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Research Article

Two New Weak Convergence Algorithms for Solving Bilevel Pseudomonotone Equilibrium Problem in Hilbert Space

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In this paper, we introduce two new subgradient extragradient algorithms to find the solution of a bilevel equilibrium problem in which the pseudomonotone and Lipschitz-type continuous bifunctions are involved in a real Hilbert space. The first method needs the prior knowledge of the Lipschitz constants of the bifunctions while the second method uses a self-adaptive process to deal with the unknown knowledge of the Lipschitz constant of the bifunctions. The weak convergence of the proposed algorithms is proved under some simple conditions on the input parameters. Our algorithms are very different from the existing related results in the literature. Finally, some numerical experiments are presented to illustrate the performance of the proposed algorithms and to compare them with other related methods.

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $g\colon H\times H\longrightarrow\mathbb{R}$ be a bifunction with $g(x,x)=0$ for all $x\in C$. The equilibrium problem (EP for short) is associated with g and C to find $z\in C$ such that

$$g(z,y)\geq 0,\quad \forall y\in C. \tag{1}$$

The solution set of (1) is denoted by $EP(g,C)$.

If $g(x,y)=\langle G(x),y-x\rangle$ for all $x,y\in H$, where G is a mapping from H into itself, then the problem (1) becomes the following variational inequality problem (VIP for short):

$$\text{find } x^{*}\in C \text{ such that } \langle G(x^{*}),y-x^{*}\rangle\geq 0,\quad \forall y\in C. \tag{2}$$

The solution set of (2) is denoted by $VI(G,C)$.

The EP (1) has a simple form and is very general in the sense that it includes, as special cases, the variational inequality problem, fixed point problem, complementarity problem, optimization problem as well as the Nash equilibrium problem; see,for example [1,2]. Many methods have been proposed for approximating a solution of the EP (1). Mastroeni [3] used the auxiliary problem principle which was first introduced for solving the optimization problems to solve EP (1) and presented the iteration algorithm in the form

$$x_0\in C, x_{n+1}=\operatorname{argmin}\left\{\lambda g(x_n,y)+\frac{1}{2}\|y-x_n\|^2\colon\ y\in C\right\}, \tag{3}$$

where the stepsize $\lambda>0$. For obtaining the convergence of this algorithm, the bifunction g is required to be strongly monotone and Lipschitz-type continuous. To avoid the hypothesis of the strong monotonicity, Quoc et al. [4] first proposed the extragradient method (or the proximal-like methods) in which two strongly convex problems are solved at each iteration. The extragradient method is as follows: $x_0\in C$ and

$$\begin{cases} y_n=\operatorname{argmin}\left\{\lambda g(x_n,y)+\frac{1}{2}\|y-x_n\|^2\colon\ y\in C\right\},\\ x_{n+1}=\operatorname{argmin}\left\{\lambda g(y_n,y)+\frac{1}{2}\|y-x_n\|^2\colon\ y\in C\right\}.\end{cases} \tag{4}$$

In 2018, Hieu [5] presented a new extragradient method for solving the EP (1.1) as follows: $x_0, y_0\in C$ and

$$\begin{cases} x_{n+1} = \operatorname{argmin}\left\{\lambda_n f(y_n, y) + \frac{1}{2}\|y - x_n\|^2 \colon y \in C\right\}, \\ y_{n+1} = \operatorname{argmin}\left\{\lambda_{n+1} f(y_n, y) + \frac{1}{2}\|y - x_{n+1}\|^2 \colon y \in C\right\}, n \geq 0, \end{cases} \tag{5}$$

where $\{\lambda_n\} \subset (0, \infty)$ is a nonincreasing sequence and f is a strongly pseudomonotone and Lipschitz-type continuous mapping.

In 2011, Censor et al. [6] proposed a new method, which is called the subgradient extragradient method, for solving the VIP (2). In 2016, Hieu [7] extended this method to the EP (1.1). In 2019, inspired by [5,7], Liu and Kong [8] introduced the following subgradient extragradient method for solving the EP (1): $x_0, y_0 \in C$ and

$$\begin{cases} x_1 = \operatorname{argmin}\left\{\lambda f(y_0, y) + \frac{1}{2}\|y - x_0\|^2 \colon y \in C\right\}, \\ y_1 = \operatorname{argmin}\left\{\lambda f(y_0, y) + \frac{1}{2}\|y - x_1\|^2 \colon y \in C\right\}, \\ x_{n+1} = \operatorname{argmin}\left\{\lambda f(y_0, y) + \frac{1}{2}\|y - x_n\|^2 \colon y \in H_n\right\}, \\ y_{n+1} = \operatorname{argmin}\left\{\lambda f(y_0, y) + \frac{1}{2}\|y - x_{n+1}\|^2 \colon y \in C\right\}, n \geq 1, \end{cases} \tag{6}$$

where $H_n = \{z \in H\colon \langle x_n - \lambda w_{n-1} - y_n, z - y_n\rangle \leq 0\}$ and $w_{n-1} \in \partial_2 f(y_{n-1}, y_n)$, and f is a pseudomonotone and Lipschitz-type continuous mapping.

The advantage of equations (5) and (6) is that only one value of f at y_n is computed at each iteration. On the recent methods for solving the EP (1), we refer the readers to [9–15].

In this paper, our interest is the bilevel equilibrium problem (BEP for short) which consists of the following:

$$\text{find } \overline{x} \in EP(g, C) \text{ such that } f(\overline{x}, y) \geq 0, \quad \forall y \in EP(g, C), \tag{7}$$

where $f\colon H \times H \longrightarrow \mathbb{R}$ with $f(x, x) = 0$ for all $x \in H$. The BEPs are the special cases of mathematical programs with equilibrium constraints and also are the generalization of variational inequality over equilibrium constraints, hierarchical minimization problems, and complementarity problems. The methods for solving BEPs have been studied extensively by many authors. Moudafi [16] introduced a proximal method and proved the weak convergence to a solution of the BEP (7). Dinh and Muu [17] proposed a penalty and gap function method for solving the BEP (7). Quy [18] introduced an algorithm by combining the proximal method with the Halpern method for solving bilevel monotone equilibrium and fixed point problem. Yuying et al. [19] presented an extragradient method as follows:

$$\begin{cases} y_n = \operatorname{argmin}\left\{\lambda_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 \colon y \in C\right\}, \\ z_n = \operatorname{argmin}\left\{\lambda_n g(y_n, y) + \frac{1}{2}\|y - x_n\|^2 \colon y \in C\right\}, \\ x_{n+1} = \eta_n x_n + (1 - \eta_n) z_n - \alpha_n \mu w_n,\ w_n \in \partial_2 f(z_n, z_n), \end{cases} \tag{8}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\} \subset [\underline{\lambda}, \overline{\lambda}]$ with $\underline{\lambda} > 0$, and $\{\eta_n\} \subset [0, 1 - \alpha_n]$. Anh and An [20] proposed the following subgradient extragradient method for solving the BEP (7):

$$\begin{cases} y_n = \operatorname{argmin}\left\{\lambda_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 \colon y \in C\right\}, \\ z_n = \operatorname{argmin}\left\{\lambda_n g(y_n, y) + \frac{1}{2}\|z - x_n\|^2 \colon z \in H_n\right\}, \\ x_{n+1} = \operatorname{argmin}\left\{\beta_n f(z_n, y) + \frac{1}{2}\|t - z_n\|^2 \colon z \in C\right\}, \end{cases} \tag{9}$$

where $H_n = \langle v \in H\colon \langle x_n - \lambda_n w_n - y_n, v - y_n\rangle \leq 0\rangle$ with $w_n \in \partial_2 g(x_n, y_n)$, $\{\lambda_n\}$ and $\{\beta_n\}$ are two nonnegative sequences.

Observe that in the works mentioned above, the bifunction g is monotone or pseudomonotone while f is strongly monotone, and then, the algorithms have a strong convergence. In this paper, inspired by [8,20], we propose two new subgradient extragradient methods for solving the BEP (7) where both the bifunction f and g are pseudomonotone. The first method needs the prior knowledge of the Lipschitz constants of the bifunctions while the second method uses a self-adaptive process to deal with the unknown knowledge of the Lipschitz constant of the bifunctions. The weak convergence of the proposed algorithms is proved under some sufficient assumptions. Finally, some numerical experiments are presented to illustrate the performance of the proposed algorithms and to compare them with other related methods.

2. Preliminaries

Let H be a real Hilbert space, $\mathbb{R}$ be the set of all real numbers, and $\mathbb{N}$ be the set of all positive integers. We list some well-known definitions and properties which will be used in our following analysis.

Definition 1. A mapping $F\colon H \longrightarrow H$ is said to be

(i) monotone if

$$\langle F(x) - F(y), x - y\rangle \geq 0, \quad \forall x, y \in H; \tag{10}$$

(ii) pseudomonotone if

$$\langle F(y), x - y\rangle \geq 0 \Rightarrow \langle F(x), x - y\rangle \geq 0, \quad \forall x, y \in H; \tag{11}$$

(iii) L-Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|F(x) - F(y)\| \leq \|x - y\|, \quad \forall x, y \in H. \tag{12}$$

Definition 2. A bifunction $f\colon H \times H \longrightarrow \mathbb{R}$ is said to be

(i) pseudomonotone on C if

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0, \quad \forall x, y \in C. \tag{13}$$

(ii) Lipschitz-type continuous on C if there exists the constants $c_1 > 0$ and $c_2 > 0$ such that

$$f(x, z) \leq f(x, y) + f(y, z) + c_1\|x - y\|^2 + c_2\|y - z\|^2, \quad \forall x, y, z \in C. \tag{14}$$

Remark 1. If F is L-Lipschitz continuous on H, then for each $x, y \in H$, $f(x, y) = \langle F(x), y - x\rangle$ is Lipschitz-type continuous with the constants $c_1 = c_2 = (L/2)$; see [21] for details.

Let C be a nonempty closed and convex subset of H. For each $x \in H$, there exists a unique point in C, denoted by $P_C x$, such that

$$P_C x = \arg\min\{\|y - x\| : y \in C\}, \tag{15}$$

P_C is said to be the metric projection from H onto C. The following lemma characterizes the property of P_C.

Lemma 1. *Let $P_C\colon H \longrightarrow C$ be the metric projection. Then,*

(i) $z = P_C x$ if and only if

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C. \tag{16}$$

(ii) for all $y \in C$ and $x \in H$,

$$\|y - P_C x\|^2 + \|P_C x - x\|^2 \leq \|x - y\|^2. \tag{17}$$

Remark 2. For any given $\overline{x} \in H$ and $v \in H$ with $v \neq 0$, let $T = \{x \in H\colon \langle v, x - \overline{x}\rangle \leq 0\}$. Then, for all $y \in H$, the projection $\Pi_T(y)$ is defined by

$$\prod_T (y) = y - \max\left\{0, \frac{\langle v, y - \overline{x}\rangle}{\|v\|^2}\right\} v. \tag{18}$$

The formula (18) gives us an explicit manner to compute the projection of any point onto a half-space; see [22] for details.

Definition 3.

(1) The normal cone N_C of C at $x \in C$ is defined by

$$N_C(x) = \{w \in H\colon \langle w, y - x\rangle \leq 0, \forall y \in C\}. \tag{19}$$

(2) The subdifferentiable of a convex function $g\colon C \longrightarrow \mathbb{R}$ at $x \in H$ is defined by

$$\partial g(x) = \{w \in H\colon g(y) - g(x) \geq \langle w, y - x\rangle, \forall y \in C\}. \tag{20}$$

Lemma 2 (see [23]). *Let $g\colon C \longrightarrow \mathbb{R}$ be a convex subdifferentiable and lower semicontinuous function on C. Then, x^* is a solution to the following convex problem:*

$$\min\{g(x)\colon x \in C\}, \tag{21}$$

if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(x^*)$ denotes the subdifferential of g and $N_C(x^*)$ is the normal cone of C at x^*.

For a proper, convex, and lower semicontinuous function: $h\colon C \longrightarrow (-\infty, +\infty]$ and $\lambda > 0$, the proximal mapping of h with λ is defined by

$$\text{prog}_{\lambda h}(x) = \text{argmin}\left\{\lambda h(y) + \frac{1}{2}\|x - y\|^2\colon y \in C\right\}, x \in C. \tag{22}$$

Lemma 3 (see [24, 25]). *For all $x, y \in C$ and $\lambda > 0$, the following inequality holds:*

$$\lambda\left(h(y) - h\left(\text{prog}_{\lambda h}(x)\right)\right) \geq \langle x - \text{prog}_{\lambda h}(x), y - \text{prog}_{\lambda h}(x)\rangle. \tag{23}$$

Remark 3. From Lemma 3, we note that if $x = \text{prog}_{\lambda h}(x)$, then

$$x \in \text{argmin}\{h(y)\colon y \in C\} = \left\{x \in C\colon h(x) = \min_{y \in C} h(y)\right\}. \tag{24}$$

Lemma 4 (see [26]). *Let $\{a_n\}$ and $\{c_n\}$ be two sequences of nonnegative real numbers satisfying the condition*

$$a_{n+1} \leq a_n + c_n, \quad \forall n \in \mathbb{N}. \tag{25}$$

If $\sum_n c_n < \infty$, then $\lim_{n\longrightarrow\infty} a_n$ exists.

3. Main Results

In this section, let $\mathbb{N}$ denotes the set of all positive integers, $N_0 = \mathbb{N} \cup \{0\}$, H be a real Hilbert space, and C be a nonempty closed convex subset of H. The notation "$\rightharpoonup$" denotes the weak converge. Let $f, g\colon H \times H \longrightarrow \mathbb{R}$ be two bifunctions satisfying the following conditions:

(A1) f and g are pseudomonotone on H

(A2) for each $y \in H$, $\limsup_{n\longrightarrow\infty} f(x_n, y) \leq f(x, y)$ and $\limsup_{n\longrightarrow\infty} g(x_n, y) \leq g(x, y)$ for every sequence $x_n \rightharpoonup x$

(A3) $f(x, \cdot)$ and $g(x, \cdot)$ are convex, lower semicontinuous, and subdifferentiable of H for each $x \in H$

(A4) g and f are Lipschitz-type continuous on H with the constants c_1, c_2 and d_1, d_2, respectively; that is, for all $x, y, z \in H$,

$$g(x,z) \le g(x,y) + g(y,z) + c_1\|x-y\|^2 + c_2\|y-z\|^2,$$
$$f(x,z) \le f(x,y) + f(y,z) + d_1\|x-y\|^2 + d_2\|y-z\|^2. \tag{26}$$

In this section, the solution set of the BEP (7) is denoted by Ω; that is, $\Omega = \{x \in E(g,C)\colon f(\overline{x}, y) \ge 0, \forall y \in E\ P(g,C)\}$, and assume that $\Omega \ne \emptyset$.

Now, we introduce the first algorithm for finding a point $\overline{x} \in \Omega$.

Remark 4. By using the notation "prog" in Section 2, u_n, t_n, x_{n+1}, and y_{n+1} may be rewritten as

$$\begin{cases} u_n = \operatorname{prog}_{\beta g(x_n,\cdot)}(x_n), \\ t_n = \operatorname{prog}_{\beta g(u_n,\cdot)}(x_n), \end{cases}$$
$$\begin{cases} x_{n+1} = \operatorname{prog}_{\lambda f(y_n,\cdot)}(x_n), \\ y_{n+1} = \operatorname{prog}_{\lambda f(y_n,\cdot)}(x_{n+1}). \end{cases} \tag{27}$$

Note that since $f(x,\cdot)$ and $g(x,\cdot)$ are convex and lower semicontinuous on H for each $x \in H$, for any given $\beta > 0$, $u \in H$, and the closed convex subset $D \subset H$, from [27], Proposition 12.15, and Definition 12.23, it follows that both

$$\operatorname{argmin}\left\{\beta g(u,y) + \frac{1}{2}\|x-y\|^2\colon y \in D\right\},$$
$$\operatorname{argmin}\left\{\beta f(u,t) + \frac{1}{2}\|x-t\|^2\colon t \in D\right\}, \tag{28}$$

are a singleton. Hence, u_n, t_n, x_{n+1}, and y_{n+1} in Algorithm 1 are obtained uniquely at each step.

Remark 5. From (A1)–(A4), it follows that (i) $EP(g,C)$ and $EP(f,C)$ are closed and convex; see [4]; (ii) $g(x,x) = 0$ and $f(x,x) = 0$ for all $x \in C$; see [28].

The following remark shows that the stop criterion in Step 3 is meaning.

Remark 6. Suppose that $x_{n+1} = y_n = x_n = u_n$ for some $n \in \mathbb{N}$. By $u_n = x_n$, the definition of u_n, and Lemma 3, we get

$$\beta(g(u_n,y) - g(x_n,u_n)) = \beta(g(x_n,y) - g(x_n,x_n)) \ge 0, \quad \forall y \in C, \tag{29}$$

which with $\beta > 0$ Remark 5 implies that $x_n \in EP(g,C)$. Similarly, by $x_{n+1} = y_n = x_n$, the definition of x_{n+1}, and Lemma 3, we can prove that $x_n \in EP(f, H_n)$. By the proof of Lemma 4, we see $EP(g,C) \subset H_n$ for all $n \in \mathbb{N}$. So $f(x_n, y) \ge 0$ for all $y \in EP(g,C)$. It follows that $x_n \in \Omega$.

Lemma 5. *Assume that $\beta \in (0, \min\{(1/2c_1), (1/2c_2)\})$. Then, $C \subset C_k$, $EP(g,C) \subset T_n \subset H_{n+1}$ for each $n \in \mathbb{N}_0$.*

Proof. We first show that $C \subset C_k$ for each $n \in \mathbb{N}_0$. By Lemma 1 and the definition of u_k, we have

$$0 \in \partial_2\left\{\beta g(x_n,y) + \frac{1}{2}\|x_n - y\|^2\right\}(u_n) + N_C(u_n). \tag{30}$$

Thus, for $w_n \in \partial_2 g(x_n,u_n)$, there exists $\overline{w}_n \in N_C(u_n)$ such that

$$\beta w_n + u_n - x_n + \overline{w}_n = 0. \tag{31}$$

So,

$$\langle x_n - u_n, y - u_n\rangle = \beta\langle w_n, y - u_n\rangle + \langle \overline{w}_n, y - u_n\rangle, \quad \forall y \in C. \tag{32}$$

Since $\overline{w}_n \in N_C(u_n)$, we have $\langle \overline{w}_n, y - u_n\rangle \le 0$ for all $y \in C$. Hence, $\beta\langle w_n, y - u_n\rangle \ge \langle x_n - u_n, y - u_n\rangle$ for all $y \in C$, which implies that $\langle x_n - \beta w_n - u_n, y - u_n\rangle \le 0$ for all $y \in C$. This shows that $C \subset C_n$ for each $\mathbb{N}_0$.

Next, we show that $EP(g,C) \subset T_n$ for each $n \in \mathbb{N}_0$. By Lemma 3 and the definition of t_n in Remark 4, we have

$$\beta(g(u_n,y) - g(u_n,t_n)) \ge \langle x_n - t_n, y - t_n\rangle, \quad \forall y \in C_n,\ \forall n \in \mathbb{N}_0. \tag{33}$$

Note, we have proved that $C \subset C_n$ for each $n \in \mathbb{N}_0$. So substituting any $x' \in EP(g,C) \subset C$ into (33), we obtain

$$\beta(g(u_n,x') - g(u_n,t_n)) \ge \langle x_n - t_n, x' - t_n\rangle,\ \forall n \in \mathbb{N}_0. \tag{34}$$

Since $u_n \in C$ and g is pseudomonotone on H, we have $g(u_n, x') \le 0$. Then, (34) implies that

$$\langle x_n - t_n, t_n - x'\rangle \ge \beta g(u_n,t_n),\ \forall n \in \mathbb{N}_0. \tag{35}$$

Now applying (A4) to g, we have

$$g(u_n,t_n) \ge g(x_n,t_n) - g(x_n,u_n) - c_1\|u_n - x_n\|^2 - c_2\|t_n - u_n\|^2, \quad \forall n \in \mathbb{N}_0. \tag{36}$$

Combining (35) and (36), we get

$$\langle x_n - t_n, t_n - x'\rangle \ge \beta\Big(g(x_n,t_n) - g(x_n,u_n) - c_1\|u_n - x_n\|^2 - c_2\|t_n - u_n\|^2\Big), \quad \forall n \in \mathbb{N}_0. \tag{37}$$

On the other hand, by the definition of $w_n \in \partial_2 g(x_n,u_n)$, we have

$$\beta(g(x_n,y) - g(x_n,u_n)) \ge \langle \beta w_n, y - u_n\rangle, \quad \forall y \in H,\ \forall n \in \mathbb{N}_0. \tag{38}$$

Since $t_n \in C_n$, we have

Initialization: Choose $x_0, y_0, y_{-1} \in C$ and the parameters $\beta > 0$ and $\lambda > 0$. Put $n = 0$.
Step 1. For given x_n, solve the strongly convex problems: $\begin{cases} u_n = \operatorname{argmin}\left\{\beta g(x_n, y) + 1/2\|x_n - y\|^2 \colon y \in C\right\}, \\ t_n = \operatorname{argmin}\left\{\beta g(u_n, t) + 1/2\|x_n - y\|^2 \colon t \in C_n\right\}, \end{cases}$
where $C_n = \{v \in H \colon \langle x_n - \beta w_n - u_n, v - u_n\rangle \le 0\}$ with $w_n \in \partial_2 g(x_n, u_n)$.
Step 2. Solve the strongly convex problems: $\begin{cases} x_{n+1} = \arg\min\left\{\lambda f(y_n, y) + 1/2\|x_n - y\|^2 \colon y \in H_n\right\}, \\ y_{n+1} = \arg\min\left\{\lambda f(y_n, y) + 1/2\|x_{n+1} - y\|^2 \colon y \in T_n\right\}, \end{cases}$
where $H_n = \{z \in H \colon \langle x_n - \lambda v_n - y_n, z - y_n\rangle \le 0\}$ with $v_n \in \partial_2 f(y_{n-1}, y_n), T_n = \{z \in H \colon \|z - t_n\| \le \|z - x_n\|\}$.
Step 3. If $x_{n+1} = y_n = x_n = u_n$, then the algorithm stops, $x_n \in \Omega$; otherwise, set $n = n + 1$ and return to Step 1.

Algorithm 1: (Extragradient-like method without prior constants).

$$\langle x_n - u_n, t_n - u_n\rangle \le \beta\langle w_n, t_n - u_n\rangle, \tag{39}$$

which with (38) implies that

$$\beta\left(g(x_n, t_n) - g(x_n, u_n)\right) \ge \langle x_n - u_n, t_n - u_n\rangle, \ \forall n \in \mathbb{N}_0. \tag{40}$$

From (37) and (40) and

$$2\langle t_n - x_n, x' - t_n\rangle = \left\|x_n - x'\right\|^2 - \left\|t_n - x_n\right\|^2 - \left\|t_n - x'\right\|^2, \tag{41}$$

it follows that

$$\begin{aligned} &\left\|x_n - x'\right\|^2 - \left\|t_n - x_n\right\|^2 - \left\|t_n - x'\right\|^2 \\ &\quad \ge 2\langle x_n - u_n, t_n - u_n\rangle - 2\beta c_1\left\|x_n - u_n\right\|^2 \\ &\qquad - 2\beta c_2\left\|u_n - t_n\right\|^2, \ \forall n \in \mathbb{N}_0. \end{aligned} \tag{42}$$

Hence,

$$\begin{aligned} \left\|t_n - x'\right\|^2 &\le \left\|x_n - x'\right\|^2 - \left\|t_n - x_n\right\|^2 \\ &\quad - 2\langle x_n - u_n, t_n - u_n\rangle + 2\beta\Big(c_1\left\|x_n - u_n\right\|^2 \\ &\quad + c_2\left\|u_n - t_n\right\|^2\Big) \\ &= \left\|x_n - x'\right\|^2 - \left\|t_n - u_n\right\|^2 - \left\|u_n - x_n\right\|^2 \\ &\quad + 2\beta\Big(c_1\left\|x_n - u_n\right\|^2 + c_2\left\|u_n - t_n\right\|^2\Big) \\ &= \left\|x_n - x'\right\|^2 - (1 - 2\beta c_1)\left\|x_n - u_n\right\|^2 \\ &\quad + (1 - 2\beta c_2)\left\|u_n - t_n\right\|^2, \quad \forall n \in \mathbb{N}_0. \end{aligned} \tag{43}$$

In particular, from $1 - 2\beta c_1 > 0$ and $1 - 2\beta c_2 > 0$, it follows that

$$\left\|t_n - x'\right\|^2 \le \left\|x_n - x'\right\|^2, \quad \forall n \in \mathbb{N}_0, \tag{44}$$

which implies that $x' \in T_n$. Since $x' \in EP(g, C)$ is arbitrary, it follows that $EP(g, C) \subset T_n$ for each $n \in \mathbb{N}_0$.

Finally, we prove that $T_n \subset H_{n+1}$ for each $n \in \mathbb{N}_0$. By the definition of y_{n+1} in Remark 4 and Lemma 3, we have

$$0 \in \lambda\partial_2 f(y_n, y_{n+1}) + y_{n+1} - x_{n+1} + N_{T_n}(y_{n+1}), \ \forall n \in \mathbb{N}_0. \tag{45}$$

Thus, for $v_n \in \partial_2 f(y_{n-1}, y_n)$, there exists $w_n \in N_{T_n}(y_{n+1})$ such that

$$\lambda v_n + y_{n+1} - x_{n+1} + w_n = 0, \quad \forall n \in \mathbb{N}_0. \tag{46}$$

It follows that

$$\begin{aligned} \langle x_{n+1} - y_{n+1}, y - y_{n+1}\rangle &= \lambda\langle v_n, y - y_{n+1}\rangle \\ &\quad + \langle w_n, y - y_{n+1}\rangle, \\ &\quad \forall y \in T_n, \ \forall n \in \mathbb{N}_0. \end{aligned} \tag{47}$$

Since $w_n \in N_{T_n}(y_{n+1})$, we have $\langle w_n, y - y_{n+1}\rangle \le 0$ for all $y \in T_n$. Hence, $\lambda\langle v_n, y - y_{n+1}\rangle \ge \langle x_{n+1} - y_{n+1}, y - y_{n+1}\rangle$ for all $y \in T_n$ and $n \in \mathbb{N}_0$, which with the definition of H_{n+1} implies that $T_n \subset H_{n+1}$ for each $n \in \mathbb{N}_0$. This completes the proof. □

Lemma 6. *Assume that $\beta \in (0, \min\{(1/2c_1), (1/2c_2)\})$ and $\lambda \in (0, (1/2d_2 + 4d_1))$. Let $\{x_n\}$ be the sequence generated by Algorithm 1. For all $x^* \in \Omega$, the limit of $\left\{\|x^* - x_n\|^2\right\}$ exists, and*

$$\begin{aligned} \lambda f(y_n, y) \ge \lambda\Big[&\langle x_n - y_n, x_{n+1} - y_n\rangle - c_1\left\|y_{n-1} - y_n\right\|^2 - c_2\left\|y_n - x_{n+1}\right\|^2\Big] \\ &+ \langle x_n - x_{n+1}, y - x_{n+1}\rangle, \ \forall y \in EP(g, C), \ \forall n \in \mathbb{N}. \end{aligned} \tag{48}$$

Proof. Since $\left\{\|x^* - x_n\|^2\right\}$, from the definition of H_n, it follows that

$$\langle x_n - \lambda v_n - y_n, x_{n+1} - y_n\rangle \le 0, \quad \forall n \in \mathbb{N}, \tag{49}$$

that is,

$$\lambda\langle v_n, x_{n+1} - y_n\rangle \geq \langle x_n - y_n, x_{n+1} - y_n\rangle, \quad \forall n \in \mathbb{N}. \tag{50}$$

By $v_n \in \partial_2 f(y_{n-1}, y_n)$ and the definition of sub-differential, we have

$$f(y_{n-1}, y) - f(y_{n-1}, y_n) \geq \langle v_n, y - y_n\rangle, \quad \forall y \in H, \forall n \in \mathbb{N}. \tag{51}$$

Replacing y in (51) with x_{n+1}, we get

$$f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) \geq \langle v_n, x_{n+1} - y_n\rangle, \quad \forall n \in \mathbb{N}. \tag{52}$$

Combining (50) and (52), we have

$$\lambda\left(f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n)\right) \geq \langle x_n - y_n, x_{n+1} - y_n\rangle, \quad \forall n \in \mathbb{N}. \tag{53}$$

By Lemma 3 and the definition of x_{n+1}, we have

$$\lambda\left(f(y_n, y) - f(y_n, x_{n+1})\right) \geq \langle x_n - x_{n+1}, y - x_{n+1}\rangle, \quad \forall y \in H_n, \forall n \in \mathbb{N}. \tag{54}$$

Substituting $y = x^* \in \Omega$ into (54), we obtain

$$\lambda\left(f(y_n, x^*) - f(y_n, x_{n+1})\right) \geq \langle x_n - x_{n+1}, x^* - x_{n+1}\rangle, \quad \forall n \in \mathbb{N}. \tag{55}$$

Note that (A1) implies that $f(y_n, x^*) \leq 0$, which with (55) leads to

$$\lambda f(y_n, x_{n+1}) \leq \langle x_n - x_{n+1}, x_{n+1} - x^*\rangle, \quad \forall n \in \mathbb{N}. \tag{56}$$

On the other hand, by the Lipschitz-type continuity of f, we have

$$f(y_n, x_{n+1}) \geq f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - d_1\|y_n - y_{n-1}\|^2 - d_2\|x_{n+1} - y_n\|^2, \forall n \in \mathbb{N}. \tag{57}$$

By (56) and (57), we obtain

$$\langle x_n - x_{n+1}, x_{n+1} - x^*\rangle \geq \lambda\Big(f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - d_1\|y_n - y_{n-1}\|^2 - d_2\|x_{n+1} - y_n\|^2\Big), \tag{58}$$

which with (53) implies that

$$\begin{aligned}\langle x_n - x_{n+1}, x_{n+1} - x^*\rangle &\geq \langle x_n - y_n, x_{n+1} - y_n\rangle - \lambda\Big(d_1\|y_n - y_{n-1}\|^2 + d_2\|x_{n+1} - y_n\|^2\Big)\\ &= \frac{1}{2}\Big(\|y_n - x_n\|^2 + \|x_{n+1} - y_n\|^2 - \|x_{n+1} - x_n\|^2\Big) - \lambda\Big(d_1\|y_n - y_{n-1}\|^2 + d_2\|x_{n+1} - y_n\|^2\Big), \quad \forall n \in \mathbb{N}.\end{aligned} \tag{59}$$

Since

$$\langle x_n - x_{n+1}, x_{n+1} - x^*\rangle = \frac{1}{2}\Big(\|x_n - x^*\|^2 - \|x_{n+1} - x_n\|^2 - \|x_{n+1} - x^*\|^2\Big), \tag{60}$$

by (59) we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2\|^2 - \|y_n - x_n\|^2 - \|x_{n+1} - y_n\|^2 + 2\lambda d_1\|y_n - y_{n-1}\|^2 + 2\lambda d_2\|x_{n+1} - y_n\|^2\\ &\leq \|x_n - x^*\|^2\|^2 - \|y_n - x_n\|^2 - \|x_{n+1} - y_n\|^2 + 2\lambda d_1\big(\|y_n - x_n\| + \|x_n - y_{n-1}\|\big)^2 + 2\lambda d_2\|x_{n+1} - y_n\|^2\\ &\leq \|x_n - x^*\|^2\|^2 - \|y_n - x_n\|^2 - \|x_{n+1} - y_n\|^2 + 4\lambda d_1\Big(\|y_n - x_n\|^2 + \|x_n - y_{n-1}\|^2\Big) + 2\lambda d_2\|x_{n+1} - y_n\|^2\\ &= \|x_n - x^*\|^2\|^2 - (1 - 4\lambda d_1)\|y_n - x_n\|^2 - (1 - 2\lambda d_2)\|x_{n+1} - y_n\|^2 + 4\lambda d_1\|x_n - y_{n-1}\|^2, \forall n \in \mathbb{N}.\end{aligned} \tag{61}$$

Fix $N \in \mathbb{N}$. For all $m \in \mathbb{N}$ with $m > N$, by (61) we have

$$\|x^* - x_{m+1}\|^2 \le \|x^* - x_N\|^2 - (1 - (4\lambda d_1 + 2\lambda d_2)) \sum_{n=N}^{m} \|x_{n+1} - y_n\|^2 - (1 - 4\lambda c_1) \sum_{n=N}^{m} \|y_n - x_n\|^2 + 4\lambda c_1 \|x_N - y_{N-1}\|^2. \quad (62)$$

Hence,

$$(1 - 4\lambda d_1 - 2\lambda d_2) \sum_{n=N}^{m} \|x_{n+1} - y_n\|^2 + (1 - 4\lambda d_1) \sum_{n=N}^{m} \|y_n - x_n\|^2 < \|x^* - x_N\|^2 < \infty, \quad \forall m > N, \quad (63)$$

which with $4\lambda d_1 + 2\lambda d_2 < 1$ leads to

$$\sum_{n=1}^{\infty} \|x_{n+1} - y_n\|^2 < \infty, \qquad \sum_{n=1}^{\infty} \|y_n - x_n\|^2 < \infty. \quad (64)$$

It follows that

$$\lim_{n\longrightarrow\infty} \|x_n - y_n\|^2 = 0, \qquad \lim_{n\longrightarrow\infty} \|x_{n+1} - y_n\|^2 = 0. \quad (65)$$

By (65) and the triangle inequality of norm, we obtain

$$\|x_n - x_{n+1}\| \le \|x_n - y_n\| + \|y_n - x_{n+1}\| \longrightarrow 0, \text{ as } n \longrightarrow \infty, \quad (66)$$

$$\|y_n - y_{n+1}\| \le \|y_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - y_{n+1}\| \longrightarrow 0, \text{ as } n \longrightarrow \infty, \quad (67)$$

and

$$\|y_{n+1} - x_n\| \le \|y_{n+1} - y_n\| + \|y_n - x_n\| \longrightarrow 0, \text{ as } n \longrightarrow \infty. \quad (68)$$

Note that (61) implies

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 + 4\lambda d_1 \|y_{n-1} - x_n\|^2, \quad \forall n \in \mathbb{N}. \quad (69)$$

From (62) and (69) and Lemma 3, it follows that the limit of $\{\|x_n - x^*\|^2\}$ exists.

Finally, by (54), (56), and (52), we get

$$\begin{aligned}
\lambda f(y_n, y) &\ge \lambda f(y_n, x_{n+1}) + \langle x_n - x_{n+1}, y - x_{n+1}\rangle \\
&\ge \lambda \left[f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - d_1 \|y_{n-1} - y_n\|^2 - d_2 \|y_n - x_{n+1}\|^2 \right] + \langle x_n - x_{n+1}, y - x_{n+1}\rangle \\
&\ge \lambda \left[\langle x_n - y_n, x_{n+1} - y_n\rangle - d_1 \|y_{n-1} - y_n\|^2 - d_2 \|y_n - x_{n+1}\|^2 \right] + \langle x_n - x_{n+1}, y - x_{n+1}\rangle, \quad \forall y \in H_n, \forall n \in \mathbb{N}.
\end{aligned} \quad (70)$$

Note that Lemma 4 has shown that $EP(g, C) \subset T_n \subset H_{n+1}$ for each $n \in \mathbb{N}_0$. So by (70), we have

$$\lambda f(y_n, y) \ge \lambda \left[\langle x_n - y_n, x_{n+1} - y_n\rangle - d_1 \|y_{n-1} - y_n\|^2 - d_2 \|y_n - x_{n+1}\|^2 \right] + \langle x_n - x_{n+1}, y - x_{n+1}\rangle, \quad \forall y \in EP(g, C), \forall n \in \mathbb{N}. \quad (71)$$

This completes the proof. □

Theorem 1. *If the parameters β and λ satisfy the conditions:*

$$\beta \in \left(0, \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}\right) \text{ and } \lambda \in \left(0, \frac{1}{2d_2 + 4d_1}\right), \tag{72}$$

then the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to the point $\overline{x} = \lim_{n\longrightarrow\infty} P_\Omega x_n$.

Proof. Since $y_{n+1} \in T_n$ for each $n \in \mathbb{N}_0$, by (66) we have

$$\|t_n - y_{n+1}\| \le \|y_{n+1} - x_n\| \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{73}$$

Furthermore, by (68) and (73), we get

$$\|t_n - x_n\| \le \|t_n - y_{n+1}\| + \|y_{n+1} - x_n\| \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{74}$$

From Lemma 5, it follows that $\{x_n\}$ is bounded. This fact with (74) implies that $\{t_n\}$ is also bounded.

Take $x' \in EP(g, C)$ and put $M = \sup_{n\in\mathbb{N}} (\|x_n - x'\| + \|t_n - x'\|)$. By (43) and (74), we have

$$\begin{aligned}\left(1 - 2\beta c_1\|x_n - u_n\|^2 + (1 - 2\beta c_2)\|u_n - t_n\|^2 \le \|x_n - x'\|^2 - \|t_n - x'\|^2 \right.\\ \le \|x_n - t_n\|\left(\|x_n - x'\| + \|t_n - x'\|\right)\\ \le M\|x_n - t_n\| \longrightarrow 0, \text{ as } n \longrightarrow \infty,\end{aligned} \tag{75}$$

which with $1 - 2\beta c_1 > 0$ and $1 - 2\beta c_2 > 0$ implies

$$\begin{aligned}\|x_n - u_n\|^2 &\longrightarrow 0,\\ \|u_n - t_n\|^2 &\longrightarrow 0, \text{ as } n \longrightarrow \infty.\end{aligned} \tag{76}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ weakly converging to $\overline{x} \in H$. By (73), we can conclude that $\{u_{n_k}\}$ also weakly converges to $\overline{x}$. Since C is closed and $\{u_n\} \subset C$ for all $n \in \mathbb{N}$, it follows that $\overline{x} \in C$. We show that $\overline{x} \in EP(g, C)$. In fact, by (33), (36), and (40), we get

$$\beta g\left(u_{n_k}, y\right) \ge \beta\left(\langle x_{n_k} - u_{n_k}, t_{n_k} - u_{n_k}\rangle - c_1\left\|u_{n_k} - x_{n_k}\right\|^2 - c_2\left\|t_{n_k} - u_{n_k}\right\|^2\right) + \langle x_{n_k} - t_{n_k}, y - t_{n_k}\rangle, \quad \forall y \in C, \forall k \in \mathbb{N}. \tag{77}$$

Letting $k \longrightarrow \infty$ in (77), by (74), (76), and (A2), we get

$$\beta g(\overline{x}, ty) \ge \limsup_{k\longrightarrow\infty} g\left(u_{n_k}, y\right) \ge 0, \forall y \in C, \tag{78}$$

which with $\beta > 0$ implies that $\overline{x} \in EP(g, C)$.

Next, we prove that $\overline{x} \in \Omega$. To end this, we need to show that

$$f(\overline{x}, y) \ge 0, \ \forall y \in EP(g, C). \tag{79}$$

In fact, by (73), we have

$$\begin{aligned}\lambda f\left(y_{n_k}, y\right) \ge \lambda\left[\langle x_{n_k} - y_{n_k}, x_{n_k+1} - y_{n_k}\rangle - d_1\left\|y_{n_k-1} - y_{n_k}\right\|^2 - d_2\left\|y_{n_k} - x_{n_k+1}\right\|^2\right]\\ + \langle x_{n_k} - x_{n_k+1}, y - x_{n_k+1}\rangle, \ \forall y \in EP(g, C), \ \forall k \in \mathbb{N}_0.\end{aligned} \tag{80}$$

Letting $k \longrightarrow \infty$ in (80), by (65)–(67) and (A2), we have

$$\lambda f(\overline{x}, y) \ge \limsup_{k\longrightarrow\infty} f\left(y_{n_k}, y\right) \ge 0, \ \forall y \in EP(g, C), \tag{81}$$

which with $\lambda > 0$ implies that $f(\overline{x}, y) \ge 0$ for all $y \in EP(g, C)$. So, $\overline{x} \in \Omega$.

Now, we prove that the whole sequence $\{x_n\}$ converges weakly to the point $\overline{x}$. Indeed, assume that there exists a different subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to $x^\dagger$ with $\overline{x} \ne x^\dagger$. By arguing similarly as above, it follows that $x^\dagger \in \Omega$. Note that in the proof of Lemma 5, we have shown that the limits of $\{\|x_n - x^\dagger\|\}$ and $\{\|x_n - \overline{x}\|\}$ exist. So by Opial's theorem [29], we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|x_n - \overline{x}\| &= \liminf_{k\longrightarrow\infty}\left\|x_{n_k} - \overline{x}\right\| < \liminf_{k\longrightarrow\infty}\left\|x_{n_k} - x^\dagger\right\|\\ &= \lim_{n\longrightarrow\infty}\left\|x_n - x^\dagger\right\| = \liminf_{i\longrightarrow\infty}\left\|x_{n_i} - x^\dagger\right\|\\ &< \liminf_{i\longrightarrow\infty}\left\|x_{n_i} - \overline{x}\right\| = \lim_{n\longrightarrow\infty}\left\|x_n - \overline{x}\right\|.\end{aligned} \tag{82}$$

It is a contradiction. Hence, $\overline{x} = x^\dagger$. Therefore, the whole sequence $\{x_n\}$ converges weakly to the point $\overline{x}$.

Finally, we prove $\overline{x} = \lim_{n\longrightarrow\infty} P_\Omega x_n$. Let $w_n = P_\Omega x_n$ for all $n \geq 1$. It is easy to see that $\{w_n\}$ is bounded from the boundedness of $\{x_n\}$. We show that $\{w_n\}$ is a Cauchy sequence. By Lemma 1 and the definition of w_{n+1}, we have

$$\|w_{n+1} - x_{n+1}\|^2 \leq \|w_n - x_{n+1}\|^2, \ \forall n \in \mathbb{N}_0. \tag{83}$$

Since $w_n \in \Omega$, replacing x^* in (69) with w_n, we get

$$\|w_n - x_{n+1}\|^2 \leq \|x_n - w_n\|^2 + 4\lambda d_1 \|y_{n-1} - x_n\|^2, \ \forall n \in \mathbb{N}. \tag{84}$$

From (83) and (84), it follows that

$$\|w_{n+1} - x_{n+1}\|^2 \leq \|x_n - w_n\|^2 + 4\lambda d_1 \|y_{n-1} - x_n\|^2, \ \forall n \in \mathbb{N}. \tag{85}$$

From (64), (85), and Lemma 3, it follows that the limit of $\{\|w_n - x_n\|^2\}$ exists. For all $m, n \in \mathbb{N}$ with $m > n$, since $w_n \in \Omega$, by (69), we deduce

$$\begin{aligned}\|x_m - w_n\|^2 &\leq \|x_{m-1} - w_n\|^2 + 4\lambda d_1 \|y_{m-2} - x_{m-1}\|^2 \\ &\leq \cdots \leq \|x_n - w_n\|^2 + 4\lambda d_1 \sum_{k=n}^{m-1} \|y_{k-1} - x_k\|^2.\end{aligned} \tag{86}$$

From $w_m = P_\Omega x_m$ and $w_n \in \Omega$, by Lemma 1 and (86), we have

$$\begin{aligned}\|w_n - w_m\|^2 &\leq \|w_n - x_m\|^2 - \|w_m - x_m\|^2 \\ &\leq \|x_n - w_n\|^2 + 4\lambda d_1 \sum_{k=n}^{m-1} \|y_{k-1} - x_k\|^2 \\ &\quad - \|w_m - x_m\|^2, \ \forall m > n.\end{aligned} \tag{87}$$

Since $\lim_{n\longrightarrow\infty} \|x_n - w_n\|^2$ exists, letting $m, n \longrightarrow \infty$ in (87), by (64), we get $\lim_{n,m\longrightarrow\infty} \|w_n - w_m\|^2 = 0$. Consequently, $\{w_n\}$ is a Cauchy sequence. Since Ω is closed, $\{w_n\}$ converges strongly to some $x' \in \Omega$. Now, we prove that $\overline{x} = x'$. In fact, it follows from Lemma 1, $w_n = P_\Omega x_n$ and $\overline{x} \in \Omega$ that $\langle \overline{x} - w_n, w_n - x_n \rangle \geq 0$. Since $w_n \longrightarrow x'$ and $x_n \rightharpoonup \overline{x}$, we have $\langle \overline{x} - x', x' - \overline{x} \rangle \geq 0$. This shows that $\overline{x} = x' = \lim_{n\longrightarrow\infty} P_\Omega x_n$. This completes the proof. □

In Algorithm 1, c_1 and c_2 need to be known as the input parameters. The following algorithm is a modification in which c_1 and c_2 do not need to be known. □

Remark 7. By (A4), we have

$$\begin{aligned} g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n) &\leq c_1 \|u_n - x_n\|^2 + c_2 \|t_n - u_n\|^2 \\ &\leq c\left(\|u_n - x_n\|^2 + \|t_n - u_n\|^2\right),\end{aligned} \tag{88}$$

where $c = \max\{c_1, c_2\}$. If $g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n) \leq 0$, then

$$\begin{aligned}\beta_{n+1} &= \min\left\{\beta_n, \frac{\mu\left(\|u_n - x_n\|^2 + \|t_n - u_n\|^2\right)}{g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n)}\right\} \\ &\geq \min\left\{\beta_n, \frac{\mu\left(\|u_n - x_n\|^2 + \|t_n - u_n\|^2\right)}{c\left(\|u_n - x_n\|^2 + \|t_n - u_n\|^2\right)}\right\} = \min\left\{\beta_n, \frac{\mu}{c}\right\} \geq \cdots \geq \min\left\{\beta_0, \frac{\mu}{c}\right\},\end{aligned} \tag{89}$$

Note that from the definition of β_{n+1}, it follows that $\beta_{n+1} \geq \min\{\beta_0, (\mu/c)\}$ still holds even if $g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n) \leq 0$. Since $\{\beta_n\}$ is nonincreasing and bounded from below by $\min\{\beta_0, (\mu/c)\}$, there exists $\beta > 0$ such that

$$\lim_{n\longrightarrow\infty} \beta_n = \beta. \tag{90}$$

Similarly, we can conclude that there exists $\lambda > 0$ such that

$$\lim_{n\longrightarrow\infty} \lambda_n = \lambda > 0. \tag{91}$$

Theorem 2. *The sequence $\{x_n\}$ generated by Algorithm 2 converges weakly to the point $\overline{x} = \lim_{n\longrightarrow\infty} P_\Omega x_n$.*

Proof. Repeating the proof of (37) and (40), we can get, for all $n \in \mathbb{N}$,

$$\lambda_n (f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n)) \geq \langle x_n - y_n, x_{n+1} - y_n \rangle, \tag{92}$$

and

$$\lambda_n f(y_n, x_{n+1}) \leq \langle x_n - x_{n+1}, x_{n+1} - x^* \rangle. \tag{93}$$

Initialization: Choose $x_0, y_{-1}, y_0 \in C$, the parameters $\beta_0, \lambda_0 > 0$, and $\mu \in (0, 1/4)$. Put $n = 0$.

Step 1. For given x_n, solve the strongly convex problems: $\begin{cases} u_n = \operatorname{argmin}\{\beta_n g(x_n, y) + 1/2\|x_n - y\|^2 : y \in C\}, \\ t_n = \operatorname{argmin}\{\beta_n g(u_n, t) + 1/2\|x_n - y\|^2 : t \in C_k\}, \end{cases}$

where $C_n = \{v \in H: \langle x_n - \beta_n w_n - u_n, v - u_n\rangle \le 0\}$ with $w_n \in \partial_2 g(x_n, u_n)$.

Step 2. Solve the strongly convex problems: $\begin{cases} x_{n+1} = \arg\min\{\lambda_n f(y_n, y) + 1/2\|x_n - y\|^2 : y \in H_n\}, \\ y_{n+1} = \arg\min\{\lambda_{n+1} f(y_n, y) + 1/2\|x_{n+1} - y\|^2 : y \in T_n\}, \end{cases}$

where $H_n = \{z \in H: \langle x_n - \lambda_n v_n - y_n, z - y_n\rangle \le 0\}$ with $v_n \in \partial_2 f(y_{n-1}, y_n), T_n = \{z \in H: \|z - t_n\| \le \|z - x_n\|\}$,

$$\lambda_{n+1} = \begin{cases} \lambda_n, & f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1}) \le 0, \\ \min\{\lambda_n, \mu(\|y_n - y_{n-1}\|^2 + \|y_n - x_{n+1}\|^2)/f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1})\}, & \text{otherwise.} \end{cases}$$

Step 3. Modify β_{n+1} by the following formula:

$$\beta_{n+1} = \begin{cases} \beta_n, & g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n) \le 0, \\ \min\{\beta_n, \mu(\|u_n - x_n\|^2 + \|t_n - u_n\|^2)/g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n)\}, & \text{otherwise.} \end{cases}$$

Step 4. If $x_{n+1} = y_{n+1} = x_n = y_n$, then the algorithm stops, $x_n \in \Omega$; otherwise, set $n = n + 1$ and return to Step 1.

Algorithm 2: (Extragradient-like method without prior constants).

By (92) and (93), we have

$$\begin{aligned} \lambda_n (f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1})) &\ge \langle x_n - y_n, x_{n+1} - y_n\rangle - \langle x_n - x_{n+1}, x_{n+1} - x^*\rangle \\ &= \frac{1}{2}\left[\left(\|y_n - x_n\|^2 + \|x_{n+1} - y_n\|^2 - \|x_{n+1} - x_n\|^2\right) - \left(\|x_n - x^*\|^2 - \|x_{n+1} - x_n\|^2 - \|x_{n+1} - x^*\|^2\right)\right], \forall n \in \mathbb{N}. \end{aligned} \tag{94}$$

By the definition of λ_{n+1}, in the case when $f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1}) > 0$, we have

$$f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1}) \le \frac{\mu\left(\|y_n - y_{n-1}\|^2 + \|y_n - x_{n+1}\|^2\right)}{\lambda_{n+1}}, \quad \forall n \in \mathbb{N}. \tag{95}$$

It is emphasized here that (95) still holds even if

$$f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1}) \le 0. \tag{96}$$

So, combining (94) with (95), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \|x_n - x^*\|^2 - \left(\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2\right) + \frac{2\mu\lambda_n}{\lambda_{n+1}}\left(\|y_n - y_{n-1}\|^2 + \|y_n - x_{n+1}\|^2\right) \\ &\le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \left(1 - \frac{2\mu\lambda_n}{\lambda_{n+1}}\right)\|x_{n+1} - y_n\|^2 + \frac{4\mu\lambda_n}{\lambda_{n+1}}\left(\|y_n - x_n\|^2 + \|x_n - y_{n-1}\|^2\right) \\ &= \|x_n - x^*\|^2 - \left(1 - \frac{4\mu\lambda_n}{\lambda_{n+1}}\right)\|x_n - y_n\|^2 - \left(1 - \frac{2\mu\lambda_n}{\lambda_{n+1}}\right)\|x_{n+1} - y_n\|^2 + \frac{4\mu\lambda_n}{\lambda_{n+1}}\|x_n - y_{n-1}\|^2, \forall n \in \mathbb{N}. \end{aligned} \tag{97}$$

Since $\lambda_n \longrightarrow \lambda > 0$, it follows that $\lim_{n\longrightarrow\infty}(4\mu\lambda_n/\lambda_{n+1}) = 4\mu < 1$. Thus, for a fixed number $\epsilon \in (4\mu, 1)$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{4\mu\lambda_n}{\lambda_{n+1}} < \epsilon, \quad \forall n \geq n_0. \tag{98}$$

By (97) and (98), we have

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - (1-\epsilon)\|x_n - y_n\|^2 - \left(1 - \frac{\epsilon}{2}\right)\|x_{n+1} - y_n\|^2 + \epsilon\|x_n - y_{n-1}\|^2, \quad \forall n \geq n_0. \tag{99}$$

which is a similar result with (61).

On the other hand, for all $x' \in EP(g, C)$, repeating the proof of (35) and (38), we have

$$\langle x_n - u_n, t_n - u_n\rangle + \langle x_n - t_n, x' - t_n\rangle \leq \beta_n[g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n)], \quad \forall n \in \mathbb{N}_0. \tag{100}$$

By the definition of β_{n+1}, if $g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n) > 0$, then

$$g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n) \leq \frac{\mu\left(\|u_n - x_n\|^2 + \|t_n - u_n\|^2\right)}{\beta_{n+1}}, \quad \forall n \in \mathbb{N}_0. \tag{101}$$

Note that the definition of β_{n+1} implies that (101) still holds even if $g(x_n, t_n)g(x_n, u_n) - g(u_n, t_n) \leq 0$. So by (100) and (101), we obtain

$$\langle x_n - u_n, t_n - u_n\rangle + \langle x_n - t_n, x' - t_n\rangle \leq \frac{\mu\beta_n\left(\|u_n - x_n\|^2 + \|t_n - u_n\|^2\right)}{\beta_{n+1}}, \forall n \in \mathbb{N}_0. \tag{102}$$

Now, by (102) and $2\langle t_n - x_n, t_n - x'\rangle = \|x_n - x'\|^2 - \|t_n - x_n\|^2 - \|t_n - x'\|^2$, we have

$$\begin{aligned}\|t_n - x'\|^2 &\leq \|x_n - x'\|^2 - \|t_n - x_n\|^2 - 2\langle x_n - u_n, t_n - u_n\rangle + \frac{2\mu\beta_n\left(\|u_n - x_n\|^2 + \|t_n - u_n\|^2\right)}{\beta_{n+1}} \\ &= \|x_n - x'\|^2 - \|t_n - u_n\|^2 - \|u_n - x_n\|^2 + \frac{2\mu\beta_n\left(\|u_n - x_n\|^2 + \|t_n - u_n\|^2\right)}{\beta_{n+1}} \\ &= \|x_n - x'\|^2 - \left(1 - \frac{2\mu\beta_n}{\beta_{n+1}}\right)\|x_n - u_n\|^2 + \left(1 - \frac{2\mu\beta_n}{\beta_{n+1}}\right)\|u_n - t_n\|^2, \forall n \in \mathbb{N}_0.\end{aligned} \tag{103}$$

Since $\beta_n \longrightarrow \beta > 0$, it follows that $\lim_{n\longrightarrow\infty}(2\mu\beta_n/\beta_{n+1}) = 2\mu < 1$. Thus, for a fixed number $\tau \in (2\mu, 1)$, there exists $m_0 \in \mathbb{N}$ such that

$$\frac{2\mu\beta_n}{\beta_{n+1}} < \tau, \ \forall n \geq m_0. \tag{104}$$

By (103) and (104), we have

$$\begin{aligned}\|t_n - x'\|^2 &\leq \|x_n - x'\|^2 - (1-\tau)\|x_n - u_n\|^2 \\ &\quad + (1-\tau)\|u_n - t_n\|^2 \\ &\leq \|x_n - x'\|^2, \ \forall n \geq m_0.\end{aligned} \tag{105}$$

Finally, by arguing similarly to the proof of Lemma 5 and Theorem 1, we can obtain the desired conclusion. This completes the proof.

As an application of the results above, we consider the following bilevel variational inequality problem (BVIP for short):

$$\text{find } \overline{x} \in VI(G,C) \text{ such that } \langle F(\overline{x}), y - \overline{x}\rangle \geq 0, \quad \forall y \in VI(G,C), \tag{106}$$

where F and G be the mappings from H into itself. We denote the solution set of (106) by Γ, that is,

$$\Gamma = \{z \in VI(G,C) \colon \langle F(z), y - z\rangle \geq 0, \ \forall y \in VI(G,C)\}. \tag{107}$$

□

Corollary 1. *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $F, G\colon H \longrightarrow H$ be the pseudomonotone and Lipschitz continuous mappings with the Lipschitz constants L_1 and L_2 satisfy the following conditions:*

$$(B) \limsup_{n\longrightarrow\infty}\langle F(x_n), y - x_n\rangle \leq \langle F(\hat{x}), y - \overline{x}\rangle \text{ and } \limsup_{n\longrightarrow\infty}\langle G(x_n), y - x_n\rangle \leq \langle G(\hat{x}), y - \overline{x}\rangle \tag{108}$$

for every sequence $\{x_n\}$ converging weakly to $\hat{x}$.

Assume that $\Gamma \neq \varnothing$. Take the parameters $\beta \in (0, (1/L_1)), \lambda \in (0, (1/3L_2))$, the initial points $x_0, y_0, y_{-1} \in H$ and generate the sequence $\{x_n\}$ in the following manner:

$$\begin{cases} u_n = P_C(x_n - \beta G(x_n)), \\ t_n = P_{C_n}(x_n - \beta G(u_n)), \\ x_{n+1} = P_{H_n}(x_n - \lambda F(y_n)), \\ y_{n+1} = P_{T_n}(x_{n+1} - \lambda F(y_n)), n \geq 0, \end{cases} \tag{109}$$

where $C_n = \{y \in H\colon \langle x_n - \beta G(x_n) - u_n, y - u_n\rangle \leq 0\}$, $H_n = \{y \in H\colon \langle x_n - \beta F(y_{n-1}) - y_n, y - y_n\rangle \leq 0\}$, and T_n is defined as in Algorithm 1. Then, the sequence $\{x_n\}$ generated by (109) converges weakly to the point $\overline{x} = \lim_{n\longrightarrow\infty} P_\Gamma x_n$.

Proof. Let $g(x, y) = \langle G(x), y - x\rangle$ and $f(x, y) = \langle F(x), y - x\rangle$ for all $x, y \in H$. Since F is pseudomonotone on H, it follows that $f(x, y) = \langle F(x), y - x\rangle \geq 0 \Rightarrow f(y, x) = \langle F(y), x - y\rangle \leq 0$. So f is pseudomonotone on H. It is obvious that f satisfies the condition (A3). In addition, if $x_n \rightharpoonup \hat{x}$, by (B), we have

$$\begin{aligned}\limsup_{n\longrightarrow\infty} f(x_n, y) &= \limsup_{n\longrightarrow\infty}\langle F(x_n), y - x_n\rangle \\ &\leq \langle F(\hat{x}), y - \hat{x}\rangle = f(\hat{x}, y).\end{aligned} \tag{110}$$

So f satisfies the condition (A2). Finally, since F is L_1-Lipschitz continuous, f is Lipschitz-type continuous with the constant $d_1 = d_2 = (L_1/2)$; see Remark 1. Thus, f satisfies the conditions (A1)–(A4). Similarly, g also satisfies the conditions (A1)–(A4). In particular, g satisfies (A4) with $c_1 = c_2 = (L_2/2)$. So, the conditions on β and λ in Lemma 5 become the ones in Corollary 1. On the other hand, by Algorithm 1,

$$u_n = \operatorname{argmin}\left\{\beta g(x_n, y) + \frac{1}{2}\|x_n - y\|^2 \colon y \in C\right\} \tag{111}$$

is equivalent to $u_n = P_C(x_n - \beta G(x_n))$. Similarly, t_n, x_{n+1}, and y_{n+1} in Algorithm 1 are equivalent to t_n, x_{n+1}, and y_{n+1} in Corollary 1, respectively. By Theorem 1, the desired conclusion can be obtained. This completes the proof.

Since the proof process of the following corollary is similar to the one of Corollary 3.1, we give the following corollary and omit the proof process. □

Corollary 2. *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $F, G\colon H \longrightarrow H$ be the pseudomonotone and Lipschitz continuous mappings with the Lipschitz constants L_1 and L_2 satisfying the condition (B) in Corollary 3.1. Assume that $\Gamma \neq \varnothing$. The parameters $\beta_0 > 0, \lambda_0 > 0$, $\mu \in (\beta_0, (1/4))$, the initial points $x_0, y_0, y_{-1} \in H$ are taken, and the sequence $\{x_n\}$ is generated by the following manner:*

$$\begin{cases} u_n = P_C(x_n - \beta_n G(x_n)), \\ t_n = P_{C_n}(x_n - \beta_n G(u_n)), \\ x_{n+1} = P_{H_n}(x_n - \lambda_n F(y_n)), \\ y_{n+1} = P_{T_n}(x_{n+1} - \lambda_{n+1} F(y_n)), n \geq 0, \end{cases} \tag{112}$$

where C_n, H_n, and T_n are defined as in Corollary 1,

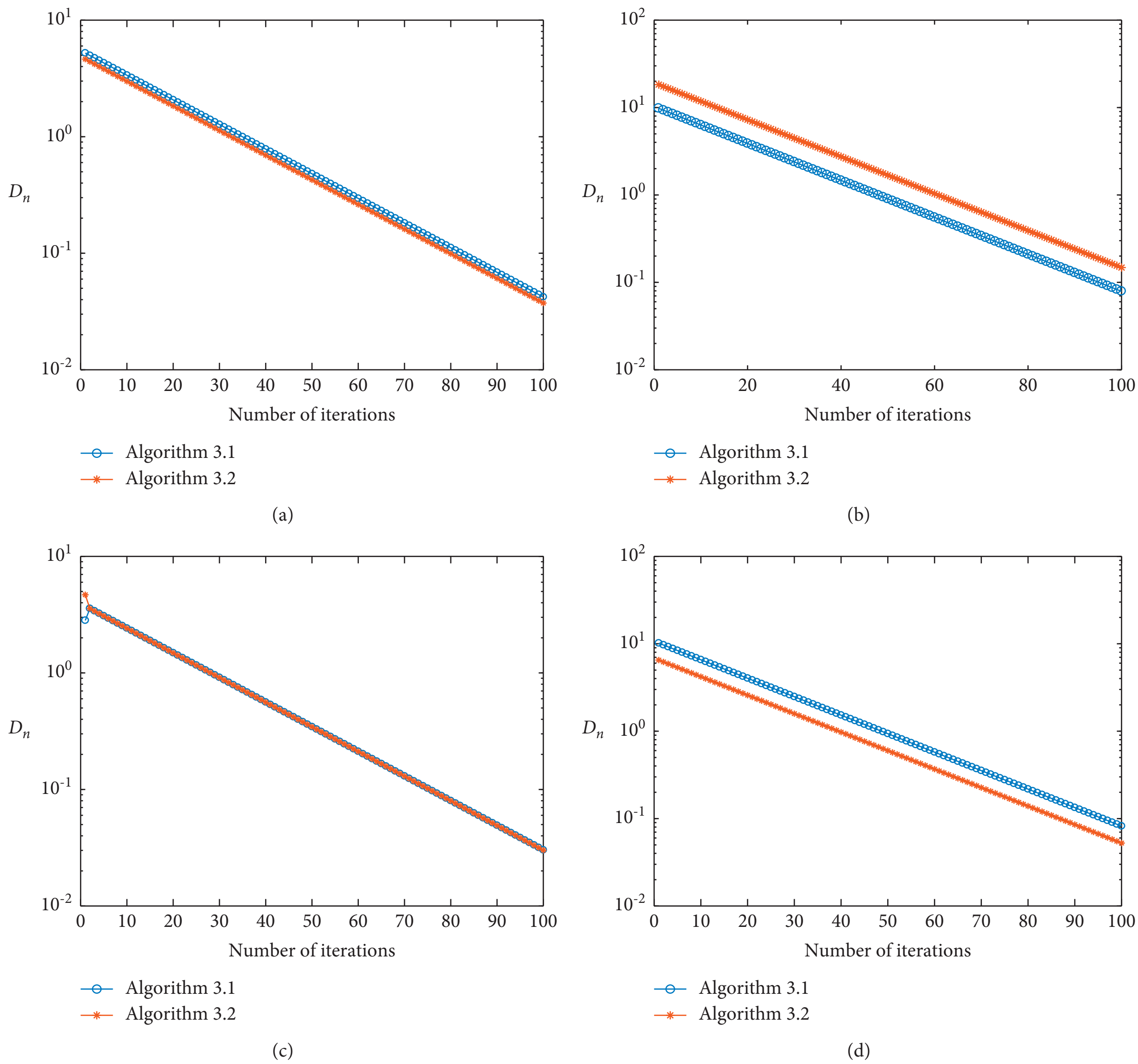

FIGURE 1: Experiment in different $\mathbb{R}^m$.

$$\lambda_{n+1} = \begin{cases} \lambda_n, & \langle F(y_{n-1}) - F(y_n), x_{n+1} - y_n \rangle \leq 0, \\ \min\left\{ \lambda_n, \dfrac{\mu\left(\|y_n - y_{n-1}\|^2 + \|y_n - x_{n+1}\|^2 \right)}{\langle F(y_{n-1}) - F(y_n), x_{n+1} - y_n \rangle} \right\}, & \text{otherwise}, \end{cases} \tag{113}$$

and β_{n+1} is modified by

$$\beta_{n+1} = \begin{cases} \beta_n, & \langle G(x_n) - G(u_n), t_n - u_n \rangle \leq 0, \\ \min\left\{ \lambda_n, \dfrac{\mu\left(\|u_n - x_n\|^2 + \|t_n - u_n\|^2 \right)}{\langle G(x_n) - G(u_n), t_n - u_n \rangle} \right\}, & \text{otherwise}. \end{cases} \tag{114}$$

Then, the sequence $\{x_n\}$ generated by (112) converges weakly to the point $\overline{x} = \lim_{n\longrightarrow\infty} P_\Gamma x_n$.

Remark 8. Since C_n, H_n, and T_n are half-spaces, from Remark 2, it follows that t_n, x_{n+1}, and y_{n+1} in Corollary 1 and 2 can be computed explicitly.

4. Numerical Examples

In this section, we give two examples to illustrate the convergence of Algorithm 1 and 2. The programs are written in Matlab 2016b, and the examples are computed on a PC Intel(R) Core (TM) i5-4260U CPU, 2.00 GHz, Ram 4.00 GB.

We first give the following example to illustrate the effectiveness of Algorithm 1 and 2.

Example 1. Let $H = \mathbb{R}^m$ and $C = \{x \in \mathbb{R}^m\colon x_1 \geq 0, x_i \geq 1, \forall i \in \{2, \ldots, m\}\}$. Let $g\colon H \times H \longrightarrow \mathbb{R}$ be defined by

$$g(x, y) = \sum_{i=2}^{m} (y_i - x_i)\|x\|, \quad \forall x = (x_1, \ldots, x_m),\ y = (y_1, \ldots, y_m) \in H. \tag{115}$$

It is known that g satisfies the conditions (A1)–(A4). In particular, g is Lipschitz-type continuous with the constants $c_1 = c_2 = 2$; see [30] for details. Let $f\colon H \times H \longrightarrow \mathbb{R}$ be defined by

$$f(x, y) = \langle Ax, y - x\rangle, \quad \forall x, y \in H, \tag{116}$$

where $Ax = ((x_1/2), (x_1 - 1/2), \ldots, (x_m - 1/2))$. It is easy to see that f satisfies the conditions (A1)–(A4). In particular, f is Lipschitz-type continuous with the constants $d_1 = d_2 = (1/4)$. The solution set of the bilevel equilibrium problem (7) in this example is found as $\Omega = \{(0, 1, \ldots, 1)\}$.

We choose the initial points x_0, y_0, y_{-1} randomly from the interval (0,5) for Algorithm 1 and 2, the input parameters $\lambda = \beta = 0.1$ for Algorithm 1, and $\lambda_0 = \beta_0 = \mu = 0.2$ for Algorithm 2. The maximum iteration of 100 as the stop criterion is used for Algorithm 1 and 2. The numerical results with the different dimensions m are shown in Figure 1. In this figures, the x-axis represents the number of iterations while the y-axis is for the value of D_n generated by Algorithm 1 and 2, where

$$D_n = \|x_n - (0, 1, \ldots, 1)\|. \tag{117}$$

From the computed results, we see the effectiveness of Algorithm 1 and 2.

The next example was ever used in [20]. Here, we use this example to illustrate the convergence of Algorithm 1 and 2 and compare the computed results with Algorithm 2.1 in [20].

Example 2. Let $C = \{x \in \mathbb{R}^5\colon -1 \leq x_i \leq 1, \forall i = 1, \ldots, 5\}$ and $f\colon \mathbb{R}^5 \times \mathbb{R}^5$ be defined by

$$f(x, y) = \langle F(x) + Qy + q, y - x\rangle, \forall x, y \in \mathbb{R}^5, \tag{118}$$

where $Q = AA^T + B + D$ with

$$A = \begin{pmatrix} -2 & 1 & 0 & 1 & -1 \\ 1 & 2 & 1 & 0 & 2 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 1 & 3 & 1 & 0 \\ 2 & 0 & 1 & -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 2 & 1 & -1 \\ -1 & 3 & 2 & 0 & 2 \\ -2 & -2 & 1 & 1 & -3 \\ -1 & 0 & -1 & 1 & 0 \\ 1 & -2 & 3 & 0 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 & 22 \end{pmatrix},$$
$$F(x) = (\xi x_1 + \xi x_2 + \sin(x_1), -\xi x_1 + \xi x_2 + \sin(x_2), (\xi - 1)x_3, (\xi - 1)x_4, (\xi - 1)x_5), \tag{119}$$

and q is a vector in $\mathbb{R}^5$.

Let $f\colon \mathbb{R}^5 \times \mathbb{R}^5$ be defined by

$$g(x, y) = \langle Px + \overline{P}y + p, y - x\rangle, \forall x, y \in \mathbb{R}^5, \tag{120}$$

where p is a vector in $\mathbb{R}^5$ and $P = 2\overline{P} + I$ with

Table 1: Computed results for Algorithm 3.1 with the different parameters.

Test prob.	ξ	β	λ	No. iter.	CPU-times (s)
1	65	$(1/3c_1)$	$(1/7d_1)$	8	1.534 5
2	69	$(1/3c_1)$	$(1/7d_1)$	17	1.711 3
3	55	$(1/5c_1)$	$(1/7d_1)$	22	2.023 4
4	55	$(1/5c_1)$	$(1/7d_1)$	13	1.743 1
5	70	$(1/5c_1)$	$(1/10d_1)$	15	1.711 3
6	80	$(1/5c_1)$	$(1/10d_1)$	24	1.988 5
7	95	$(1/8c_1)$	$(1/15d_1)$	35	2.233 1
8	45	$(1/8c_1)$	$(1/15d_1)$	21	1.993 5
9	100	$(1/10c_1)$	$(1/10d_1)$	19	2.100 3
10	150	$(1/10c_1)$	$(1/10d_1)$	22	2.002 3

Table 2: Computed results for Algorithm 3.2 with the different parameters.

Test prob.	ξ	β_0	λ_0	μ	No. iteration	CPU times (s)
1	65	$(1/3c_1)$	$(1/7d_1)$	0.2	12	4.324 3
2	69	$(1/3c_1)$	$(1/7d_1)$	0.2	23	4.001 2
3	55	$(1/5c_1)$	$(1/7d_1)$	0.15	22	3.023 4
4	55	$(1/5c_1)$	$(1/7d_1)$	0.15	32	5.111 3
5	70	$(1/5c_1)$	$(1/10d_1)$	0.1	25	4.875 0
6	80	$(1/5c_1)$	$(1/10d_1)$	0.1	29	5.114 6
7	95	$(1/8c_1)$	$(1/15d_1)$	0.1	33	6.143 7
8	45	$(1/8c_1)$	$(1/15d_1)$	0.12	31	5.887 5
9	100	$(1/10c_1)$	$(1/10d_1)$	0.09	29	4.997 3
10	150	$(1/10c_1)$	$(1/10d_1)$	0.09	31	5.880 3

Table 3: Computed results for Algorithm 2.1 in [20] with the different parameters.

Test prob.	ξ	β_n	λ_n	No. iteration	CPU times (s)
1	65	$(2\eta/2d_1^2(n^2+2))$	$(1/2c_1+100n)$	8	1.500 2
2	69	$(2\eta/2d_1^2(2n^2+12))$	$(1/2c_1+200n)$	7	0.887 2
3	55	$(2\eta/2d_1^2(2n^2+20))$	$(1/2c_1+500n)$	12	1.687
4	55	$(2\eta/2d_1^2(2n^2+15))$	$(1/2c_1+600n)$	13	1.400 3
5	70	$(2\eta/2d_1^2(2n^2+20))$	$(1/2c_1+1000n)$	5	1.711 3
6	80	$(2\eta/2d_1^2(2n^2+30))$	$(1/2c_1+1000n)$	5	1.422 2
7	95	$(2\eta/2d_1^2(n^2+100))$	$(1/2c_1+500n)$	7	0.983 1
8	45	$(2\eta/2d_1^2(n^2+150))$	$(1/2c_1+500n)$	9	0.800 5
9	100	$(2\eta/2d_1^2(n^2+200))$	$(1/2c_1+1000n)$	7	0.965 9
10	150	$(2\eta/2d_1^2(2n^2+1))$	$(1/2c_1+200n)$	14	2.110 3

$$\overline{P} = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 1 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 5.5 \end{pmatrix}. \tag{121}$$

It is known that f and g satisfy all the conditions required in [20] and Section 3 of this paper. In particular, f is Lipschitz-type continuous with the constants $d_1 = d_2 = (1/2)(\sqrt{2(2\xi^2+2\xi+1)} + \|Q\|)$, where $\|Q\| = 58.9677$ and g is Lipschitz-type continuous with the constants $c_1 = c_2 = (1/2)\|\overline{P}+I\|$; see [20].

We choose the initial point $x_0 = y_0 = y_{-1} = (1,1,1,1,1)$ for Algorithm 1 and 2 and $x_0 = (1,0,0,1,1)$ for Algorithm 2.1 in [20]. The stop criterion is $D_n \leq 10^{-3}$, where

$$D_n = \max\left\{\left\|x_n - y_n\right\|, \left\|x_{n+1} - y_n\right\|\right\} \tag{122}$$

for the three algorithms. The computed results are presented in Tables 1–3 for Algorithm 1, 2, and Algorithm 2.1 in [20], respectively. In Table 3, $\eta = \xi - 1 - \|Q\|$.

From the computed results, we see that Algorithm 2 needs more CPU times and iterations over Algorithm 1 and Algorithm 2.1 in [20]. The course may be that Algorithm 2 involves a self-adaptive process of computing the values of β_{n+1} and λ_{n+1}.

5. Conclusion

We have proposed two iterative algorithms for finding the solution of a bilevel equilibrium problem in a real Hilbert space. The sequence generated by our algorithms converges weakly to the solution. Furthermore, we reported some numerical results to support our algorithms. How to obtain the strong convergence of Algorithm 1 and 2 without the additional assumptions is our future investigation.

Data Availability

The data used to support the findings of this study are available from the author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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