Generalizations of Fuzzy q-Ideals of BCI-Algebras

G. Muhiuddin,1 D. Al-Kadi,2 A. Mahboob,3 A. Assiry,4 and Abdullah Alsubhi1

1Department of Mathematics, Faculty of Science, University of Tabuk, P.O. Box 741, Tabuk 71491, Saudi Arabia
2Department of Mathematics and Statistic, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia
3Department of Mathematics, Madanapalle Institute of Technology & Science, Madanapalle 517325, India
4Department of Mathematical Sciences, College of Applied Science, Umm Al-Qura University, Makkah 21955, Saudi Arabia

Correspondence should be addressed to G. Muhiuddin; chishtygm@gmail.com

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In this paper, we introduce the notion of \((\epsilon, \epsilon\vee (\kappa, q_\kappa))\)-fuzzy q-ideals of BCI-algebras to propose a more general form of fuzzy q-ideals of BCI-algebras. We prove that \((\epsilon, \epsilon\vee q)\)-fuzzy q-ideals and \((\epsilon\vee (\kappa, q_\kappa), \epsilon\vee (\kappa, q_{\kappa}))\)-fuzzy q-ideals are \((\epsilon, \epsilon\vee (\kappa, q_\kappa))\)-fuzzy q-ideals, but the converse assertion is not valid and examples are given to support this. It is proved that every \((\epsilon, \epsilon\vee (\kappa, q_\kappa))\)-fuzzy q-ideal is an \((\epsilon, \epsilon\vee (\kappa, q_\kappa))\)-fuzzy ideal, but the converse need not be true in general and an example is provided. In addition, correspondence between \((\epsilon, \epsilon\vee (\kappa, q_\kappa))\)-fuzzy q-ideals and q-ideals of BCI-algebras is considered.

1. Introduction

A fuzzy set, as defined by Zadeh [1], is a powerful methodology for dealing with possibilistic complexity related to expectations, state imprecision, and preferences. Fuzzy set theory has become an essential study subject in research disciplines such as operation research, statistics, graph theory, social science, management, medical science, computer science, machine learning, multicriteria decision-making, information processing, and optimization. Imai and Iseki proposed the notions of BCK- and BCI-algebras in 1966 [2, 3]. Since then, a large number of studies have been published concerning the theory of BCK/BCI-algebras. Many authors, especially Liu et al. [4], Khalid and Ahmad [5], Jun et al. [6–8], Muhiuddin et al. [9, 10], and Al-Masarwah and Ahmad [11], studied different aspects of BCK/BCI-algebras based on ideal theory.

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, as stated in [12], was fundamental in the development of various types of fuzzy subgroups, known as \((\alpha, \beta)\)-fuzzy subgroups, as defined by Bhakat and Das in [13]. The concepts of \((\alpha, \beta)\)-fuzzy subalgebras and \((\alpha, \beta)\)-fuzzy ideals in BCK/BCI-algebras are also important and useful generalizations of fuzzy subalgebras and fuzzy ideals, which were introduced and studied by Jun [14, 15]. Zhang et al. [16] introduced the concepts of \((\epsilon, \epsilon\vee q)\)-fuzzy p-ideals, \((\epsilon, \epsilon\vee q)\)-fuzzy q-ideals, and \((\epsilon, \epsilon\vee q)\)-fuzzy a-ideals in BCI-algebras by using the idea of a quasi-coincidence of a fuzzy point with a fuzzy set in the ideal theory of BCI-algebras. Ma et al. [17] proposed and investigated the concepts of (positive implicative, implicative, and commutative) \((\epsilon, \epsilon\vee q)\)-interval-valued fuzzy ideals of BCI-algebras. Ma et al. [18] also proposed the concepts of \((\epsilon, \epsilon\vee q)\)-interval-valued fuzzy \((p\ and q)\)-a-ideals of BCI-algebras. Al-Masarwah et al. [19] proposed a new system of m-polar \((\alpha, \beta)\)-fuzzy ideals and m-polar \((\alpha, \beta)\)-fuzzy commutative ideals in BCK/BCI-algebras by extending the concept of fuzzy point to m-polar fuzzy sets. Muhiuddin et al. [20] established the concept of m-polar \((\epsilon, \epsilon)\)-fuzzy q-ideals in BCI-algebras and explored the characteristics of m-polar \((\alpha, \beta)\)-fuzzy q-ideals and m-polar \((\alpha, \beta)\)-fuzzy ideals/subalgebras. Many researchers have also extended the fuzzy set theory and related concepts to different algebras and other structures (see, for e.g., [10, 21–27]).

It is obvious to provide a generalized version of the existing fuzzy ideals of BCI-algebras. To do this, we first review some fundamental concepts from the sequel in Section 2. The notions of \((\epsilon, \epsilon\vee (\kappa, q_\kappa))\)-fuzzy q-ideals and...
(εv(κ*, qκ), εv(κ*, qκ))-fuzzy q-ideals are then introduced and associated properties are investigated in Section 3. Moreover, correspondence between (ε, εv(κ*, qκ))-fuzzy q-ideals and q-ideals of BCI-algebras is presented.

2. Preliminaries

An algebra ((a, *, 0), 0) of type (2, 0) is a BCI-algebra if

1. ((a * Θ) * (a * 0)) * (a * Θ) = 0
2. (a * (a * Θ)) * Θ = 0
3. a * 0 = 0
4. (Θ * a) * 0 = a = Θ

Any BCI-algebra A satisfies the following:

1. a * 0 = 0
2. (a * Θ) * a = (a * a) * Θ

Define an order ≤ on A as a ≤ Θ := Θ * a = 0.

Let A be a BCI-algebra. A mapping T: A → [0, 1] is called a fuzzy subset (briefly, FS) of A.

Definition 1. Let a ∈ A and δ ∈ (0, 1]. An ordered fuzzy point (briefly, OFP) aδ of A is defined as

\[ a_δ(α) = \begin{cases} δ, & \text{if } a ∈ (a], \\ 0, & \text{if } a ∉ (a], \\ \end{cases} \]

Consequently, aδ is a FS of A. For a FS T of A, we write aδ ≤ T as aδ ∈ T in the sequel. So, aδ ∈ T if aδ(T) ≥ δ.

Definition 2. A FS T of A is called an (ε, εv(κ*, qκ))-fuzzy ideal (briefly, (ε, εv(κ*, qκ))-FI) of A if qδ ∈ T and qδ ∈ T imply (q * α)δ q ∈ εv(κ*, qκ)T for all δ, ε ∈ (0, 1] and q, α ∈ A.

Lemma 1. Let T be a FS of A. Then, qδ ∈ T implies qδ ∈ εv(κ*, qκ)

\[ \forall q ∈ T, (0) ≥ T (q) \land (κ^* - κ/2). \]

Lemma 2. Let T be an (ε, εv(κ*, qκ))-fuzzy ideal of A such that q ≤ δ. Then, T (q) ≥ T (0) \land (κ^* - κ/2).

Lemma 3. Let T be an (ε, εv(κ*, qκ))-fuzzy ideal of A. Then, for any q, ω, Θ ∈ A, q * ω ≤ Θ implies T (q) ≥ T (ω) \land T (Θ) \land (κ^* - κ/2).

3. (ε, εv(κ*, qκ))-Fuzzy q-Ideals

Definition 3. Let aδ be an OFP of A and κ* ∈ (0, 1]. Then, aδ is said to be (κ*, qκ)-quasi-coincident with a FS T of A, written as aδ(κ*, qκ)T, if

\[ T (a) + δ > κ^*. \]

Let 0 ≤ k < κ* ≤ 1. For an OFP qδ, we write

1. \( q_δ(κ^*, qκ)T \) if \( T (q) + δ > κ^* \)
2. \( q_δ(κ^*, qκ)T \) if \( q_δ ∈ T \) or \( q_δ(κ^*, qκ)T \)
3. \( q_δ(κ^*, qκ)T \) if \( q_δ(κ^*, qκ)T \) does not hold for \( a ∈ (κ^*, qκ), εv(κ^*, qκ) \)

Definition 4. A FS T of A is called an (ε, εv(κ*, qκ))-fuzzy q-ideal (briefly, (ε, εv(κ*, qκ))-FQI) of A if

1. \( q_δ ∈ T \) implies \( q_δ ∈ εv(κ^*, qκ)T \)
2. \( (a * (Θ * q))δ ∈ T \) and \( Θ ∈ T \) imply \( (a * q)δεv(κ^*, qκ)T \)
3. \( εv(κ^*, qκ)T, q ∈ A \) and \( δ, ε ∈ (0, 1] \).

Example 1. Consider A = [0, 1, j, a, Θ] as a BCI-algebra under the operation (⋆) which is defined in Table 1.

Define a FS T on A as

\[ T (a) = \begin{cases} 0.6 \text{ if } a = 0, \\ 0.1 \text{ if } a ∈ [1, Θ], \\ 0.3 \text{ if } a = j, \\ 0.2 \text{ if } a = q. \end{cases} \]

Choose κ* = 0.8 and κ = 0.4. It is straightforward to show that T is an (ε, εv(κ*, qκ))-FQI of A.

Definition 5. A FS T of A is called an (ε, εvq)-fuzzy q-ideal (briefly, (ε, εvq)-FQI) of A if

1. \( q_δ ∈ T \) implies \( q_δ ∈ εvqT \)
2. \( (a * (Θ * q))δ ∈ T \) and \( Θ ∈ T \) imply \( (a * q)δεvqT \)
3. \( εvqT, q ∈ A \) and \( δ, ε ∈ (0, 1] \).

Example 2. Take a BCI-algebra A = [0, 1, j, a] with operation (⋆) which is described in Table 2.

Define a FS T on A as

\[ T (q) = \begin{cases} 0.6 \text{ if } q = 0, \\ 0.1 \text{ if } q ∈ [1, j]. \end{cases} \]

It is straightforward to show that T is an (ε, εvq)-FQI of A.

Lemma 4. In A, every (ε, εvq)-FQI is (ε, εv(κ*, qκ))-FQI.

Proof. Let T be an (ε, εvq)-FQI of A. Take any qδ ∈ T for q ∈ A and δ ∈ (0, 1]. Then, by hypothesis, qδ ∈ εvqT. It implies that \( T (0) ≥ δ \) or \( T (0) + δ ≥ 1 \), and thus \( T = T (0) + δ > κ^* \). Therefore, \( q_δ ∈ εv(κ^*, qκ)T \). Next, take any \( (a * (Θ * q))δ ∈ T \) and \( Θ ∈ T \). So, \( (a * q)δεvqT \) implies \( T (a * q) ≥ δεvq \) or \( T (a * q) + δεvq > 1 \). Therefore, \( T (a * q) + δεvq > κ^* \). Thus, \( (a * q)δεv(κ^*, qκ)T \). Hence, T is an (ε, εv(κ*, qκ))-FQI of A.

In general, the converse of Lemma 4 is not valid, as illustrated by the following example.
Example 3. Take a BCI-algebra \( \mathcal{A} = \{0, 1, j, q\} \) with the operation \((*)\) described in Table 3.

Define a FS \( \mathcal{T} \) on \( \mathcal{A} \) as

\[
\mathcal{T}(\bar{a}) = \begin{cases} 
0.7 & \text{if } \bar{a} = 0, \\
0.6 & \text{if } \bar{a} = 1, \\
0.4 & \text{if } \bar{a} \in \{j, q\}.
\end{cases}
\]

Choose \( \kappa' = 0.82 \) and \( \kappa = 0.02 \). It is straightforward to show that \( \mathcal{T} \) is an \((\epsilon', \text{ev}(\kappa', q_j))-\text{FQI}\) of \( \mathcal{A} \) but not an \((\epsilon, \text{ev}(\kappa', q_j))-\text{FQI}\) because \( q \neq 0 \) for some \( \epsilon \in \{0, 1\} \).

Definition 6. A FS \( \mathcal{T} \) of \( \mathcal{A} \) is called an \((\epsilon, \text{ev}(\kappa', q_j))-\text{fuzzy q-ideal} \) (briefly, \((\epsilon, \text{ev}(\kappa', q_j))-\text{FQI}\)) of \( \mathcal{A} \) if

1. \( \epsilon \neq 0 \) if \( \bar{a} = 0 \),
2. \( \epsilon \neq 1 \) if \( \bar{a} = 1 \),
3. \( \epsilon \neq \bar{a} \) if \( \bar{a} \in \{j, q\} \).

Consider \( \kappa' = 0.2 \) and \( \kappa = 0.1 \). It is easy to check that \( \mathcal{T} \) is an \((\epsilon, \text{ev}(\kappa', q_j))-\text{FI}\) of \( \mathcal{A} \).

Lemma 5. In \( \mathcal{A} \), every \((\epsilon, \text{ev}(\kappa', q_j))-\text{FQI}\) is \((\epsilon, \text{ev}(\kappa', q_j))-\text{FI}\).

Proof. Let \( \mathcal{T} \) be any \((\epsilon, \text{ev}(\kappa', q_j))-\text{FQI}\) of \( \mathcal{A} \). Take any \( \bar{a}, \bar{b}, \bar{c} \in \mathcal{T} \) for \( \bar{a}, \bar{b}, \bar{c} \in \mathcal{A} \) and \( \delta \in \{0, 1\} \). Then, \( \bar{a}, \bar{b}, \bar{c} \in \mathcal{A} \) and \( \delta \in \{0, 1\} \).

Example 4. Take a BCI-algebra \( \mathcal{A} = \{0, 1, j, q\} \) with operation \((*)\) which is described in Table 4.

Define \( \mathcal{T}: \mathcal{A} \rightarrow [0, 1] \) by

\[
\mathcal{T}(\bar{a}) = \begin{cases} 
0.9 & \text{if } \bar{a} = 0, \\
0.3 & \text{if } \bar{a} \in \{1, j, q\}.
\end{cases}
\]

Consider \( \kappa' = 0.2 \) and \( \kappa = 0.1 \). It is easy to check that \( \mathcal{T} \) is an \((\epsilon, \text{ev}(\kappa', q_j))-\text{FI}\) of \( \mathcal{A} \).

Lemma 5. In \( \mathcal{A} \), every \((\epsilon, \text{ev}(\kappa', q_j))-\text{FQI}\) is \((\epsilon, \text{ev}(\kappa', q_j))-\text{FI}\).

Proof. Let \( \mathcal{T} \) be any \((\epsilon, \text{ev}(\kappa', q_j))-\text{FQI}\) of \( \mathcal{A} \). Take any \( \bar{a}, \bar{b}, \bar{c} \in \mathcal{T} \) for \( \bar{a}, \bar{b}, \bar{c} \in \mathcal{A} \) and \( \delta \in \{0, 1\} \). Then, \( \bar{a}, \bar{b}, \bar{c} \in \mathcal{A} \) and \( \delta \in \{0, 1\} \).

In general, the converse of Lemma 5 is not valid, as illustrated in the following example.

Example 5. Take a BCI-algebra \( \mathcal{A} = \{0, 1, j, q, \Theta\} \) with the operation \((*)\) described in Table 5.

Define \( \mathcal{T}: \mathcal{A} \rightarrow [0, 1] \) by

\[
\mathcal{T}(\bar{a}) = \begin{cases} 
0.4 & \text{if } \bar{a} = 0, \\
0.6 & \text{if } \bar{a} \in \{1, j, q\}, \\
0.1 & \text{if } \bar{a} = \Theta.
\end{cases}
\]

Consider \( \kappa = 0 \) and \( \kappa' = 0.7 \). Then, \( \mathcal{T} \) is an \((\epsilon, \text{ev}(\kappa', q_j))-\text{FQI}\) of \( \mathcal{A} \), but it is not an \((\epsilon, \text{ev}(\kappa', q_j))-\text{FQI}\) of \( \mathcal{A} \) as

\[
\mathcal{T}(\bar{a}) = \begin{cases} 
0.4 & \text{if } \bar{a} = 0, \\
0.6 & \text{if } \bar{a} \in \{1, j, q\}, \\
0.1 & \text{if } \bar{a} = \Theta.
\end{cases}
\]

Lemma 6. Let \( \mathcal{T} \) be a FS of \( \mathcal{A} \). Then, \((\bar{a} \oplus (\Theta \oplus q_j))_\delta \in \mathcal{T} \) implies \((\bar{a} \oplus (\Theta \oplus q_j))_\delta \in \mathcal{T} \) if \( \delta \in \{0, 1\} \) and \( \Theta \in \mathcal{A} \) and \( \delta \in \{0, 1\} \).

Proof. On the contrary, suppose that \( \mathcal{T}(\bar{a} \oplus (\Theta \oplus q_j))_\delta \notin \mathcal{T} \) for some \( \bar{a}, \Theta, q_j \in \mathcal{A} \). Choose \( \delta \in \{0, 1\} \). Then, \( \mathcal{T}(\bar{a} \oplus (\Theta \oplus q_j))_\delta \notin \mathcal{T} \). Then, \( \mathcal{T}(\bar{a} \oplus (\Theta \oplus q_j))_\delta \notin \mathcal{T} \). Therefore, \( \mathcal{T}(\bar{a} \oplus (\Theta \oplus q_j))_\delta \notin \mathcal{T} \). Hence, \( \mathcal{T}(\bar{a} \oplus (\Theta \oplus q_j))_\delta \notin \mathcal{T} \).
Example 6. Take a BCI-algebra $\bar{A} = [0, 1, j, q, \Theta]$ with the operation $(\ast)$ described in Table 6.

Define a FS $\mathcal{T}$ on $\bar{A}$ as

$$\mathcal{T}(\bar{a}) = \begin{cases} 0.4 & \text{if } \bar{a} = 0, \\ 0.2 & \text{if } \bar{a} = 1, \\ 0 & \text{if } \bar{a} \in \{ j, q, \Theta \}. \end{cases}$$

Choose $\kappa = 0.1$ and $\kappa^* = 0.9$. It is straightforward to check that $\mathcal{T}$ is an $(\epsilon, \epsilon \vee (\kappa^*, q_\lambda))$-FI of $\bar{A}$, but it is not an $(\epsilon, \epsilon \vee (\kappa^*, q_\lambda))$-FQI because $0 = \mathcal{T}((\Theta \ast j) \ast \mathcal{T}(\Theta \ast (0 \ast j)) \vee \mathcal{T}(0)) \vee 0.4 = \mathcal{T}(0) = 0.4$.

Theorem 3. If $\mathcal{T}$ is an $(\epsilon, \epsilon \vee (\kappa^*, q_\lambda))$-FI of $\bar{A}$, then the following statements are equivalent:

(1) $\mathcal{T}$ is an $(\epsilon, \epsilon \vee (\kappa^*, q_\lambda))$-FQI of $\bar{A}$

(2) $\mathcal{T}(\ast) \geq \mathcal{T}(\ast \ast) \vee \mathcal{T}(\Theta) \vee (\kappa^* - \kappa)/2)$, $\forall \bar{a}, \bar{b} \in \bar{A}$

(3) $\mathcal{T}(\ast) \geq \mathcal{T}(\ast \ast) \vee \mathcal{T}(\Theta) \vee (\kappa^* - \kappa)/2)$, $\forall \bar{a}, \bar{b} \in \bar{A}$

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{T}$ be an $(\epsilon, \epsilon \vee (\kappa^*, q_\lambda))$-FQI of $\bar{A}$. Then, $\forall \bar{a}, \bar{b} \in \bar{A}$, we have

$$\mathcal{T}(\bar{a} \ast \bar{b}) \geq \mathcal{T}(\bar{a} \ast (\Theta \ast \bar{b})) \vee \mathcal{T}(\Theta) \vee (\kappa^* - \kappa)/2).$$

Replacing $\bar{a}$ by $\bar{b}$ and $\Theta$ by $\Theta$, we get

$$\mathcal{T}(\bar{b} \ast \bar{a}) \geq \mathcal{T}(\bar{b} \ast (\Theta \ast \bar{a})) \vee \mathcal{T}(\Theta) \vee (\kappa^* - \kappa)/2).$$

Substitute $\bar{b}$ by 0, to obtain

$$\mathcal{T}(\bar{0} \ast \bar{a}) \geq \mathcal{T}(\bar{0} \ast (\Theta \ast \bar{a})) \vee \mathcal{T}(\Theta) \vee (\kappa^* - \kappa)/2).$$

(2) $\Rightarrow$ (3). Let $\bar{a}, \bar{b} \in \bar{A}$. Then,

$$\mathcal{T}(\bar{a} \ast \bar{b}) \geq \mathcal{T}(\bar{a} \ast (0 \ast \bar{b})) \vee \mathcal{T}(0) \vee (\kappa^* - \kappa)/2).$$

Now, we have

$$((\bar{a} \ast \bar{b}) \ast (0 \ast \bar{b})) = ((\bar{a} \ast (\Theta \ast \bar{b})) \ast (0 \ast \bar{b}))$$

$$\leq ((\bar{a} \ast (\Theta \ast \bar{b})) \ast (0 \ast \bar{b})) = ((\Theta \ast \bar{b}) \ast (0 \ast \bar{b}))$$

$$= (0 \ast \bar{b}) \ast (0 \ast \bar{b})$$

$$= 0.$$

By Lemma 3, we have

$$\mathcal{T}(\bar{a} \ast \bar{b}) \geq \mathcal{T}(\bar{a} \ast (\Theta \ast \bar{b})) \vee \mathcal{T}(0) \vee (\kappa^* - \kappa)/2).$$

From (16) and (18), we obtain $\mathcal{T}(\bar{a} \ast \bar{b}) \geq \mathcal{T}(\Theta \ast (\Theta \ast \bar{b})) \vee (\kappa^* - \kappa)/2).$

(3) $\Rightarrow$ (1). As $\mathcal{T}$ is an $(\epsilon, \epsilon \vee (\kappa^*, q_\lambda))$-FI of $\bar{A}$, $\forall \bar{a}, \bar{b} \in \bar{A}$, we have

$$\mathcal{T}(\bar{a} \ast \bar{b}) \geq \mathcal{T}(\Theta \ast (\Theta \ast \bar{b})) \vee \mathcal{T}(0) \vee (\kappa^* - \kappa)/2).$$

$$= \mathcal{T}(\bar{a} \ast (\Theta \ast \bar{b})) \vee \mathcal{T}(0) \vee (\kappa^* - \kappa)/2).$$
Lemma 7. Let $T$ be an $(\varepsilon,v(\kappa^*,q_\delta))$-FQI of $\mathcal{A}$. Then, $T(0) \geq T(q) \wedge (\kappa^* - \kappa/2)\forall q \in A$.

Proof. Assume that $T$ is an $(\varepsilon,v(\kappa^*,q_\delta))$-FQI of $\mathcal{A}$. Then, $\forall \omega,\Theta,\varrho \in A$, we have

$$T(\omega \ast \varrho) \geq T(\omega \ast (\Theta \ast \varrho)) \wedge T(\Theta) \wedge \frac{\kappa^* - \kappa}{2}.$$  (20)

Substitute $\omega$ by 0 and $\Theta$ by $q$, and we have

$$T(0 \ast q) \geq T(0 \ast (q \ast q)) \wedge T(q) = T(q) \wedge \frac{\kappa^* - \kappa}{2}.$$  (21)

Definition 7. Let $T$ be a FS of $\mathcal{A}$. The set

$$T_{\delta} = \{q \in A | T(q) \geq \delta\},$$

where $\delta \in (0, 1]$ is called the level subset of $T$.

Theorem 4. Let $T$ be a FS of $\mathcal{A}$. Then, $T$ is an $(\varepsilon,v(\kappa^*,q_\delta))$-FQI of $\mathcal{A}$ if and only if $T$ is a q-ideal of $\mathcal{A}$, $\forall \delta \in (0, 1]$.

Proof. (⇒) Let $\delta \in (0, (\kappa^* - \kappa/2)]$ with $T_{\delta} \neq \emptyset$. From Theorem 1, we have

$$T(0) \geq T(q) \wedge \frac{\kappa^* - \kappa}{2}.$$  (23)

with $q \in T_{\delta}$. It implies that $T(0) \geq \delta \wedge (\kappa^* - \kappa/2) = \delta$. Therefore, $0 \in T_{\delta}$.

Next, assume that $(\omega \ast (\Theta \ast q)) \in T_{\delta}$ and $\Theta \in T_{\delta}$. Then, $T(\omega \ast (\Theta \ast q)) \geq \delta$ and $T(\Theta) \geq \delta$. Again, by Theorem 1, we have

$$T(\omega \ast q) \geq T(\omega \ast (\Theta \ast q)) \wedge T(\Theta) \wedge \frac{\kappa^* - \kappa}{2} \geq \delta \wedge \frac{\kappa^* - \kappa}{2} = \delta.$$  (24)

Therefore, $\omega \ast q \in T_{\delta}$. Hence, $T_{\delta}$ is a q-ideal of $\mathcal{A}$.

(⇐) Assume that $T_{\delta}$ is a q-ideal of $\mathcal{A}$, $\forall \delta \in (0, (\kappa^* - \kappa/2)]$. If $T(0) < T(q) \wedge (\kappa^* - \kappa/2)$ for some $q \in A$, then $\exists \Theta \in (0, (\kappa^* - \kappa/2)]$ such that $T(0) < \delta \leq T(q) \wedge (\kappa^* - \kappa/2)$. It follows that $q \in T_{\delta}$ but $0 \notin T_{\delta}$, a contradiction. Therefore, $T(0) \geq T(q) \wedge (\kappa^* - \kappa/2)$. Also, if $T(\omega \ast q) < T(\omega \ast (\Theta \ast q)) \wedge T(\Theta) \wedge (\kappa^* - \kappa/2)$ for some $\omega, \Theta, q \in A$, then $\exists \delta \in (0, (\kappa^* - \kappa/2)]$ such that

$$T(\omega \ast q) < \delta \leq T(\omega \ast (\Theta \ast q)) \wedge T(\Theta) \wedge \frac{\kappa^* - \kappa}{2}.$$  (25)

It follows that $(\omega \ast (\Theta \ast q)) \in T_{\delta}$ and $\Theta \in T_{\delta}$ but $\omega \ast q \notin T_{\delta}$, which is again a contradiction. Therefore, $T(\omega \ast q) \geq T(\omega \ast (\Theta \ast q)) \wedge T(\Theta) \wedge (\kappa^* - \kappa/2)$. Hence, $T$ is an $(\varepsilon,v(\kappa^*,q_\delta))$-FQI of $\mathcal{A}$. □

Definition 8. Let $T$ be a FS of $\mathcal{A}$. The set

$$[T]_\delta = \{q \in \mathcal{A} | q_\delta \varepsilon v(\kappa^*,q_\delta) \wedge T(q) \geq \delta\},$$

where $\delta \in (0, 1]$, is called an $(\varepsilon,v(\kappa^*,q_\delta))$-level subset of $T$.

Theorem 5. Let $T$ be a FS of $\mathcal{A}$. Then, $T$ is an $(\varepsilon,v(\kappa^*,q_\delta))$-FQI of $\mathcal{A}$ if and only if $T$ is a q-ideal of $\mathcal{A}$, $\forall \delta \in (0, 1]$.

Proof. (⇒) Suppose $T$ is an $(\varepsilon,v(\kappa^*,q_\delta))$-FQI of $\mathcal{A}$. Take any $q \in [T]_\delta$. Then, $q_\delta \varepsilon v(\kappa^*,q_\delta)$. So, $T(q) \geq \delta$ or $T(q) + \delta \geq \kappa^* - \kappa$. Now, by Theorem 1, we have $T(0) \geq T(q) \wedge (\kappa^* - \kappa/2)$. Thus, $T(0) \geq \delta \wedge (\kappa^* - \kappa/2)$ when $T(q) \geq \delta$. If $\delta > (\kappa^* - \kappa/2)$, then $T(0) \geq (\kappa^* - \kappa/2)$ implies $0 \in [T]_\delta$. Also, if $\delta \leq (\kappa^* - \kappa/2)$, then $T(0) \geq \delta$ implies $0 \in [T]_\delta$. Similarly, $0 \in [T]_\delta$ when $T(q) + \delta > \kappa^* - \kappa$.

Next, take any $(\omega \ast (\Theta \ast q)) \in [T]_\delta$ and $\Theta \in [T]_\delta$. Then, $(\omega \ast (\Theta \ast q)) \varepsilon v(\kappa^*,q_\delta)$ and $\Theta \varepsilon v(\kappa^*,q_\delta)$, i.e., either $T(\omega \ast (\Theta \ast q)) \geq \delta$ or $T(\omega \ast (\Theta \ast q)) + \delta > \kappa^* - \kappa$ and either $T(\Theta) \geq \delta$ or $T(\Theta) + \delta > \kappa^* - \kappa$. By assumption, $T(q) \geq T(\omega \ast (\Theta \ast q)) \wedge T(\Theta) \wedge (\kappa^* - \kappa/2)$. Thus, the following cases arise. □

Case 1. Let $T(\omega \ast (\Theta \ast q)) \geq \delta$ and $T(\Theta) \geq \delta$. If $\delta > (\kappa^* - \kappa/2)$, then

$$T(\omega \ast q) \geq T(\omega \ast (\Theta \ast q)) \wedge T(\Theta) \wedge \frac{\kappa^* - \kappa}{2} \geq \delta \wedge \frac{\kappa^* - \kappa}{2} = \frac{\kappa^* - \kappa}{2}.$$  (27)

and so, $(\omega \ast q)_\delta \in (\kappa^*,q_\delta)T$. If $\delta \leq (\kappa^* - \kappa/2)$, then

$$T(\omega \ast q) \geq T(\omega \ast (\Theta \ast q)) \wedge T(\Theta) \wedge \frac{\kappa^* - \kappa}{2} \geq \delta \wedge \frac{\kappa^* - \kappa}{2} = \delta.$$  (28)

Case 2. Let $T(\omega \ast (\Theta \ast q)) \geq \delta$ and $T(\Theta) + \delta \geq \kappa^* - \kappa$. If $\delta > (\kappa^* - \kappa/2)$, then

$$T(\omega \ast q) \geq T(\omega \ast (\Theta \ast q)) \wedge T(\Theta) \wedge \frac{\kappa^* - \kappa}{2} \geq \delta \wedge \frac{\kappa^* - \kappa}{2} = \frac{\kappa^* - \kappa}{2}.$$  (29)

i.e., $T(\omega \ast q) + \delta > \kappa^* - \kappa$, and thus, $(\omega \ast q)_\delta \in (\kappa^*,q_\delta)T$. If $\delta \leq (\kappa^* - \kappa/2)$, then
The main aim of the present paper is to introduce the concept of \( (\varepsilon, \epsilon\mathcal{V}(\kappa^*, q_\delta)) \)-FQI in BCI-algebras. We provided some equivalent conditions and different characterizations of the \( (\varepsilon, \epsilon\mathcal{V}(\kappa^*, q_\delta)) \)-FQI in terms of level subsets and \( (\epsilon\mathcal{V}(\kappa^*, q_\delta)) \)-level subsets of BCI-algebras. It has been shown that in any BCI-algebras, the \( (\varepsilon, \epsilon\mathcal{V}(\kappa^*, q_\delta)) \)-FQI is \( (\varepsilon, \epsilon\mathcal{V}(\kappa^*, q_\delta)) \)-FI but the converse does not hold and an example is provided to support this. Furthermore, relation between \( (\varepsilon, \epsilon\mathcal{V}(\kappa^*, q_\delta)) \)-FQI and \( \kappa^*-\)ideal of BCI-algebras has been considered. In this investigation, we get to the following conclusions:

(1) If we choose \( \kappa^* = 1 \) and \( k = 0 \), then \( (\varepsilon, \epsilon\mathcal{V}(\kappa^*, q_\delta)) \)-FQI reduces to the notion \( (\epsilon\mathcal{V}(\kappa^*, q_\delta)) \)-FQI of X as established in [16]

(2) If we choose \( \kappa^* = 1 \) and \( k = k \), then \( (\varepsilon, \epsilon\mathcal{V}(\kappa^*, q_\delta)) \)-fuzzy subalgebras and \( (\varepsilon, \epsilon\mathcal{V}(\kappa^*, q_\delta)) \)-FQI reduce to the concepts \( (\epsilon\mathcal{V}(\kappa^*, q_\delta)) \)-fuzzy subalgebras and \( (\epsilon, \epsilon\mathcal{V}(\kappa^*, q_\delta)) \)-FQI of X as introduced in [28]

Consequently, the notions introduced in this paper are more general than the existing notions. In future study, these ideas may be extended to other algebraic structures such as rings, hemirings, LA-semigroups, semi-hypergroups, semi-hyper-rings, BL-algebras, MTL-algebras, R0-algebras, EQ-algebras, MV-algebras, and lattice implication algebras.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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**References**


