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Research Article

Modeling and Analysis of MHD Oscillatory Flows of Generalized Burgers' Fluid in a Porous Medium Using Fourier Transform

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In this article, exact solutions of unsteady oscillatory generalized Burgers' fluid are proposed for three different cases using Fourier transform approach. The fluid is electrically conducting under the influence of uniform transverse magnetic field and passing through the porous medium. MHD flows are induced by imposed periodic pressure gradients with smaller oscillations. Closed form solutions are obtained using Fourier sine transform, and several existing results are recovered as limiting cases. Furthermore, effects of different fluid parameters on the velocity profile are studied graphically. Analysis reveals that magnetic field and porosity parameter increases the velocity profile in case of Oldroyd-B and generalized Burgers' fluid. It is also observed that magnetic field has more prominent effect on Burgers' fluid as compared to Oldroyd-B fluid, while porosity parameter showed noticeable effect on Oldroyd-B fluid as compared to Burgers' fluid.

1. Introduction

The mechanics of non-Newtonian fluids is much complicated and more nonlinear in comparison with that of the Newtonian fluids. Many problems dealing with the flow of non-Newtonian fluids have been studied by engineers and mathematicians in various geometrical configurations. Examples of these fluids are paste, gel, shampoo, soap, bead dough etc. The analysis of such flows finds important applications in engineering practice, particularly in chemical industries. A huge variety of consumer goods are now made from injection-molded plastics which often contain high concentrations of glass or carbon fibers. Many modern paints and lubricants contain polymer additives which are added to enhance their flow properties or the quality of the finished products. Also, many food stuffs (e.g. tomato sauce) and biological fluids (e.g. blood) are non-Newtonian.

Due to variety of fluids, there are several constitutive equations of non-Newtonian fluids in the literature. Among these the viscoelastic fluids have acquired the special status. In the literature, numerous works have been done to investigate the behavior of viscoelastic fluids. Some interesting flows in this direction are discussed by Rajagopal [1], Shamsuddin et al. [2, 3], Anwar et al. [4], Tan and Masuoka [5], Venkatesan and Ganesan [6], Fetecau and Vien [7], Fetecau and Fetecau [8], Lee [9], Fetecau and Agop [10], Ali et al. [11], Hayat et al. [12–14], and Nanganthran et al. [15].

All the above investigations have dealt with second grade and Oldroyd-B fluid models. Very little work have been reported to flow of a Burgers' fluid. In light of the above literature review, it is noted that generalized Burgers' fluid model with oscillatory motion in porous medium has not been solved using Fourier transform method. Novelty of this study is further elaborated through Table 1. Moreover, recent studies on Burgers' fluid flow are also being considered in literature. Gangadhar et al. [16] have investigated Burgers' fluid with convective heating. Generalized Burgers' nanofluid with Cattaneo–Christov Relations is studied by Shahzad et al. [17] using Galerkin finite element method. Hayat et al. [18, 19] reported some analytical results for flows of a Burgers' fluid under varying conditions. Khan et al. [20] investigated Burgers' fluid under magnetization with stagnation point flow. The Burgers' fluid is a viscoelastic fluid which has been used to characterize cheese, soil, asphalt etc [21–25]. This model has also been used in calculating the transient creep properties

Table 1: Present study in comparison with existing work in literature.

	Generalized Burgers' fluid	MHD	Porous medium	Oscillatory flow	Fourier transform
Fetecau and Vieru [32]	No	Yes	Yes	No	No
Akram et al. [33]	No	No	No	Yes	No
Ali et al. [34]	No	Yes	Yes	Yes	No
Hayat et al. [35]	Yes	No	No	No	Yes
Present	Yes	Yes	Yes	Yes	Yes

of the earth's mantle and in the modeling of high temperature viscoelasticity of fine-grained poly-crystalline olivine [26, 27]. The Burgers' fluid model has also been generalized and Waqas et al. [28] discussed some heat flux models to flow in this direction.

In order to solve fluid models various methodologies are used in literature. Fourier transform method is employed in this study to obtain exact solutions of the modeled problems. The Fourier transform can express an arbitrary a-periodic function as an infinite integral over a continuous range of frequencies. Firstly it was used in the treatment of single pulse phenomena by electrical engineers. The Fourier transform and the related operations of convolution and correlation has applications in optics, acoustics, scattering and diffraction of x-rays, neutrons and electrons, and a-periodic effects in electrical circuits. In various recent studies, Fourier transform method is used to solve mathematical problems effectively. Abro et al. [29] studied thermal effects on micro polar fluid with MHD and porosity by utilizing Fourier sine transform scheme. Numerical solution of Burgers' equation was presented by Egidi et al. [30] by using Fourier transforms. Liu et al. [31] employed different transforms to study MHD flow and heat transfer of generalized Burgers' fluid.

The main objective of this paper is to analyze the unsteady flows of generalized Burgers' fluid induced by the cosine and sine oscillations of an infinite plate in a porous medium. The flow caused by an oscillating pressure gradient is also considered. The fluid is electrically conducting and occupies the porous structure. Graphical analysis for fluid velocity is given against varying fluid parameters. Generalization of fluid model into different fluids such as Newtonian, second grade, Maxwell, Oldroyd-B, and generalized fluid with varying material constants is depicted in tabular form. Such flows in porous media are important in enhanced oil recovery, paper and textile coating, and composite manufacturing processes. The whole analysis is given in the presence of a constant magnetic field which is very important. Finally, the results are discussed with the help of several graphs.

2. Governing Equations

The Cauchy stress T in a generalized Burgers' fluid is [28].

$$T = -pI + S, \tag{1}$$

$$\left(1 + \lambda_1 \frac{\delta}{\delta t} + \lambda_2 \frac{\delta^2}{\delta t^2}\right) S = \mu \left(1 + \lambda_3 \frac{\delta}{\delta t} + \lambda_4 \frac{\delta^2}{\delta t^2}\right) A, \tag{2}$$

where $-pI$ indicates the indeterminate spherical stress, $A = L + L^T$ is first Rivlin-Ericksen tensor, μ is the dynamic viscosity, L is the velocity gradient, λ_1 and $\lambda_3 (<\lambda_1)$ are the relaxation and retardation times, S is the extra stress tensor, λ_2, λ_4 are material constants and $\delta/\delta t$ is the upper convected time derivative defined by

$$\frac{\delta \mathbf{S}}{\delta t} = \frac{d\mathbf{S}}{dt} + \mathrm{LS} - \mathrm{SL}^T, \tag{3}$$

where d/dt is the material derivative and

$$\frac{\delta^2 \mathbf{S}}{\delta t^2} = \frac{\delta}{\delta t}\left(\frac{\delta \mathbf{S}}{\delta t}\right). \tag{4}$$

The flows under consideration have the following properties

Unsteady velocity in y-direction and fluid is at rest at $t = 0$.

(i) Uniform magnetic field is assumed along with Darcian Porous medium

The fluid velocity vector in this case is

$$V = u(y, t) i, \tag{5}$$

where i and u are the unit vector and velocity parallel to the x-axis, respectively. The velocity field (5) automatically satisfies the incompressibility condition. Since u is a function of y and t, the stress field will also depend upon y and t. Now, Eq.(2) together with the initial condition (the fluid being at rest up to the moment t=0)

$$S(y, 0) = 0 \tag{6}$$

yields $S_{xz} = S_{yz} = S_{yy} = S_{zz} = 0$ and

$$\begin{aligned} &S_{xx} + \lambda_1 \left(\frac{\partial S_{xx}}{\partial t} - 2 S_{xy} \frac{\partial u}{\partial y}\right) + \lambda_2 \left(\frac{\partial^2 S_{xx}}{\partial t^2} - 4 \frac{\partial S_{xy}}{\partial t}\frac{\partial u}{\partial y} - 2 S_{xy} \frac{\partial^2 u}{\partial t \partial y}\right) \\ &\quad = -2\mu\lambda_3 \left(\frac{\partial u}{\partial y}\right)^2 - 6\mu\lambda_4 \left(\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial t \partial y}, \\ &\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) S_{xy} = \mu \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) \frac{\partial u}{\partial y}. \end{aligned} \tag{7}$$

The balance of linear momentum for MHD fluid in a porous medium is

$$\rho \frac{dV}{dt} = -\nabla p + \text{div } S + J \times B + r, \tag{8}$$

where ρ is the fluid density, J is the current density, $B(= B_0 + b)$ is the total magnetic field, B_0 is applied magnetic field, b is the induced magnetic fields and r is the Darcy resistance for a generalized Burgers' fluid in the porous medium. Neglecting the displacement current, the Maxwell equations and Ohms' law are

$$\begin{gathered} \text{div } B = 0, \quad \text{curl } B = \mu_m J, \quad \text{curl } E = -\frac{\partial B}{\partial t}, \\ J = \sigma(E + V \times B), \end{gathered} \tag{9}$$

where E is the electric field, μ is the magnetic permeability and σ is the electrical conductivity. For small magnetic Reynolds number, the induced magnetic field is neglected. It is also assumed that $E = 0$.

The Darcy's law holds for flows of viscous fluid with low speed, in an unbounded porous medium. This law gives relation among velocity and pressure drop induced by frictional drag while ignoring boundary effects on the flow. A direct proportionality between induced pressure and Darcian velocity is observed by this law. Brinkman proposed an equation describing the locally averaged flow for the porous medium with boundaries. In literature, there are various modified Darcy's law applied for viscous flows. Very little attention has been given to macroscopic models for viscoelastic flows in a porous medium. The following law for both relaxation and retardation phenomenon in an unbounded porous medium holds [5]:

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \nabla p = -\frac{\mu}{k} \left(1 + \lambda_3 \frac{\partial}{\partial t}\right) V_D, \tag{10}$$

in which k is permeability, $V_D(= \phi V)$ is the Darcian velocity and ϕ is the porosity of porous medium. Note that for $\lambda_1 = \lambda_3 = 0$, Eq. (10) reduces to well known Darcy's law of viscous fluid.

By the analogy with constitutive Eq (2), the following law for unidirectional flow of a generalized Burgers' fluid has been suggested:

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \nabla p = -\frac{\mu\phi}{k} \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) V. \tag{11}$$

The pressure gradient in above equation can also be interpreted as a measure of resistance to flow in the bulk of porous medium and r is a measure of flow resistance offered by the solid matrix. Thus r can be inferred from Eq.(11) to satisfy the following equation:

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) r = -\frac{\mu\phi}{k} \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) V. \tag{12}$$

Upon use of the stated assumptions, Eq.(8) yields

$$\begin{aligned} \left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \frac{\partial u}{\partial t} = {} & \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) \frac{\partial^2 u}{\partial y^2} \\ & - \frac{\sigma B_0^2}{\rho} \left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) u \\ & - \frac{\nu\phi}{k} \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) u, \end{aligned} \tag{13}$$

where the pressure gradient in x direction has been ignored and ν is the kinematic viscosity.

3. Stokes' Second Problem

This section deals with the MHD flow of a generalized Burgers' fluid in a porous space $y > 0$.The fluid is bounded by a rigid boundary at $y = 0$. Initially, both fluid and boundary are at rest. For $y = 0$, the boundary starts to oscillate in its own plane. In absence of pressure gradient, the equation which governs the flow is (13). The appropriate boundary and initial conditions are

$$u(0, t) = U_0 \cos \omega t \text{or } u(0, t) = U_0 \sin \omega t, \quad t > 0, \tag{14}$$

$$u(y, y) \longrightarrow 0, \frac{\partial u}{\partial y} \longrightarrow 0 \text{ as } y \longrightarrow \infty, \quad t \geq 0 \tag{15}$$

$$u(y, 0) = \frac{\partial u(y, 0)}{\partial t} = \frac{\partial^2 u(y, 0)}{\partial t^2} = 0, \quad y \geq 0 \tag{16}$$

in which ω is the imposed frequency.

In order to find the solution we define the Fourier sine transform pair as

$$\begin{aligned} \tilde{u}(\xi, t) &= \sqrt{\frac{2}{\pi}} \int_0^\infty u(y, t) \sin(\xi y)\, dy, \\ u(y, t) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{u}(\xi, t) \sin(\xi y)\, d\xi. \end{aligned} \tag{17}$$

3.1. When $u(0, t) = U_0 \cos \omega t$ *and* $\lambda_2 \neq 0$. Taking Fourier sine transform of Eqs. (13)–(15) and then solving them in the ξ-plane, we get the expression for the starting solution as follows

$$\bar{u}(\xi, t) = \bar{u}_t(\xi, t) + \bar{u}_s(\xi, t), \tag{18}$$

where $\bar{u}_t(\xi, t)$ and $\bar{u}_s(\xi, t)$ indicate the transient and steady state solutions, respectively, and are given by

$$\bar{u}_t(\xi,t)=\frac{-\sqrt{2/\pi}U_0\xi\nu\left(\left(m_2m_3-\omega^2\right)\left(F_0-\omega^2F_1\right)-\omega^2(m_2+m_3)F_2\right)e^{m_1t}}{(m_1-m_2)(m_1-m_3)F_3}+\frac{\sqrt{2/\pi}U_0\xi\nu\left(\left(\omega^2-m_1m_3\right)\left(F_0-\omega^2F_1\right)+\omega^2(m_1+m_3)F_2\right)e^{m_2t}}{(m_2-m_1)(m_2-m_3)F_3}-\frac{\sqrt{2/\pi}U_0\xi\nu\left(\left(m_1m_2-\omega^2\right)\left(F_0-\omega^2F_1\right)-\omega^2(m_1+m_2)F_2\right)e^{m_3t}}{(m_3-m_1)(m_3-m_2)F_3}, \tag{19}$$

$$\bar{u}_s(\xi,t)=\sqrt{\frac{2}{\pi}}U_0\nu\omega\sin\omega t\frac{\xi F_2}{F_3}+\sqrt{\frac{2}{\pi}}U_0\nu\cos\omega t\frac{\xi\left(F_0-\omega^2F_1\right)}{F_3}, \tag{20}$$

where

$$F_0=\nu\xi^2E_1+\frac{\sigma B_0^2}{\rho}+\frac{\nu\phi}{k}, F_1=-E_2+\frac{\sigma B_0^2}{\rho}E_3+\frac{\nu\phi}{k}E_4,$$

$$F_2=E_5-\frac{\sigma B_0^2}{\rho}E_2, F_3=\left(\nu\xi^2+\frac{\sigma B_0^2}{\rho}+\frac{\nu\phi}{k}-\omega^2E_6\right)^2+\omega^2E_7^2,$$

$$E_1=\left(\left(1-\lambda_4\omega^2\right)^2+(\lambda_3\omega)^2\right), E_2=\left(\lambda_3-\lambda_1-\lambda_2\lambda_3\omega^2+\lambda_1\lambda_4\omega^2\right),$$

$$E_3=\left(\lambda_2+\lambda_4-\lambda_1\lambda_3\omega^2-\lambda_2\lambda_4\omega^2\right), E_4=\left(2\lambda_4-\lambda_4^2\omega^2-\lambda_3^2\right),$$

$$E_5=\left(1-\omega^2(\lambda_2+\lambda_4)+\omega^2\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\right), E_6=\left(\lambda_1+\beta\xi^2+\frac{\sigma B_0^2}{\rho}\lambda_2+\frac{\phi}{k}\beta\right),$$

$$E_7=\left(1+\alpha\xi^2+\frac{\sigma B_0^2}{\rho}\lambda_1+\frac{\phi}{k}\alpha-\lambda_2\omega^2\right),$$

$$m_1=-\frac{d}{3\lambda_2}-\frac{2^{1/3}\left(-d^2+3e\lambda_2\right)}{3\lambda_2\left(-2d^3+9de\lambda_2-27f\lambda_2^2+\sqrt{4\left(-d^2+3e\lambda_2\right)^3+\left(-2d^3+9de\lambda_2-27f\lambda_2^2\right)}\right)^{1/3}}+\frac{\left(-2d^3+9de\lambda_2-27f\lambda_2^2+\sqrt{4\left(-d^2+3e\lambda_2\right)^3+\left(-2d^3+9de\lambda_2-27f\lambda_2^2\right)}\right)^{1/3}}{3(2)^{1/3}\lambda_2},$$

$$m_2=-\frac{d}{3\lambda_2}-\frac{\left(1+i\sqrt{3}\right)\left(-d^2+3e\lambda_2\right)}{3(2)^{2/3}\lambda_2\left(-2d^3+9de\lambda_2-27f\lambda_2^2+\sqrt{4\left(-d^2+3e\lambda_2\right)^3+\left(-2d^3+9de\lambda_2-27f\lambda_2^2\right)}\right)^{1/3}}-\frac{\left(1-i\sqrt{3}\right)\left(-2d^3+9de\lambda_2-27f\lambda_2^2+\sqrt{4\left(-d^2+3e\lambda_2\right)^3+\left(-2d^3+9de\lambda_2-27f\lambda_2^2\right)}\right)^{1/3}}{6(2)^{1/3}\lambda_2},$$

$$m_3=-\frac{d}{3\lambda_2}+\frac{\left(1-i\sqrt{3}\right)\left(-d^2+3e\lambda_2\right)}{3(2)^{2/3}\lambda_2\left(-2d^3+9de\lambda_2-27f\lambda_2^2+\sqrt{4\left(-d^2+3e\lambda_2\right)^3+\left(-2d^3+9de\lambda_2-27f\lambda_2^2\right)}\right)^{1/3}}-\frac{\left(1+i\sqrt{3}\right)\left(-2d^3+9de\lambda_2-27f\lambda_2^2+\sqrt{4\left(-d^2+3e\lambda_2\right)^3+\left(-2d^3+9de\lambda_2-27f\lambda_2^2\right)}\right)^{1/3}}{6(2)^{1/3}\lambda_2},$$

$$d=\lambda_1+\beta\xi^2+\frac{\sigma B_0^2}{\rho}\lambda_2+\frac{\phi}{k}\beta, e=1+\alpha\xi^2+\frac{\sigma B_0^2}{\rho}\lambda_1+\frac{\phi}{k}\alpha,$$

$$f=\nu\xi^2+\frac{\sigma B_0^2}{\rho}+\frac{\phi}{k}\beta, \alpha=\nu\lambda_3, \beta=\nu\lambda_4. \tag{21}$$

Fourier inversion of Eqs. (19) and (20) yields

$$u_t(y,t)=-\frac{2}{\pi}U_0\nu\int_0^\infty\frac{\xi\left(\left(m_2m_3-\omega^2\right)\left(F_0-\omega^2F_1\right)-\omega^2(m_2+m_3)F_2\right)e^{m_1t}\sin(\xi y)\,d\xi}{(m_1-m_2)(m_1-m_3)F_3}+\frac{2}{\pi}U_0\nu\int_0^\infty\frac{\xi\left(\left(\omega^2-m_1m_3\right)\left(F_0-\omega^2F_1\right)+\omega^2(m_1+m_3)F_2\right)e^{m_2t}\sin(\xi y)\,d\xi}{(m_2-m_1)(m_2-m_3)F_3}-\frac{2}{\pi}U_0\nu\int_0^\infty\frac{\xi\left(\left(m_1m_2-\omega^2\right)\left(F_0-\omega^2F_1\right)-\omega^2(m_1+m_2)F_2\right)e^{m_3t}\sin(\xi y)\,d\xi}{(m_3-m_1)(m_3-m_2)F_3}, \tag{22}$$

$$u_s(y,t)=\frac{2}{\pi}U_0\nu\omega\sin\omega t\int_0^\infty\frac{\xi F_2\sin(\xi y)\,d\xi}{F_3}+\frac{2}{\pi}U_0\nu\cos\omega t\int_0^\infty\frac{\xi\left(F_0-\omega^2F_1\right)\sin(\xi y)\,d\xi}{F_3}. \tag{23}$$

Note that for large times $u_t(y,t)\longrightarrow 0$ and $u_s(y,t)$ becomes [see appendix]

$$u_s(y,t)=U_0\exp(-Ay)\cos(\omega t-By), \tag{24}$$

$$2A^2=\sqrt{\left(\frac{\omega^2E_2+(\phi/k)\nu E_1+\left(\sigma B_0^2/\rho\right)E_5}{\nu E_1}\right)^2+\nu^2\omega^2\left(\frac{\left(E_5-\left(\sigma B_0^2/\rho\right)E_2\right)}{\nu^2E_1}\right)^2}+\frac{\omega^2E_2+(\phi/k)\nu E_1+\left(\sigma B_0^2/\rho\right)E_5}{\nu E_1}, \tag{25}$$

$$2B^2=\sqrt{\left(\frac{\omega^2E_2+(\phi/k)\nu E_1+\left(\sigma B_0^2/\rho\right)E_5}{\nu E_1}\right)^2+\nu^2\omega^2\left(\frac{\left(E_5-\left(\sigma B_0^2/\rho\right)E_2\right)}{\nu^2E_1}\right)^2}-\frac{\omega^2E_2+(\phi/k)\nu E_1+\left(\sigma B_0^2/\rho\right)E_5}{\nu E_1}. \tag{26}$$

Introducing the following dimensionless quantities

$$\tilde{y}=\sqrt{\frac{\omega}{2\nu}}y, \tilde{u}=\frac{u}{U_0}, \tilde{t}=\omega t, \tilde{\lambda}_1=\lambda_1\omega, \tilde{\lambda}_2=\lambda_2\omega$$
$$\tilde{\lambda}_3=\lambda_3\omega, \tilde{\lambda}_4=\lambda_4\omega, M^2=\frac{\sigma B_0^2}{\rho\omega}, \frac{1}{K}=\frac{\nu\phi}{k\omega} \tag{27}$$

Eq.(24) takes the following form

$$\tilde{u}_s(y,t) = \exp\left(-\tilde{A}\tilde{y}\right)\cos\left(\tilde{t} - \tilde{B}\tilde{y}\right). \tag{28}$$

where

$$\begin{aligned}
\tilde{A} &= \left(\sqrt{\left(\frac{\tilde{E}_2 + (1/K)\tilde{E}_1 + M^2\tilde{E}_5}{\tilde{E}_1}\right)^2 + \left(\frac{\tilde{E}_5 + M^2\tilde{E}_2}{\tilde{E}_1}\right)^2} + \left(\frac{\tilde{E}_2 + (1/K)\tilde{E}_1 + M^2\tilde{E}_5}{\tilde{E}_1}\right)\right)^{1/2},\\
B &= \left(\sqrt{\left(\frac{\tilde{E}_2 + (1/K)\tilde{E}_1 + M^2\tilde{E}_5}{\tilde{E}_1}\right)^2 + \left(\frac{\tilde{E}_5 + M^2\tilde{E}_2}{\tilde{E}_1}\right)^2} - \left(\frac{\tilde{E}_2 + (1/K)\tilde{E}_1 + M^2\tilde{E}_5}{\tilde{E}_1}\right)\right)^{1/2}.\\
\tilde{E}_1 &= \left(\left(1-\tilde{\lambda}_4\right)^2 + \tilde{\lambda}_3^2\right), E_2 = \left(\tilde{\lambda}_3 - \tilde{\lambda}_1\right) + \left(\tilde{\lambda}_1\tilde{\lambda}_4 - \tilde{\lambda}_2\tilde{\lambda}_3\right),\\
\tilde{E}_5 &= \left(1 - \left(\tilde{\lambda}_2 + \tilde{\lambda}_4\right) + \left(\tilde{\lambda}_1\tilde{\lambda}_3 + \tilde{\lambda}_2\tilde{\lambda}_4\right)\right)
\end{aligned} \tag{29}$$

3.2. *For $u(0,t) = U_0 \cos \omega t$ and $\lambda_2 = 0$.*

For this case we have the following expression

$$\begin{aligned}
u_t(y,t) &= \frac{2}{\pi}U_0\nu\int_0^\infty \frac{\xi\begin{pmatrix} r_2\left(F_0 - \omega^2\left(-F_4 + \left(\sigma B_0^2/\rho\right)F_5 + (\nu\phi/k)F_6\right)\right) \\ -\omega^2\left(F_7 - \left(\sigma B_0^2/\rho\right)F_4\right)\end{pmatrix} e^{r_1 t}\sin(\xi y)\,d\xi}{(r_1 - r_2)\left((F_8 - \omega^2F_9)^2 + \omega^2F_{10}^2\right)}\\
&+ \frac{2}{\pi}U_0\nu\int_0^\infty \frac{\xi\begin{pmatrix} r_1\left(F_0 - \omega^2\left(-F_4 + \left(\sigma B_0^2/\rho\right)F_5 + (\nu\phi/k)F_6\right)\right) \\ -\omega^2\left(F_7 - \left(\sigma B_0^2/\rho\right)F_4\right)\end{pmatrix} e^{r_2 t}\sin(\xi y)\,d\xi}{(r_2 - r_1)\left((F_8 - \omega^2F_9)^2 + \omega^2F_{10}^2\right)},
\end{aligned} \tag{30}$$

$$\begin{aligned}
u_s(y,t) &= \frac{2}{\pi}U_0\nu\omega\sin\omega t\int_0^\infty \frac{\xi\left(F_7 - \left(\sigma B_0^2/\rho\right)F_4\right)\sin(\xi y)\,d\xi}{\left((F_8 - \omega^2F_9)^2 + \omega^2F_{10}^2\right)}\\
&+ \frac{2}{\pi}U_0\nu\cos\omega t\int_0^\infty \xi\frac{F_0 - \omega^2\left(-F_4 + \left(\sigma B_0^2/\rho\right)F_5 + (\nu\phi/k)F_6\right)}{\left((F_8 - \omega^2F_9)^2 + \omega^2F_{10}^2\right)}\sin(\xi y)\,d\xi,
\end{aligned} \tag{31}$$

where

$$r_1 = \frac{-F_{10} + \sqrt{F_{10}^2 - 4F_8F_9}}{2F_9}, \tag{32}$$

$$r_2 = \frac{-F_{10} - \sqrt{F_{10}^2 - 4F_8F_9}}{2F_9}, \tag{33}$$

$$F_4 = \left(\lambda_3 - \lambda_1 + \lambda_1\lambda_4\omega^2\right), F_5 = \lambda_4 - \lambda_1\lambda_3, F_6 = \left(2\lambda_4 - \lambda_4^2\omega^2 - \lambda_3^2\right), \tag{34}$$

$$\begin{aligned}
F_7 &= \left(1 - \omega^2\lambda_4 + \omega^2\lambda_1\lambda_3\right),\\
F_8 &= \nu\xi^2 + \frac{\sigma B_0^2}{\rho} + \frac{\nu\phi}{k},\\
F_9 &= \lambda_1 + \beta\xi^2 + \frac{\phi}{k}\beta,
\end{aligned} \tag{35}$$

$$F_{10} = 1 + \alpha\xi^2 + \frac{\sigma B_0^2}{\rho}\lambda_1 + \frac{\phi}{k}\alpha. \tag{36}$$

Adopting the same methodology of solution as for Eqs. (23) and (31), gives

$$\tilde{u}_{s1}(y,t) = \exp\left(-\tilde{A}_1\tilde{y}\right)\cos\left(\tilde{t} - \tilde{B}_1\tilde{y}\right), \tag{37}$$

where

$$\begin{aligned}
\tilde{A}_1 &= \left(\sqrt{\left(\frac{\tilde{F}_4 + (1/K)\tilde{E}_1 + M^2\tilde{F}_7}{\tilde{E}_1}\right)^2 + \left(\frac{\tilde{F}_7 - M^2\tilde{F}_4}{\tilde{E}_1}\right)^2} + \left(\frac{\tilde{F}_4 + (1/K)\tilde{E}_1 + M^2\tilde{F}_7}{\tilde{E}_1}\right)\right)^{1/2},\\
\tilde{B}_1 &= \left(\sqrt{\left(\frac{\tilde{F}_4 + (1/K)\tilde{E}_1 + M^2\tilde{F}_7}{\tilde{E}_1}\right)^2 + \left(\frac{\tilde{F}_7 - M^2\tilde{F}_4}{\tilde{E}_1}\right)^2} - \left(\frac{\tilde{F}_4 + (1/K)\tilde{E}_1 + M^2\tilde{F}_7}{\tilde{E}_1}\right)\right)^{1/2},\\
\tilde{F}_4 &= \left(\tilde{\lambda}_3 - \tilde{\lambda}_1\right) + \left(\tilde{\lambda}_1\tilde{\lambda}_4\right), \tilde{F}_7 = \left(1 - \tilde{\lambda}_4 + \tilde{\lambda}_1\tilde{\lambda}_3\right).
\end{aligned} \tag{38}$$

It is worth mentioning to note that for $\lambda_4 = M = \phi = 0$ the Eq. (37) reduces to the solutions for an Oldroyd-B fluid. Moreover, Eq. (37) recovers the results of second grade fluid [8] when $\lambda_1 = \lambda_4 = M = \phi = 0$ and $\nu\lambda_3 = \alpha_1/\rho$ (α_1 is the material parameter of second grade fluid).

3.3. *For $u(0,t) = U_0 \sin \omega t$ and $\lambda_2 \neq 0$.*

Employing the similar procedure as for the case of $U_0 \cos \omega t$, the transient and steady state solutions are

$$\begin{aligned}
u_t(y,t) &= -\frac{2}{\pi}U_0\nu\omega\int_0^\infty \frac{\xi\left(\left(\omega^2 - m_2m_3\right)F_2 - (m_2 + m_3)\left(F_0 - \omega^2F_1\right)\right)e^{m_1 t}\sin(\xi y)\,d\xi}{(m_1 - m_2)(m_1 - m_3)F_3}\\
&+ \frac{2}{\pi}U_0\nu\omega\int_0^\infty \frac{\xi\left(\left(m_1m_3 - \omega^2\right)F_2 + (m_1 + m_3)\left(F_0 - \omega^2F_1\right)\right)e^{m_2 t}\sin(\xi y)\,d\xi}{(m_2 - m_1)(m_2 - m_3)F_3}\\
&- \frac{2}{\pi}U_0\nu\omega\int_0^\infty \frac{\xi\left(\left(\omega^2 - m_1m_2\right)F_2 - (m_1 + m_2)\left(F_0 - \omega^2F_1\right)\right)e^{m_3 t}\sin(\xi y)\,d\xi}{(m_3 - m_1)(m_3 - m_2)F_3},
\end{aligned} \tag{39}$$

$$\begin{aligned}
u_s(y,t) &= \frac{2}{\pi}U_0\nu\omega\cos\omega t\int_0^\infty \frac{\xi F_2\sin(\xi y)\,d\xi}{F_3}\\
&+ \frac{2}{\pi}U_0\nu\sin\omega t\int_0^\infty \frac{\xi\left(F_0 - \omega^2F_1\right)\sin(\xi y)\,d\xi}{F_3},
\end{aligned} \tag{40}$$

The Eq. (40) in dimensionless variables now gives

$$\tilde{u}_s(y,t) = \exp\left(-\tilde{A}\tilde{y}\right)\sin\left(\tilde{t} - \tilde{B}\tilde{y}\right). \tag{41}$$

3.4. For $u(0,t) = U_0 \sin \omega t$ and $\lambda_2 = 0$. Here we have

$$u_t(y,t) = -\frac{2}{\pi}U_0\nu\omega\int_0^{\infty}\frac{\xi\begin{pmatrix} r_2\left(F_7 - \left(\sigma B_0^2/\rho\right)F_4\right) + \\ F_0 - \omega^2\left(-F_4 + \left(\sigma B_0^2/\rho\right)F_5 + \left(\nu\phi/k\right)F_6\right)\end{pmatrix}e^{r_1 t}\sin(\xi y)\,d\xi}{(r_1 - r_2)\left(\left(F_8 - \omega^2 F_9\right)^2 + \omega^2 F_{10}^2\right)} + \frac{2}{\pi}U_0\nu\omega\int_0^{\infty}\frac{\xi\begin{pmatrix} r_1\left(F_7 - \left(\sigma B_0^2/\rho\right)F_4\right) + \\ \left(F_0 - \omega^2\left(-F_4 + \left(\sigma B_0^2/\rho\right)F_5 + \left(\nu\phi/k\right)F_6\right)\right)\end{pmatrix}e^{r_2 t}\sin(\xi y)\,d\xi}{(r_1 - r_2)\left(\left(F_8 - \omega^2 F_9\right)^2 + \omega^2 F_{10}^2\right)},$$

$$u_s(y,t) = -\frac{2}{\pi}U_0\nu\omega\cos\omega t\int_0^{\infty}\frac{\xi\left(F_7 - \left(\sigma B_0^2/\rho\right)F_4\right)\sin(\xi y)\,d\xi}{\left(\left(F_8 - \omega^2 F_9\right)^2 + \omega^2 F_{10}^2\right)} + \frac{2}{\pi}U_0\nu\sin\omega t\int_0^{\infty}\frac{\xi\left(F_0 - \omega^2\left(-F_4 + \left(\sigma B_0^2/\rho\right)F_5 + \left(\nu\phi/k\right)F_6\right)\right)\sin(\xi y)\,d\xi}{\left(\left(F_8 - \omega^2 F_9\right)^2 + \omega^2 F_{10}^2\right)}, \tag{42}$$

After finding the above integral, the solution in dimensionless variables is obtained as follows

$$\tilde{u}_{s1}(y,t) = \exp\left(-\tilde{A}_1\tilde{y}\right)\sin\left(\tilde{t} - \tilde{B}_1\tilde{y}\right) \tag{43}$$

The above equation also reduces to the result of second grade fluid [8] for $\lambda_1 = \lambda_4 = M = \phi = 0$ and $\nu\lambda_3 = \alpha_1/\rho$.

4. Modified Stokes' Second Problem

Here, we consider the MHD flow between two infinite plates distance d apart. The lower plate at $y = 0$ oscillates in its own plane for $t > 0$ while the upper plate at $y = d$ is stationary. The problem which governs the flow consists of Eqs. (13) and (14)

$$u(d,t) = 0; \quad t \in R, \tag{44}$$

$$u(y,0) = \frac{\partial u(y,0)}{\partial t} = \frac{\partial^2 u(y,0)}{\partial t^2} = 0, \quad 0 < y < d \tag{45}$$

Following the same method of solution as in the previous section we have.

4.1. For $u(0,t) = U_0 \cos \omega t$ and $\lambda_2 \neq 0$.

$$u_t(y,t) = -\frac{2}{d}U_0\nu\sum_{n=1}^{\infty}\frac{\lambda_n\left(\left(n_2 n_3 - \omega^2\right)\left(F_{11} - \omega^2 F_1\right) - \omega^2(n_2 + n_3)F_2\right)e^{n_1 t}\sin(\lambda_n y)}{(n_1 - n_2)(n_1 - n_3)F_{12}} + \frac{2}{d}U_0\nu\sum_{n=1}^{\infty}\frac{\lambda_n\left(\left(\omega^2 - n_1 n_3\right)\left(F_{11} - \omega^2 F_1\right) + \omega^2(n_1 + n_3)F_2\right)e^{n_2 t}\sin(\lambda_n y)}{(n_2 - n_1)(n_2 - n_3)F_{12}} - \frac{2}{\pi}U_0\nu\sum_{n=1}^{\infty}\frac{\lambda_n\left(\left(n_1 n_2 - \omega^2\right)\left(F_{11} - \omega^2 F_1\right) - \omega^2(n_1 + n_2)F_2\right)e^{n_3 t}\sin(\lambda_n y)}{(n_3 - n_1)(n_3 - n_2)F_{12}},$$

$$u_s(y,t) = \frac{2}{d}U_0\nu\omega\sin\omega t\sum_{n=1}^{\infty}\frac{\lambda_n F_2\sin(\lambda_n y)}{F_{12}} + \frac{2}{d}U_0\nu\cos\omega t\sum_{n=1}^{\infty}\frac{\lambda_n\left(F_{11} - \omega^2 F_1\right)\sin(\lambda_n y)}{F_{12}} \tag{46}$$

where

$$F_{11} = \nu\lambda_n^2 E_1 + \frac{\sigma B_0^2}{\rho} + \frac{\nu\phi}{k},$$

$$F_{12} = \left(\left(\nu\lambda_n^2 + \frac{\sigma B_0^2}{\rho} + \frac{\nu\phi}{k} - \omega^2 E_9\right)^2 + \omega^2 E_{10}^2\right),$$

$$E_9 = \left(\lambda_1 + \beta\lambda_n^2 + \frac{\sigma B_0^2}{\rho}\lambda_2 + \frac{\phi}{k}\beta\right),$$

$$E_{10} = \left(1 + \alpha\lambda_n^2 + \frac{\sigma B_0^2}{\rho}\lambda_1 + \frac{\phi}{k}\alpha - \lambda_2\omega^2\right)$$

$$n_1 = -\frac{d}{3\lambda_2} - \frac{2^{1/3}\left(-d_1^2 + 3e_1\lambda_2\right)}{3\lambda_2\begin{pmatrix} -2d_1^3 + 9d_1e_1\lambda_2 - 27f_1\lambda_2^2 \\ +\sqrt{4\left(-d_1^2 + 3e_1\lambda_2\right)^3 + \left(-2d_1^3 + 9d_1e_1\lambda_2 - 27f_1\lambda_2^2\right)}\end{pmatrix}^{1/3}} + \frac{\begin{pmatrix} -2d_1^3 + 9d_1e_1\lambda_2 - 27f_1\lambda_2^2 \\ +\sqrt{4\left(-d_1^2 + 3e_1\lambda_2\right)^3 + \left(-2d_1^3 + 9d_1e_1\lambda_2 - 27f_1\lambda_2^2\right)}\end{pmatrix}^{1/3}}{3(2)^{1/3}\lambda_2},$$

$$n_2 = -\frac{d_1}{3\lambda_2} + \frac{\left(1 + i\sqrt{3}\right)\left(-d_1^2 + 3e_1\lambda_2\right)}{3(2)^{2/3}\lambda_2\begin{pmatrix} -2d_1^3 + 9d_1e_1\lambda_2 - 27f_1\lambda_2^2 \\ +\sqrt{4\left(-d_1^2 + 3e_1\lambda_2\right)^3 + \left(-2d_1^3 + 9d_1e_1\lambda_2 - 27f_1\lambda_2^2\right)}\end{pmatrix}^{1/3}} - \frac{\left(1 - i\sqrt{3}\right)\begin{pmatrix} -2d_1^3 + 9d_1e_1\lambda_2 - 27f_1\lambda_2^2 \\ +\sqrt{4\left(-d_1^2 + 3e_1\lambda_2\right)^3 + \left(-2d_1^3 + 9d_1e_1\lambda_2 - 27f_1\lambda_2^2\right)}\end{pmatrix}^{1/3}}{6(2)^{1/3}\lambda_2},$$

$$n_3 = -\frac{d_1}{3\lambda_2} + \frac{\left(1 - i\sqrt{3}\right)\left(-d_1^2 + 3e_1\lambda_2\right)}{3(2)^{2/3}\lambda_2\begin{pmatrix} -2d_1^3 + 9d_1e_1\lambda_2 - 27f_1\lambda_2^2 \\ +\sqrt{4\left(-d_1^2 + 3e_1\lambda_2\right)^3 + \left(-2d_1^3 + 9d_1e_1\lambda_2 - 27f_1\lambda_2^2\right)}\end{pmatrix}^{1/3}} - \frac{\left(1 + i\sqrt{3}\right)\begin{pmatrix} -2d_1^3 + 9d_1e_1\lambda_2 - 27f_1\lambda_2^2 \\ +\sqrt{4\left(-d_1^2 + 3e_1\lambda_2\right)^3 + \left(-2d_1^3 + 9d_1e\lambda_2 - 27f_1\lambda_2^2\right)}\end{pmatrix}^{1/3}}{6(2)^{1/3}\lambda_2},$$

$$d_1 = \lambda_1 + \beta\lambda_n^2 + \frac{\sigma B_0^2}{\rho}\lambda_2 + \frac{\phi}{k}\beta, e_1 = 1 + \alpha\lambda_n^2 + \frac{\sigma B_0^2}{\rho}\lambda_1 + \frac{\phi}{k}\alpha, f_1 = \nu\lambda_n^2 + \frac{\sigma B_0^2}{\rho} + \frac{\phi}{k}\nu.$$

$$E_9 = \left(\lambda_1 + \beta\lambda_n^2 + \frac{\sigma B_0^2}{\rho}\lambda_2 + \frac{\phi}{k}\beta\right), E_{10} = \left(1 + \alpha\lambda_n^2 + \frac{\sigma B_0^2}{\rho}\lambda_1 + \frac{\phi}{k}\alpha - \lambda_2\omega^2\right). \tag{47}$$

4.2. For $u(0,t)=U_0\cos\omega t$ and $\lambda_2=0$. we have

$$u_t(y,t)=\frac{2}{d}U_0\nu\sum_{n=1}^{\infty}\frac{\lambda_n\begin{pmatrix}r_4\left(F_{11}-\omega^2\left(-F_4+\left(\sigma B_0^2/\rho\right)F_5+\left(\nu\phi/k\right)F_6\right)\right)\\-\omega^2\left(F_7-\left(\sigma B_0^2/\rho\right)F_4\right)\end{pmatrix}e^{r_3t}\sin\left(\lambda_n y\right)}{\left(r_3-r_4\right)\left(\left(F_{13}-\omega^2F_{14}\right)^2+\omega^2F_{15}^2\right)}+\frac{2}{d}U_0\nu\sum_{n=1}^{\infty}\frac{\lambda_n\begin{pmatrix}r_3\left(F_{11}-\omega^2\left(-F_4+\left(\sigma B_0^2/\rho\right)F_5+\left(\nu\phi/k\right)F_6\right)\right)\\-\omega^2\left(F_7-\left(\sigma B_0^2/\rho\right)F_4\right)\end{pmatrix}e^{r_4t}\sin\left(\lambda_n y\right)}{\left(r_4-r_3\right)\left(\left(F_{13}-\omega^2F_{14}\right)^2+\omega^2F_{15}^2\right)}, \tag{48}$$

$$u_s(y,t)=\frac{2}{d}U_0\nu\omega\sin\omega t\sum_{n=1}^{\infty}\frac{\lambda_n\left(F_7-\left(\sigma B_0^2/\rho\right)F_4\right)\sin\left(\lambda_n y\right)}{\left(\left(F_{13}-\omega^2F_{14}\right)^2+\omega^2F_{15}^2\right)}+\frac{2}{d}U_0\nu\cos\omega t\sum_{n=1}^{\infty}\cdot\frac{\lambda_n\left(F_{11}-\omega^2\left(-F_4+\left(\sigma B_0^2/\rho\right)F_5+\left(\nu\phi/k\right)F_6\right)\right)\sin\left(\lambda_n y\right)}{\left(\left(F_{13}-\omega^2F_{14}\right)^2+\omega^2F_{15}^2\right)}, \tag{49}$$

where

$$\begin{aligned}r_3&=\frac{-F_{15}+\sqrt{F_{15}^2-4F_{13}F_{14}}}{2F_{14}},\\ r_4&=\frac{-F_{15}-\sqrt{F_{15}^2-4F_{13}F_{14}}}{2F_{14}},\\ F_{13}&=\nu\lambda_n^2+\frac{\sigma B_0^2}{\rho}+\frac{\nu\phi}{k},\\ F_{14}&=\lambda_1+\beta\lambda_n^2+\frac{\phi}{k}\beta,\\ F_{15}&=1+\alpha\lambda_n^2+\frac{\sigma B_0^2}{\rho}\lambda_1+\frac{\phi}{k}\alpha.\end{aligned} \tag{50}$$

Note that the results of second grade fluid can be obtained by choosing $\lambda_1=\lambda_4=M=\phi=0$ and $\nu\lambda_3=\alpha_1/\rho$ in Eq. (49).

4.3. For $u(0,t)=U_0\sin\omega t$ and $\lambda_2\neq 0$. we obtain

$$\begin{aligned}u_t(y,t)=&-\frac{2}{d}U_0\nu\omega\sum_{n=1}^{\infty}\frac{\lambda_n\left(\left(\omega^2-n_2n_3\right)F_2-\left(n_2+n_3\right)\left(F_{11}-\omega^2F_1\right)\right)e^{n_1t}\sin\left(\lambda_n y\right)}{\left(n_1-n_2\right)\left(n_1-n_3\right)F_{12}}\\&+\frac{2}{d}U_0\nu\omega\sum_{n=1}^{\infty}\frac{\lambda_n\left(\left(n_1n_3-\omega^2\right)F_2+\left(n_1+n_3\right)\left(F_{11}-\omega^2F_1\right)\right)e^{n_2t}\sin\left(\lambda_n y\right)}{\left(n_2-n_1\right)\left(n_2-n_3\right)F_{12}}\\&-\frac{2}{d}U_0\nu\omega\sum_{n=1}^{\infty}\frac{\lambda_n\left(\left(\omega^2-n_1n_2\right)F_2-\left(n_1+n_2\right)\left(F_{11}-\omega^2F_1\right)\right)e^{n_3t}\sin\left(\lambda_n y\right)}{\left(n_3-n_1\right)\left(n_3-n_2\right)F_{12}},\end{aligned}$$

$$u_s(y,t)=-\frac{2}{d}U_0\nu\omega\cos\omega t\sum_{n=1}^{\infty}\frac{\lambda_nF_2\sin\left(\lambda_n y\right)}{F_{12}}+\frac{2}{d}U_0\nu\sin\omega t\sum_{n=1}^{\infty}\frac{\lambda_n\left(F_{11}-\omega^2F_1\right)\sin\left(\lambda_n y\right)}{F_{12}}. \tag{51}$$

4.4. For $u(0,t)=U_0\sin\omega t$ and $\lambda_2=0$. we get

$$u_t(y,t)=-\frac{2}{d}U_0\nu\omega\sum_{n=1}^{\infty}\frac{\lambda_n\left(r_4\left(F_7-\left(\sigma B_0^2/\rho\right)F_4\right)+F_{11}-\omega^2\left(-F_4+\left(\sigma B_0^2/\rho\right)F_5+\left(\nu\phi/k\right)F_6\right)\right)\times e^{r_3t}\sin\left(\lambda_n y\right)}{\left(r_3-r_4\right)\left(\left(F_{13}-\omega^2F_{14}\right)^2+\omega^2F_{15}^2\right)}+\frac{2}{d}U_0\nu\omega\sum_{n=1}^{\infty}\frac{\lambda_n\left(r_3\left(F_7-\left(\sigma B_0^2/\rho\right)F_4\right)+F_{11}-\omega^2\left(-F_4+\left(\sigma B_0^2/\rho\right)F_5+\left(\nu\phi/k\right)F_6\right)\right)\times e^{r_4t}\sin\left(\xi y\right)}{\left(r_3-r_4\right)\left(\left(F_{13}-\omega^2F_{14}\right)^2+\omega^2F_{15}^2\right)},$$

$$u_s(y,t)=-\frac{2}{d}U_0\nu\omega\cos\omega t\sum_{n=1}^{\infty}\frac{\lambda_n\left(F_7-\left(\sigma B_0^2/\rho\right)F_4\right)\sin\left(\lambda_n y\right)}{\left(\left(F_{13}-\omega^2F_{14}\right)^2+\omega^2F_{15}^2\right)}+\frac{2}{d}U_0\nu\sin\omega t\sum_{n=1}^{\infty}\cdot\frac{\lambda_n\left(F_{11}-\omega^2\left(-F_4+\left(\sigma B_0^2/\rho\right)F_5+\left(\nu\phi/k\right)F_6\right)\right)}{\left(\left(F_{13}-\omega^2F_{14}\right)^2+\omega^2F_{15}^2\right)}\sin\left(\lambda_n y\right). \tag{52}$$

The above equation gives the solution of second grade fluid [8] for $\lambda_1=\lambda_4=M=\phi=0$ and $\nu\lambda_3=\alpha_1/\rho$.

5. Time-Periodic Plane Poiseuille Flow

In this section the flow between the two stationary plates is induced by an oscillating pressure gradient in the x-direction. Initially the fluid and plates are at rest. The pressure gradient is of the following form

$$\frac{\partial p}{\partial x}=-\rho Q\cos\omega t\text{or}\ \frac{\partial p}{\partial x}=-\rho Q\sin\omega t. \tag{53}$$

The flow is governed by Eq. (45) and

$$\begin{aligned}&\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\frac{\partial u}{\partial t}+\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\frac{\partial p}{\partial x}\\&\quad=\nu\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_3\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2u}{\partial y^2}\\&\quad-\frac{\sigma B_0^2}{\rho}\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)u-\frac{\nu\phi}{k}\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_3\frac{\partial^2}{\partial t^2}\right)u,\\&u(0,t)=u(d,t)=0\,;t\in R.\end{aligned} \tag{54}$$

The solutions here are given by.

5.1. When $\partial p/\partial x=-\rho Q\cos\omega t$ and $\lambda_2\neq 0$. then

$$\begin{aligned}u_t(y,t)=&-\frac{4}{d}Q\sum_{n=1}^{\infty}\frac{\left(\left(q_2q_3-\omega^2\right)F_{16}-\omega^2\left(q_2+q_3\right)F_{17}\right)e^{q_1t}\sin\left(\lambda_{2n-1}y\right)}{\left(q_1-q_2\right)\left(q_1-q_3\right)F_{18}\lambda_{2n-1}}\\&+\frac{4}{d}Q\sum_{n=1}^{\infty}\frac{\left(\left(\omega^2-q_1q_3\right)F_{16}+\omega^2\left(q_1+q_3\right)F_{17}\right)e^{q_2t}\sin\left(\lambda_{2n-1}y\right)}{\left(q_2-q_1\right)\left(q_2-q_3\right)F_{18}\lambda_{2n-1}}\\&-\frac{4}{d}Q\sum_{n=1}^{\infty}\frac{\left(\left(q_1q_2-\omega^2\right)F_{16}-\omega^2\left(q_1+q_2\right)F_{17}\right)e^{q_3t}\sin\left(\lambda_{2n-1}y\right)}{\left(q_3-q_1\right)q_3-q_2F_{18}\lambda_{2n-1}},\end{aligned}$$

$$u_s(y,t)=\frac{4Q}{d}\cos\omega t\sum_{n=1}^{\infty}\frac{F_{16}\sin\left(\lambda_{2n-1}y\right)}{\lambda_{2n-1}F_{18}}+\frac{4\omega Q}{d}\sin\omega t\sum_{n=1}^{\infty}\frac{F_{17}\sin\left(\lambda_{2n-1}y\right)}{\lambda_{2n-1}F_{18}} \tag{55}$$

where

$$q_1=-\frac{d_2}{3\lambda_2}-\frac{2^{1/3}\left(-d_2^2+3e_2\lambda_2\right)}{3\lambda_2\left(\begin{array}{c}-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\\+\sqrt{4\left(-d_2^2+3e_2\lambda_2\right)^3+\left(-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\right)}\end{array}\right)^{1/3}}+\frac{\left(\begin{array}{c}-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\\+\sqrt{4\left(-d_2^2+3e_2\lambda_2\right)^3+\left(-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\right)}\end{array}\right)^{1/3}}{3(2)^{1/3}\lambda_2},$$

$$q_2=-\frac{d_2}{3\lambda_2}-\frac{\left(1+i\sqrt{3}\right)\left(-d_2^2+3e_2\lambda_2\right)}{3(2)^{2/3}\lambda_2\left(\begin{array}{c}-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\\+\sqrt{4\left(-d_2^2+3e_2\lambda_2\right)^3+\left(-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\right)}\end{array}\right)^{1/3}}-\frac{\left(1-i\sqrt{3}\right)\left(\begin{array}{c}-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\\+\sqrt{4\left(-d_2^2+3e_2\lambda_2\right)^3+\left(-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\right)}\end{array}\right)^{1/3}}{6(2)^{1/3}\lambda_2},$$

$$q_3=-\frac{d_2}{3\lambda_2}+\frac{\left(1-i\sqrt{3}\right)\left(-d_2^2+3e_2\lambda_2\right)}{3(2)^{2/3}\lambda_2\left(\begin{array}{c}-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\\+\sqrt{4\left(-d_2^2+3e_2\lambda_2\right)^3+\left(-2d_2^3+9d_2e_1\lambda_2-27f_2\lambda_2^2\right)}\end{array}\right)^{1/3}}-\frac{\left(1+i\sqrt{3}\right)\left(\begin{array}{c}-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\\+\sqrt{4\left(-d_2^2+3e_2\lambda_2\right)^3+\left(-2d_2^3+9d_2e_2\lambda_2-27f_2\lambda_2^2\right)}\end{array}\right)^{1/3}}{6(2)^{1/3}\lambda_2},$$

$$F_{16}=\left(\nu\lambda_{2n-1}^2E_5+\frac{\sigma B_0^2}{\rho}E_{11}+\frac{\nu\phi E_5}{k}\right),$$

$$F_{17}=\left(\nu\lambda_{2n-1}^2E_2+E_{11}+\frac{\nu\phi}{k}E_2\right),$$

$$F_{18}=\left(\nu\lambda_{2n-1}^2+\frac{\sigma B_0^2}{\rho}+\frac{\nu\phi}{k}-\omega^2E_{12}\right)^2+\omega^2E_{13}^2$$

$$E_{13}=\left(1-\omega^2\lambda_2\right)^2-\lambda_1^2\omega^2,$$

$$E_{12}=\left(\lambda_1+\beta\lambda_{2n-1}^2+\frac{\sigma B_0^2}{\rho}\lambda_2+\frac{\phi}{k}\beta\right),$$

$$E_{13}=\left(1+\alpha\lambda_{2n-1}^2+\frac{\sigma B_0^2}{\rho}\lambda_1+\frac{\phi}{k}\alpha-\lambda_2\omega^2\right)$$

$$d_2=\lambda_1+\beta\lambda_{2n-1}^2+\frac{\sigma B_0^2}{\rho}\lambda_2+\frac{\phi}{k}\beta,$$

$$e_2=1+\alpha\lambda_{2n-1}^2+\frac{\sigma B_0^2}{\rho}\lambda_1+\frac{\phi}{k}\alpha,$$

$$f_2=\nu\lambda_{2n-1}^2+\frac{\sigma B_0^2}{\rho}+\frac{\phi}{k}\nu. \tag{56}$$

5.2. When $\partial p/\partial x=-\rho Q\cos\omega t$ and $\lambda_2=0$. then

$$u_t(y,t)=\frac{4}{d}Q\sum_{n=1}^{\infty}\frac{\left(\begin{array}{c}r_6\left(\nu\lambda_{2n-1}^2F_7+\left(\sigma B_0^2/\rho\right)F_{19}+\left(\nu\phi/k\right)F_7\right)\\-\omega^2\left(\nu\lambda_{2n-1}^2F_4+F_{19}+\left(\nu\phi/k\right)F_4\right)\end{array}\right)e^{r_5t}\sin\left(\lambda_{2n-1}y\right)}{\left(r_5-r_6\right)\left(\left(F_{20}-\omega^2F_{21}\right)^2+\omega^2F_{22}^2\right)\lambda_{2n-1}}+\frac{4}{d}Q\sum_{n=1}^{\infty}\frac{\left(\begin{array}{c}r_5\left(\nu\lambda_{2n-1}^2F_7+\left(\sigma B_0^2/\rho\right)F_{19}+\left(\nu\phi/k\right)F_7\right)\\-\omega^2\left(\nu\lambda_{2n-1}^2F_4+F_{19}+\left(\nu\phi/k\right)F_4\right)\end{array}\right)e^{r_6t}\sin\left(\lambda_{2n-1}y\right)}{\left(r_6-r_5\right)\left(\left(F_{20}-\omega^2F_{21}\right)^2+\omega^2F_{22}^2\right)\lambda_{2n-1}}, \tag{57}$$

$$u_s(y,t)=\frac{4Q}{d}\cos\omega t\sum_{n=1}^{\infty}\cdot\frac{\left(\nu\lambda_{2n-1}^2F_7+\left(\sigma B_0^2/\rho\right)F_{19}+\left(\nu\phi/k\right)F_7\right)}{\lambda_{2n-1}\left(\left(F_{20}-\omega^2F_{21}\right)^2+\omega^2F_{22}^2\right)}\sin\left(\lambda_{2n-1}y\right)+\frac{4\omega Q}{d}\sin\omega t\sum_{n=1}^{\infty}\frac{\left(\nu\lambda_{2n-1}^2F_4+F_{19}+\left(\nu\phi/k\right)F_4\right)\sin\left(\lambda_{2n-1}y\right)}{\lambda_{2n-1}\left(\left(F_{20}-\omega^2F_{21}\right)^2+\omega^2F_{22}^2\right)}, \tag{58}$$

where

$$r_5=\frac{-F_{22}+\sqrt{F_{22}^2-4F_{21}F_{20}}}{2F_{21}},\ r_4=\frac{-F_{22}-\sqrt{F_{22}^2-4F_{21}F_{20}}}{2F_{21}}$$

$$F_{19}=1+\lambda_1^2\omega^2,\ F_{20}=\nu\lambda_{2n-1}^2+\frac{\sigma B_0^2}{\rho}+\frac{\nu\phi}{k},$$

$$F_{21}=\lambda_1+\beta\lambda_{2n-1}^2+\frac{\phi}{k}\beta,\ F_{22}=1+\alpha\lambda_{2n-1}^2+\frac{\sigma B_0^2}{\rho}\lambda_1+\frac{\phi}{k}\alpha. \tag{59}$$

The solution of second grade fluid [8] can be deduced from Eq. (58) by taking $\lambda_1=\lambda_4=M=\phi=0$ and $\nu\lambda_3=\alpha_1/\rho$.

5.3. For $\partial p/\partial x=-\rho Q\sin\omega t$ and $\lambda_2\neq 0$. we have

$$u_t(y,t)=-\frac{4}{d}\omega Q\sum_{n=1}^{\infty}\cdot\frac{\left(\left(\omega^2-q_2q_3\right)F_{17}-\left(q_2+q_3\right)F_{16}\right)e^{q_1t}\sin\left(\lambda_{2n-1}y\right)}{\left(q_1-q_2\right)\left(q_1-q_3\right)F_{18}\lambda_{2n-1}}+\frac{4}{d}\omega Q\sum_{n=1}^{\infty}\cdot\frac{\left(\left(q_2q_3-\omega^2\right)F_{17}-\left(q_2+q_3\right)F_{16}\right)e^{q_2t}\sin\left(\lambda_{2n-1}y\right)}{\left(q_2-q_1\right)\left(q_2-q_3\right)F_{18}\lambda_{2n-1}}-\frac{4}{d}\omega Q\sum_{n=1}^{\infty}\cdot\frac{\left(\left(\omega^2-q_1q_2\right)F_{17}-\left(q_1+q_2\right)F_{16}\right)e^{q_1t}\sin\left(\lambda_{2n-1}y\right)}{\left(q_3-q_1\right)\left(q_3-q_1\right)F_{18}\lambda_{2n-1}}, \tag{60}$$

$$u_s(y,t)=\frac{4Q}{d}\sin\omega t\sum_{n=1}^{\infty}\frac{F_{16}\sin\left(\lambda_{2n-1}y\right)}{\lambda_{2n-1}F_{18}}+\frac{4\omega Q}{d}\cos\omega t\sum_{n=1}^{\infty}\frac{F_{17}\sin\left(\lambda_{2n-1}y\right)}{\lambda_{2n-1}F_{18}}.$$

5.4. For $\partial p/\partial x = -\rho Q \sin \omega t$ and $\lambda_2 = 0$. we get

$$u_t(y,t) = \frac{4}{d}Q\sum_{n=1}^{\infty}\frac{\begin{pmatrix} r_6\left(\nu\lambda_{2n-1}^2F_4 + \left(\sigma B_0^2/\rho\right)F_{19} + (\nu\phi/k)F_4\right) \\ +\left(\nu\lambda_{2n-1}^2F_7 + F_{19} + (\nu\phi/k)F_7\right)\end{pmatrix} e^{r_5 t}\sin\left(\lambda_{2n-1}y\right)}{\left(r_5 - r_6\right)\left(\left(F_{20} - \omega^2F_{21}\right)^2 + \omega^2F_{22}^2\right)\lambda_{2n-1}}$$
$$+ \frac{4}{d}Q\sum_{n=1}^{\infty}\frac{\begin{pmatrix} r_5\left(\nu\lambda_{2n-1}^2F_4 + \left(\sigma B_0^2/\rho\right)F_{19} + (\nu\phi/k)F_4\right) \\ +\left(\nu\lambda_{2n-1}^2F_7 + F_{19} + (\nu\phi/k)F_7\right)\end{pmatrix} e^{r_6 t}\sin\left(\lambda_{2n-1}y\right)}{\left(r_5 - r_6\right)\left(\left(F_{20} - \omega^2F_{21}\right)^2 + \omega^2F_{22}^2\right)\lambda_{2n-1}},$$

$$u_s(y,t) = \frac{4Q}{d}\sin\omega t\sum_{n=1}^{\infty}\frac{\left(\nu\lambda_{2n-1}^2F_7 + \left(\sigma B_0^2/\rho\right)F_{19} + (\nu\phi/k)F_7\right)}{\lambda_{2n-1}\left(\left(F_{20} - \omega^2F_{21}\right)^2 + \omega^2F_{22}^2\right)}\sin\left(\lambda_{2n-1}y\right)$$
$$- \frac{4\omega Q}{d}\cos\omega t\sum_{n=1}^{\infty}\frac{\left(\nu\lambda_{2n-1}^2F_4 + F_{19} + (\nu\phi/k)F_4\right)\sin\left(\lambda_{2n-1}y\right)}{\lambda_{2n-1}\left(\left(F_{20} - \omega^2F_{21}\right)^2 + \omega^2F_{22}^2\right)}. \quad (61)$$

The above solution yields the result of second grade fluid [8] when $\lambda_1 = \lambda_4 = M = \phi = 0$ and $\nu\lambda_3 = \alpha_1/\rho$.

6. Results and Discussion

In this section we discuss the graphical results of velocity profiles due to the oscillations of the plate at $t > 0$.

6.1. Analysis of Results. The difference between velocity profiles of Oldroyd-B fluid and generalized Burgers' fluid is shown for different values of M and K. Figure 1 is constructed to describe the effects of M on the velocity profiles in two fluid cases. It is evident from this figure that the fluid velocity decreases in Oldroyd-B fluid and generalized Burgers' fluids by increasing M. Figure 2 shows the influence of permeability of porous medium K on the velocity profile in the presence of magnetic field parameter M. By increasing K the velocity profiles for both fluid increase. Figure 3 is displayed for the variation of K in the absence of M. It can be seen from this figure that the velocity also increases for both cases of Oldroyd-B and generalized Burgers' fluid. The comparison of steady state solution u_s in various fluid models is shown in Table 2.

6.2. Discussion of Results. It is observed from Figure 1 that influence of magnetic field is more prominent on the velocity profiles in Oldroyd -B fluid when compared with a generalized Burgers' fluid. The effects of K on the velocity in presence of magnetic field M, in Figure 2 are more significant in Oldroyd-B fluid case as compared to generalized Burgers' fluid. Similarly, when K is varied in absence of magnetic field M in Figure 3, Oldroyd-B fluid shows more notable behavior in comparison with generalized Burgers' fluid. Hence it is concluded that Oldroyd-B fluid shows significant increase in steady state velocity when porosity is increased in presence of magnetic field and similar behavior is observed when magnetic field is neglected.

In Table 2 it is clearly seen that the value of steady state solution u_s in Oldroyd-B fluid is greatest and smallest in viscous fluid when $M \neq 0$ and $K \neq 0$. Moreover, the value of u_s in Maxwell fluid is greater than that of second grade and generalized Burgers' fluids. However, u_s is maximum for

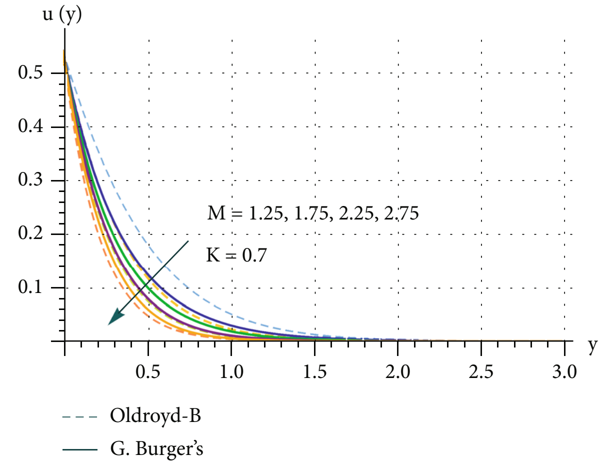

FIGURE 1: Profile of normalized steady state velocity $u(y)$ for various values of M.

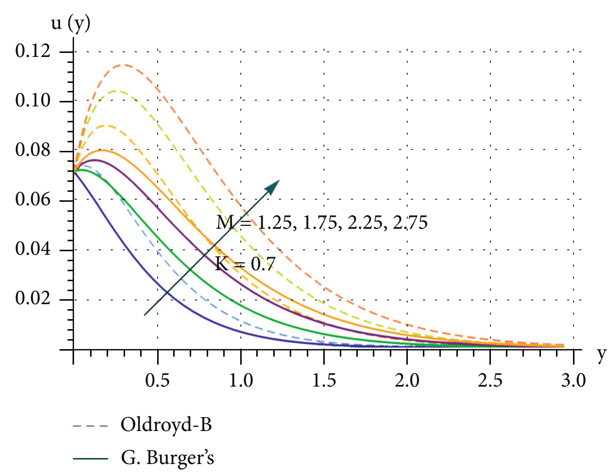

FIGURE 2: Profile of normalized steady state velocity $u(y)$ for various values of K.

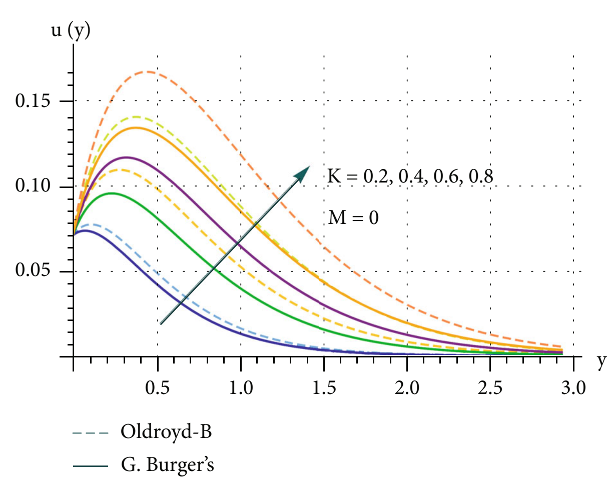

FIGURE 3: Profile of normalized steady state velocity $u(y)$ for various values of K at $M = 0$.

Table 2: Comparison of velocity in different fluids when $t = 1.5$ and $y = 0.5$.

Types of fluid	Material constants	u_s for $M = 1$, $K = 0.4$	u_s for $M = 0$, $K \longrightarrow \infty$	u_s for $M = 0$, $K = 0.4$	u_s for $M = 1$, $K \longrightarrow \infty$
Newtonian	$\lambda_i = 0, i = 1, 2, 3, 4$	0.131357	0.32771	0.158758	0.227262
Second grade	$\lambda_1 = \lambda_2 = \lambda_4 = 0, \lambda_3 = 1.5$	0.156116	0.542339	0.184418	0.726136
Maxwell	$\lambda_2 = \lambda_3 = \lambda_4 = 0, \lambda_1 = 1$	0.195815	0.488669	0.198766	0.322845
Oldroyd-B	$\lambda_2 = \lambda_4 = 0, \lambda_1 = 1, \lambda_3 = 1.5$	0.333288	0.338183	0.209639	0.444708
G. Burgers'	$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 1.5, \lambda_4 = 3$	0.136538	0.301306	0.186846	0.2033335

hydrodynamic second grade fluid and minimum in hydrodynamic generalized Burgers' fluid when permeability of the porous medium is much larger. In this case, u_s for Maxwell fluid is large when compared with Newtonian and Oldroyd-B fluid. For $M = 0$ and $K \neq 0$ the behavior of u_s is similar to the case of $M \neq 0$ and $K \neq 0$. However, it is found that u_s for $M = 0, K \neq 0$ is large for all fluids except Oldroyd-B fluid when compared with $M \neq 0, K \neq 0$. For $M = 1$ and $K \longrightarrow \infty$ the behavior of u_s is similar to that of $M = 1$ and $K \longrightarrow \infty$. As a result, it can be concluded that steady state velocity gives highest value for Oldroyd B fluid in all cases when compared with Newtonian or Maxwell fluid, whereas generalized Burgers' fluid shows minimum value of u_s.

7. Concluding Remarks

In current study MHD flow of generalized Burgers' fluid is mathematically modeled in a porous medium. The exact solutions for three flow problems are developed. The results for various fluids in a porous space can be obtained as a special case of the present analysis by choosing appropriate values of the involved parameters. The comparison of the steady state velocity has been shown for five different fluids. Key conclusions of this study are:

(i) In case of Oldroyd-B fluid decrease in u_s is more prominent when magnetic field is increased

(ii) Steady state velocity increases more prominently with increasing K incase of Oldroyd-B fluid in presence of magnetic field when compared with generalized Burgers' fluid

Oldroyd-B fluid shows higher steady state velocity as compared to Newtonian or Maxwell fluid for all values of M and K.

Generalized Burgers' fluid shows lower steady state velocity as compared to Newtonian or Maxwell fluid for all values of M and K.

The existing results of second grade fluid [8] can be deduced by selecting $\lambda_1 = \lambda_2 = \lambda_4 = M = \phi = 0$ and $\nu\lambda_3 = \alpha_1/\rho$. Moreover, this work can be extended in future by considering different flow geometries and boundary conditions for various non-Newtonian fluids.

Appendix

In order to obtain the equality (24) from (23) we have to use the following two integrals [36]:

$$\int_0^\infty \frac{x \sin(ax)}{\left(x^2 + \epsilon b^2\right)^2 + c^2} dx = \frac{\pi}{2c} \exp(-aA) \sin(aB),$$
$$\int_0^\infty \frac{x\left(x^2 + \epsilon b^2\right) \sin(ax)}{\left(x^2 + \epsilon b^2\right)^2 + c^2} dx = \frac{\pi}{2} \exp(-aA) \cos(aB) \tag{62}$$

where

$$2A^2 = \sqrt{b^4 + c^2} + \epsilon b^2, 2B^2 = \sqrt{b^4 + c^2} + \epsilon b^2, \tag{63}$$

and

$$\epsilon = \pm 1. \tag{64}$$

Straightforward computations show that the integrals from (23) can be written under forms

$$u_s(y,t) = \frac{2}{\pi} U_0 \frac{\nu\omega \begin{pmatrix} 1 - (\lambda_2 + \lambda_4)\omega^2 + \left(\lambda_1\lambda_3 + \lambda_2\lambda_4\omega^2\right)\omega^2 + \\ \left(\sigma B_0^2/\rho\right)\left((\lambda_1 - \lambda_3) + (\lambda_2\lambda_3 + \lambda_1\lambda_4)\omega^2\right) \end{pmatrix}}{\nu^2\left(1 - \lambda_4\omega^2\right)^2 + \nu^2\omega^2\lambda_3^2} \sin\omega t \int_0^\infty \cdot \frac{\xi \sin(a\xi)\, d\xi}{\left(\xi^2 + b^2\right)^2 + c^2} + \frac{2}{\pi} U_0 \cos\omega t \int_0^\infty \frac{\xi\left(\xi^2 + b^2\right) \sin(a\xi)\, d\xi}{\left(\xi^2 + b^2\right)^2 + c^2}, \tag{65}$$

or

$$u_s(y,t) = \frac{2}{\pi} U_0 c \sin\omega t \int_0^\infty \frac{\xi \sin(a\xi)\, d\xi}{\left(\xi^2 + b^2\right)^2 + c^2} + \frac{2}{\pi} U_0 \cos\omega t \int_0^\infty \frac{\xi\left(\xi^2 + b^2\right) \sin(a\xi)\, d\xi}{\left(\xi^2 + b^2\right)^2 + c^2}, \tag{66}$$

or

$$u_s(y,t)=\frac{2}{\pi}U_0 c\sin\omega t\frac{\pi}{2c}\exp(-Ay)\sin(yB)+\frac{2}{\pi}U_0\cos\omega t\frac{\pi}{2c}\exp(-Ay)\cos(yB), \tag{67}$$

or

$$u_s(y,t)=U_0\exp(-Ay)\cos(\omega t-yB), \tag{68}$$

where

$$\begin{aligned}
b^2&=\frac{1}{\nu(1-\lambda_4\omega^2)^2+\nu\omega^2\lambda_3^2}\begin{pmatrix}\omega^2(\lambda_3-\lambda_1)+\omega^4(\lambda_1\lambda_4-\lambda_2\lambda_3)\\ +\frac{\phi}{k}\nu\left((1-\lambda_4\omega^2)^2+\omega^2\lambda_3^2\right)\\ +\frac{\sigma B_0^2}{\rho}\left(1-(\lambda_2+\lambda_4)\omega^2+\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\omega^2\right)\end{pmatrix},\\
c^2&=\frac{\nu^2\omega^2}{\left(\nu^2(1-\lambda_4\omega^2)^2+\nu^2\omega^2\lambda_3^2\right)^2}\begin{pmatrix}1-(\lambda_2+\lambda_4)\omega^2+\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\omega^2\\ +\frac{\sigma B_0^2}{\rho}\left((\lambda_1-\lambda_3)+(\lambda_2\lambda_3+\lambda_1\lambda_4)\omega^2\right)\end{pmatrix}^2,\\
2A^2&=\sqrt{\begin{aligned}&\left(\frac{\omega^2(\lambda_3-\lambda_1)+\omega^4(\lambda_1\lambda_4-\lambda_2\lambda_3)+(\phi/k)\nu\left((1-\lambda_4\omega^2)^2+\omega^2\lambda_3^2\right)+\left(\sigma B_0^2/\rho\right)\left(1-(\lambda_2+\lambda_4)\omega^2+\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\omega^2\right)}{\nu(1-\lambda_4\omega^2)^2+\nu\omega^2\lambda_3^2}\right)^2\\ &+\nu^2\omega^2\left(\frac{\left(1-(\lambda_2+\lambda_4)\omega^2+(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2)\omega^2+\left(\sigma B_0^2/\rho\right)\left((\lambda_1-\lambda_3)+(\lambda_2\lambda_3+\lambda_1\lambda_4)\omega^2\right)\right)}{\left(\nu^2(1-\lambda_4\omega^2)^2+\nu^2\omega^2\lambda_3^2\right)}\right)^2\end{aligned}}\\
&\quad+\frac{\omega^2(\lambda_3-\lambda_1)+\omega^4(\lambda_1\lambda_4-\lambda_2\lambda_3)+(\phi/k)\nu\left((1-\lambda_4\omega^2)^2+\omega^2\lambda_3^2\right)+\left(\sigma B_0^2/\rho\right)\left(1-(\lambda_2+\lambda_4)\omega^2+\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\omega^2\right)}{\nu(1-\lambda_4\omega^2)^2+\nu\omega^2\lambda_3^2},\\
2B^2&=\sqrt{\begin{aligned}&\left(\frac{\omega^2(\lambda_3-\lambda_1)+\omega^4(\lambda_1\lambda_4-\lambda_2\lambda_3)+(\phi/k)\nu\left((1-\lambda_4\omega^2)^2+\omega^2\lambda_3^2\right)+\left(\sigma B_0^2/\rho\right)\left(1-(\lambda_2+\lambda_4)\omega^2+\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\omega^2\right)}{\nu(1-\lambda_4\omega^2)^2+\nu\omega^2\lambda_3^2}\right)^2\\ &+\nu^2\omega^2\left(\frac{\left(1-(\lambda_2+\lambda_4)\omega^2+(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2)\omega^2+\left(\sigma B_0^2/\rho\right)\left((\lambda_1-\lambda_3)+(\lambda_2\lambda_3+\lambda_1\lambda_4)\omega^2\right)\right)}{\left(\nu^2(1-\lambda_4\omega^2)^2+\nu^2\omega^2\lambda_3^2\right)}\right)^2\end{aligned}}\\
&\quad-\frac{\omega^2(\lambda_3-\lambda_1)+\omega^4(\lambda_1\lambda_4-\lambda_2\lambda_3)+(\phi/k)\nu\left((1-\lambda_4\omega^2)^2+\omega^2\lambda_3^2\right)+\left(\sigma B_0^2/\rho\right)\left(1-(\lambda_2+\lambda_4)\omega^2+\left(\lambda_1\lambda_3+\lambda_2\lambda_4\omega^2\right)\omega^2\right)}{\nu(1-\lambda_4\omega^2)^2+\nu\omega^2\lambda_3^2}.
\end{aligned} \tag{69}$$

Similar computations show us that Eq. (31) is equivalent to (32).

Nomenclature

- T: Cauchy stress tensor
- S: Extra stress tensor
- p: Pressure ($kgm^{-1}s^{-2}$)
- A: First Rivilin-Erickson tensor
- V: Velocity (ms^{-1})
- u: Velocity component (ms^{-1})
- y: Cartesian coordinate (m)
- t: Time (s)
- σ: Electric conductivity (Sm^{-1})
- V_D: Darcian velocity (ms^{-1})
- ν: Kinematic viscosity (m^2s^{-1})
- α_1: Material parameter of second grade fluid
- λ_1: Relaxation time (s)
- λ_3: Retardation time (s)

λ_2, λ_4: Material constants
ρ: Density (kgm^{-3})
J: Current density (Am^{-2})
B_0: Applied magnetic field (Am^{-1})
r: Darcy resistance
μ: Magnetic permeability (NA^{-2})
k: Permeability
ϕ: Porosity
ω: Frequency (s^{-1}).

Data Availability

All the related data is within the manuscript.

Conflicts of Interest

Authors have no conflict of interest on the publication of this article.

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