediol across cell membrane," *Biochemical Engineering Journal*, vol. 38, no. 1, pp. 22–32, 2008.

[18] J. L. Yuan, X. Zhang, X. Zhu, E. M. Feng, H. C. Yin, and Z. L. Xiu, "Modelling and pathway identification involving the transport mechanism of a complex metabolic system in batch culture," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 6, pp. 2088–2103, 2014.

[19] H. Kitano, "Towards a theory of biological robustness," *Molecular Systems Biology*, vol. 3, no. 1, p. 137, 2007.

[20] J. Hale and D. P. Hou, *Ordinary Differential Equations*, People's Education Press, Beijing, China, 1980.