Research Article

A Complete Normed Space of a Class of Guage Integrable Functions

Yassin Alzubaidi

Department of Mathematical Sciences, Umm Al-Qura University, Makkah Almukarramah, Saudi Arabia

Correspondence should be addressed to Yassin Alzubaidi; yazubaidi@uqu.edu.sa

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The space of all gauge integrable functions equipped with the Alexiewicz norm is not a Banach space. This article defines a class of gauge integrable functions that pose a complete normed space under a suitable norm. First, we discuss how the introduced space evolves naturally from the properties of the gauge integral. Then, we show that this space properly contains the space of all Lebesgue integrable functions. Finally, we prove that it is a Banach space. This result opens the gate for further studies, such as investigating the validity of convergence theorems on the presented space and studying its other properties.

1. Introduction

The gauge integral was formulated and developed separately by Ralph Henstock and Jaroslav Kurzweil in [1, 2], respectively. Therefore, the gauge integral is referred to as Henstock–Kurzweil integral in much literature. Their integration method outweighs Riemann and Lebesgue integrals in different areas and shows great potential. In the first section of this article, we will introduce some basic definitions and important results regarding the gauge integral. Throughout this article, \([a, b]\) will denote a nondegenerate finite closed interval.

Definition 1. A gauge on \([a, b]\) is a strictly positive function \(\delta: [a, b] \rightarrow (0, \infty)\).

Definition 2. Let 
\[P = \left\{ ([x_{i-1}, x_i], t_i) \right\}_{i=1}^{n}\]
be a tagged partition of \([a, b]\), and let \(\delta\) be a gauge on \([a, b]\). We say that \(P\) is \(\delta\)-fine if for all \(i = 1, \ldots, n\), and we have
\[x_{i-1}, x_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i)).\] (1)

If \(\delta\) is any gauge on \([a, b]\), then Cousin’s lemma (see [3]) guarantees the existence of a tagged partition \(P\) of \([a, b]\), such that \(P\) is \(\delta\)-fine. This result allows us to introduce the following main definition.

Definition 3. A function \(f: [a, b] \rightarrow R\) is said to be gauge integrable if there exists a real number \(A\) such that for all \(\varepsilon > 0\), there exists a gauge \(\delta_{\varepsilon}\) on \([a, b]\) satisfying the following: if 
\[P = \left\{ ([x_{i-1}, x_i], t_i) \right\}_{i=1}^{n}\]
is a tagged partition of \([a, b]\) and \(P\) is \(\delta_{\varepsilon}\)-fine, then
\[\left| \sum_{i=1}^{n} f(t_i) (x_i - x_{i-1}) - A \right| < \varepsilon.\] (2)

In this case, we write
\[\int_{a}^{b} f \, dx = A.\] (3)

It is clear that the gauge integral is a generalization of the Riemann Integral, where the constant \(\delta\) in Riemann’s definition is being replaced with a positive function (gauge) \(\delta(t)\). This slight yet crucial change makes it possible to integrate a much broader class of functions. In fact, every Lebesgue integrable function is a gauge integrable function, but the reverse is not valid. There exist functions that are gauge integrable, and at the same time, they are not Lebesgue integrable nor Riemann integrable.

A significant difference between the gauge integral and the Lebesgue integral appears in regard to the fundamental theorem of calculus, where the gauge integral allows a more general form of the fundamental theorem of calculus. The
Theorem 1. Let $F: [a, b] \rightarrow \mathbb{R}$ be a continuous function, which is differentiable nearly everywhere and $F' = f$, then

(i) $f$ is gauge integrable on $[a, b]$

(ii) $f$ is Lebesgue integrable on $[a, b]$ if and only if $F$ is absolutely continuous

Another superior feature of the gauge integral is concerning the improper integral, where a function is gauge integrable if and only if its improper gauge integral exists. This feature is well known as Hake’s theorem (see [5]), which can be stated as follows.

Theorem 2. Let $F: [a, b] \rightarrow \mathbb{R}$ be a function, then $f$ is gauge integrable on $[a, b]$ if and only if $f$ is gauge integrable on $[a, c]$ for all $c \in (a, b)$ and

$$
\lim_{c \searrow b} \int_{a}^{c} f.
$$

exists. In this case,

$$
\int_{a}^{b} f = \lim_{c \searrow b} \int_{a}^{c} f.
$$

Definition 4. Let $f$ be a gauge integrable function, then

$$
\|f\|_{A} = \sup_{a \leq x \leq b} \left|\int_{a}^{x} f\right|.
$$

The above norm is called Alexiewicz’s norm, and it was introduced in [6]. Despite many attempts to define different norms on the class of all gauge integrable functions, Alexiewicz’s norm became the standard norm and got the most attention. It is a seminorm, but since $\|f - g\|_{A} = 0 \Leftrightarrow f = g$ a.e., $\|\cdot\|_{A}$ is considered a norm once we identify the functions which are equal almost everywhere.

Definition 5. $G = G([a, b]) = \{f: [a, b] \rightarrow \mathbb{R}: \|f\|_{A} < \infty\}$

$(G, \|\cdot\|_{A})$ is a normed space, but unfortunately, it is not complete; it is not a Banach space. Its completion is described as a subspace of distributions (see [3]). This lack of completeness opens many areas of research. One of these areas is the search for complete subspaces of $G([a, b])$, which strictly contain the space of all Lebesgue integrable functions.

2. Construction and Results

As it is mentioned above, every Lebesgue integrable function is a gauge integrable function. However, if $f$ is a gauge integrable function that is not Lebesgue integrable on an interval $I$, then there exists a closed and nowhere dense subset $D \subset I$ such that $f$ is not Lebesgue integrable on any subinterval $J$ whenever $J \cap D \neq \emptyset$. $D$ is called the set of nonsummability points or the set of singularities of the Henstock integral. Furthermore,

$$
\int f = \int_{D} f + \sum_{i=1}^{\infty} \int_{J_{i}} f,
$$

for some disjoint open subintervals $J_{i}$ satisfy

$$
I - D = \bigcup_{i=1}^{\infty} J_{i}.
$$

In addition, $f$ is Lebesgue integrable on any $[c, d] \subset J_{i}$ for all $i$. That is, $f$ might only have nonsummable points at the limits of the intervals $J_{i}$ (see [4]). This argument motivates us to consider the local case, particularly when $f$ has possible points of singularity at the limit points only.

Remark 1. Without loss of generality, we can assume that $f$ could only attain a nonsummable point at the right limit. Note that if $f$ has singularities at both limits, we can split the integral into two integrals using the fact that (see [7])

$$
\int_{a}^{b} f = \int_{a}^{c} f + \int_{c}^{b} f, \quad \text{for } c \in (a, b).
$$

Moreover, if $f$ has a singularity at the left limit, then using a change of variables will transfer the singularity to the right limit where we have

$$
\int_{a}^{b} f(x) = \int_{a}^{b} f(a + b - x).
$$

Let $f$ be a gauge integrable function on $[a, b]$ with a possible singularity at $b$; then, the indefinite integral

$$
F(x) = \int_{a}^{x} f,
$$

is a continuous function on $[a, b]$ (see [4]). Therefore, $F$ is both gauge and Lebesgue integrable. Integrating by parts in the gauge integral sense yields that

$$
\int_{a}^{b} F = \int_{a}^{b} \int_{a}^{x} f = \int_{a}^{x} \int_{a}^{b} f - \int_{a}^{b} x f = \int_{a}^{b} (b - x) f.
$$

We notice that the integrand on the left-hand side is Lebesgue integrable, while the integrand on the right-hand side is not necessarily Lebesgue integrable (it is gauge integrable). For example, consider the function

$$
f(x) = \frac{1}{(b - x)^{2}} \cos \frac{1}{(b - x)},
$$

on $[a, b]$, which we will discuss in Proposition 2. This observation motivates us to study the class of all gauge integrable functions on $[a, b]$ which at the same time satisfy the condition that $(b - x)f$ is Lebesgue integrable on $[a, b]$. So, we start by introducing the following basic proposition.

Proposition 1. If $(b - x)f$ is Lebesgue integrable on $[a, b]$, then the only possible point of nonsummability for $f$ on $[a, b]$ is $b$. 

Proof. It is clear that \((b-x)f\) is Lebesgue integrable on \([a,c]\) for all \(c \in [a,b]\). Also, we have in the Lebesgue sense that
\[
\int_a^c |f| = \int_a^c \frac{1}{b-x} (b-x)f \leq \frac{1}{b-c} \int_a^b |(b-x)f| < \infty. \tag{14}
\]

Thus, \(f\) is Lebesgue integrable on \([a,c]\) for all \(c \in [a,b]\), and hence, the only possible point of nonsummability is \(b\). \(\square\)

Let \(L = L([a,b])\) denote as usual the space of all Lebesgue integrable functions on \([a,b]\) equipped with the norm \(\|f\|_L = \int_a^b |f|\). In view of the above discussion, we introduce the following subspace of gauge integrable functions.

**Definition 6.** \(G^1 = G^1([a,b]) = \{ f \in G([a,b]): (b-x)f \in L([a,b]) \}\).

Now, in an obvious way, we define a norm on the space \(G^1\).

**Definition 7.** \(\|f\|_{G^1} = \|f\|_A + \| (b-x)f \|_L \).

In the above definition, \(f\) represents the class of all functions equal to \(f\) almost everywhere. Indeed, \(\| \cdot \|_{G^1}\) is a norm on \(G^1\) since it is a linear combination of norms, and its properties are inherited from the norms \(\| \cdot \|_A\) and \(\| \cdot \|_L\).

**Proposition 2.** \(L([a,b]) \subseteq G^1([a,b]) \subseteq G([a,b])\).

Proof. The inclusion \(L([a,b]) \subseteq G^1\) follows from the facts that \(L \subseteq G\) and the uniform boundedness of the term \((b-x)\), while the inclusion \(G^1 \subseteq G\) is clear from the definition of \(G^1\). To see that these inclusions are proper, consider the following example:

\[
F(x) = \begin{cases} 
(b-x)^p \sin \frac{1}{(b-x)^q} & \text{if } x \in [a,b), \\
0 & \text{if } x = b,
\end{cases}
\]

where \(1 \leq p \leq q\). This function \(F\) satisfies the following properties (see [8]):

(i) \(F\) is not of bounded variation on the interval \([a,b]\), and hence, it is not absolutely continuous on \([a,b]\).

(ii) \(F\) is differentiable on \([a,b]\), and

\[
F'(x) = \frac{1}{(b-x)^{q+1-p}} \cos \frac{1}{(b-x)^q} + p(b-x)^{q-p} \sin \frac{1}{(b-x)^q} \text{ if } x \in [a,b).
\]

Thus, by Theorem 1, \(F'\) is gauge integrable, but it is not Lebesgue integrable. Note that the term \(h = p(b-x)^{q-p} \sin 1/(b-x)^q\) is Lebesgue integrable since it is bounded and measurable. Choosing \(p = 3\) and \(q = 3\), the function

\[
f_1 = \frac{1}{(b-x)} \cos \frac{1}{(b-x)^3}
\]

is gauge integrable since \(f_1 = F' - h\). Also, it is clear that \((b-x)f_1 \in L([a,b])\). In contrast, \(f_1\) is not Lebesgue integrable since if it is Lebesgue integrable, and this will imply that \(F' = f_1 + h\) is also in \(L([a,b])\), which is a contradiction. Therefore, \(L([a,b]) \subseteq G^1([a,b])\). To show that \(G^1 \subseteq G^1\) is nonempty, choose \(p = 1\) and \(q = 3\), then the function

\[
f_2 = \frac{1}{(b-x)^2} \cos \frac{1}{(b-x)^3},
\]

is gauge integrable, but it is not in \(G^1\) since \((b-x)f_2 = f_1 \notin L\). \(\square\)

**Theorem 3.** \(G^1([a,b])\) is a complete normed space under the norm \(\| \cdot \|_{G^1}\).

Proof. Let \((f_n)\) be a Cauchy sequence on the space \((G^1, \| \cdot \|_{G^1})\). Then, \((f_n)\) and \((b-x)f_n\) are Cauchy sequences on the spaces \((G, \| \cdot \|_A)\) and \((L, \| \cdot \|_L)\), respectively. By the completeness of the Lebesgue space, there exists \(g \in L\) such that

\[
\lim_{n \to \infty} (b-x)f_n = g \text{ in } L. \tag{19}
\]

Define \(f = g/(b-x)\). By Proposition 1, \(f\) is Lebesgue integrable on \([a,r]\) for all \(r \in [a,b]\). Also, we have

\[
\lim_{n \to \infty} \int_a^r \frac{f_n - f}{b-x} = \lim_{n \to \infty} \frac{1}{b-r} \int_a^r (b-x)f_n - g = 0. \tag{20}
\]

Hence, \(f_n\) converges to \(f\) in \(L([a,r])\) for all \(r \in [a,b]\), and we get

\[
\int_a^r f = \lim_{n \to \infty} \int_a^r f_n. \tag{21}
\]

On the other hand and since \((f_n)\) is a Cauchy sequence on the space \((G, \| \cdot \|_A)\), for all \(\epsilon > 0\), there exists \(N_\epsilon \in \mathbb{N}\) such that

\[
\| f_n - f_m \|_A < \epsilon \text{ for all } n, m > N_\epsilon. \tag{22}
\]

That is,

\[
\sup_{a \leq r \leq b} \left| \int_a^r f_n - \int_a^r f_m \right| < \epsilon \text{ for all } n, m > N_\epsilon. \tag{23}
\]

Therefore, the sequence

\[
\left( \int_a^r f_n \right). \tag{24}
\]

is a uniformly Cauchy sequence on the real line \(\mathbb{R}\) with respect to the variable \(r\). Considering in (23) the particular case \(r = b\), we get by the completeness of \(\mathbb{R}\) that there exists \(A \in \mathbb{R}\) satisfying

\[
\lim_{n \to \infty} \int_a^b f_n = A. \tag{25}
\]
Also, for all \( r \in [a, b) \), the convergence in (21) is uniform. Hence, we can interchange the limits in the following way:
\[
\lim_{r \to b^-} \int_a^r f = \lim_{n \to \infty} \int_a^b f_n = \lim_{n \to \infty} \lim_{r \to b^-} \int_a^r f_n \tag{26}
\]

Thus, applying Theorem 2 (Hake’s Theorem) and using the result in (25), we get
\[
\lim_{r \to b^-} \int_a^r f = \lim_{n \to \infty} \int_a^b f_n = A. \tag{27}
\]

Using Hake’s theorem again, we conclude that \( f \) is gauge integrable and
\[
\int_a^b f = \lim_{n \to \infty} \int_a^b f_n. \tag{28}
\]

We combine the uniform convergence in (21) and the result in (28) to get that
\[
\sup_{a \leq r \leq b} \left| \int_a^r f_n - \int_a^r f \right| \to 0 \quad \text{as} \quad n \to \infty, \tag{29}
\]

which implies that
\[
\lim_{n \to \infty} f_n = f \text{ in } G. \tag{30}
\]

Also, it is clear from the definition of \( f \) that
\[
\lim_{n \to \infty} (b - x)f_n = (b - x)f \text{ in } L. \tag{31}
\]

Therefore, \((f_n)\) converges to \( f \) in \( G^1 \), which completes the proof. \( \square \)

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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