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Research Article

Generalizations of Fuzzy q-Ideals of *BCI*-Algebras

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In this paper, we introduce the notion of $(\in, \in\vee(\kappa^*, q_\kappa))$-fuzzy q-ideals of *BCI*-algebras to propose a more general form of fuzzy q-ideals of *BCI*-algebras. We prove that $(\in, \in\vee q)$-fuzzy q-ideals and $(\in\vee(\kappa^*, q_\kappa), \in\vee(\kappa^*, q_\kappa))$-fuzzy q-ideals are $(\in, \in\vee(\kappa^*, q_\kappa))$-fuzzy q-ideals, but the converse assertion is not valid and examples are given to support this. It is proved that every $(\in, \in\vee(\kappa^*, q_\kappa))$-fuzzy q-ideal is an $(\in, \in\vee(\kappa^*, q_\kappa))$-fuzzy ideal, but the converse need not be true in general and an example is provided. In addition, correspondence between $(\in, \in\vee(\kappa^*, q_\kappa))$-fuzzy q-ideals and q-ideals of *BCI*-algebras is considered.

1. Introduction

A fuzzy set, as defined by Zadeh [1], is a powerful methodology for dealing with possibilistic complexity related to expectations, state imprecision, and preferences. Fuzzy set theory has become an essential study subject in research disciplines such as operation research, statistics, graph theory, social science, management, medical science, computer science, machine learning, multicriteria decision-making, information processing, and optimization. Imai and Iseki proposed the notions of *BCK*- and *BCI*-algebras in 1966 [2, 3]. Since then, a large number of studies have been published concerning the theory of *BCK/BCI*-algebras. Many authors, especially Liu et al. [4], Khalid and Ahmad [5], Jun et al. [6–8], Muhiuddin et al. [9, 10], and Al-Masarwah and Ahmad [11], studied different aspects of *BCK/BCI*-algebras based on ideal theory.

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, as stated in [12], was fundamental in the development of various types of fuzzy subgroups, known as (α, β)-fuzzy subgroups, as defined by Bhakat and Das in [13]. The concepts of (α, β)-fuzzy subalgebras and (α, β)-fuzzy ideals in *BCK/BCI*-algebras are also important and useful generalizations of fuzzy subalgebras and fuzzy ideals, which were introduced and studied by Jun [14, 15]. Zhang et al. [16] introduced the concepts of $(\in, \in\vee q)$-fuzzy p-ideals, $(\in, \in\vee q)$-fuzzy q-ideals, and $(\in, \in\vee q)$-fuzzy a-ideals in *BCI*-algebras by using the idea of a quasi-coincidence of a fuzzy point with a fuzzy set in the ideal theory of *BCI*-algebras. Ma et al. [17] proposed and investigated the concepts of (positive implicative, implicative, and commutative) $(\in, \in\vee q)$-interval-valued fuzzy ideals of *BCI*-algebras. Ma et al. [18] also proposed the concepts of $(\in, \in\vee q)$-interval-valued fuzzy (p and q) a-ideals of *BCI*-algebras. Al-Masarwah et al. [19] proposed a new system of m-polar (α, β)-fuzzy ideals and m-polar (α, β)-fuzzy commutative ideals in *BCK/BCI*-algebras by extending the concept of fuzzy point to m-polar fuzzy sets. Muhiuddin et al. [20] established the concept of m-polar $(\in, \in)$-fuzzy q-ideals in *BCI*-algebras and explored the characteristics of m-polar (α, β)-fuzzy q-ideals and m-polar (α, β)-fuzzy ideals/subalgebras. Many researchers have also extended the fuzzy set theory and related concepts to different algebras and other structures (see, for e.g., [10, 21–27]).

It is obvious to provide a generalized version of the existing fuzzy ideals of *BCI*-algebras. To do this, we first review some fundamental concepts from the sequel in Section 2. The notions of $(\in, \in\vee(\kappa^*, q_\kappa))$-fuzzy q-ideals and

$(\in\vee(\kappa^*,q_\kappa),\in\vee(\kappa^*,q_\kappa))$-fuzzy q-ideals are then introduced and associated properties are investigated in Section 3. Moreover, correspondence between $(\in,\in\vee(\kappa^*,q_\kappa))$-fuzzy q-ideals and q-ideals of *BCI*-algebras is presented.

2. Preliminaries

An algebra $\tilde{\mathscr{A}}=(\tilde{\mathscr{A}};*,0)$ of type $(2,0)$ is a *BCI*-algebra if

(1) $((\varpi*\Theta)*(\varpi*\varrho))*(\varrho*\Theta)=0$

(2) $(\varpi*(\varpi*\Theta))*\Theta=0$

(3) $\varpi*\varpi=0$

(4) $\varpi*\Theta=0$ and $\Theta*\varpi=0\Rightarrow\varpi=\Theta$

$\forall\varpi,\Theta,\varrho\in\tilde{\mathscr{A}}$

Any *BCI*-algebra $\tilde{\mathscr{A}}$ satisfies the following:

(1) $\varpi*0=\varpi$

(2) $(\varpi*\Theta)*\varrho=(\varpi*\varrho)*\Theta$

Define an order $\leq$ on $\tilde{\mathscr{A}}$ as $\varpi\leq\Theta\Leftrightarrow\varpi*\Theta=0$.

Let $\tilde{\mathscr{A}}$ be a *BCI*-algebra. A mapping $\mathscr{T}:\tilde{\mathscr{A}}\longrightarrow[0,1]$ is called a fuzzy subset (briefly, FS) of $\tilde{\mathscr{A}}$.

Definition 1. Let $a\in\tilde{\mathscr{A}}$ and $\delta\in(0,1]$. An *ordered fuzzy point* (briefly, OFP) a_δ of $\tilde{\mathscr{A}}$ is defined as

$$a_\delta(\varpi)=\begin{cases}\delta, & \text{if }\varpi\in(a],\\ 0, & \text{if }\varpi\notin(a],\end{cases}\quad\forall\varpi\in\tilde{\mathscr{A}}. \tag{1}$$

Consequently, a_δ is a FS of $\tilde{\mathscr{A}}$. For a FS $\mathscr{T}$ of $\tilde{\mathscr{A}}$, we write $a_\delta\subseteq\mathscr{T}$ as $a_\delta\in\mathscr{T}$ in the sequel. So, $a_\delta\in\mathscr{T}\Leftrightarrow\mathscr{T}(a)\geq\delta$.

Definition 2. A *FS* $\mathscr{T}$ of $\tilde{\mathscr{A}}$ is called an $(\in,\in\vee(\kappa^*,q_\kappa))$-fuzzy ideal (briefly, $(\in,\in\vee(\kappa^*,q_\kappa))$-FI) of $\tilde{\mathscr{A}}$ if $\varrho_\delta\in\mathscr{T}$ and $\varpi_\epsilon\in\mathscr{T}$ imply $(\varrho*\varpi)_{\delta\wedge\epsilon}\in\vee(\kappa^*,q_\kappa)\mathscr{T}$ for all $\delta,\varepsilon\in(0,1]$ and $\varrho,\varpi\in\tilde{\mathscr{A}}$.

Lemma 1. *Let $\mathscr{T}$ be a FS of $\tilde{\mathscr{A}}$. Then, $\varrho_\delta\in\mathscr{T}$ implies* $0_\delta\in\vee(\kappa^*,q_\kappa)$ $\mathscr{T}\Leftrightarrow\forall\varrho\in\tilde{\mathscr{A}},\mathscr{T}(0)\geq\mathscr{T}(\varrho)\wedge(\kappa^*-\kappa/2)$.

Lemma 2. *Let $\mathscr{T}$ be an $(\in,\in\vee(\kappa^*,q_\kappa))$-fuzzy ideal of $\tilde{\mathscr{A}}$ such that $\varrho\leq\varpi$. Then, $\mathscr{T}(\varrho)\geq\mathscr{T}(\varpi)\wedge(\kappa^*-\kappa/2)$.*

Lemma 3. *Let $\mathscr{T}$ be an $(\in,\in\vee(\kappa^*,q_\kappa))$-fuzzy ideal of $\tilde{\mathscr{A}}$. Then, for any $\varrho,\varpi,\Theta\in\tilde{\mathscr{A}}$, $\varrho*\varpi\leq\Theta\Rightarrow\mathscr{T}(\varrho)\geq\mathscr{T}(\varpi)\wedge\mathscr{T}(\Theta)\wedge(\kappa^*-\kappa/2)$.*

3. $(\in,\in\vee(\kappa^*,q_\kappa))$-Fuzzy q-Ideals

Definition 3. Let a_δ be an OFP of $\tilde{\mathscr{A}}$ and $\kappa^*\in(0,1]$. Then, a_δ is said to be (κ^*,q)*-quasi-coincident* with a FS $\mathscr{T}$ of $\tilde{\mathscr{A}}$, written as $a_\delta(\kappa^*,q)\mathscr{T}$, if

$$\mathscr{T}(a)+\delta>\kappa^*. \tag{2}$$

Let $0\leq\kappa<\kappa^*\leq1$. For an OFP ϱ_δ, we write

(1) $\varrho_\delta(\kappa^*,q_\kappa)\mathscr{T}$ if $\mathscr{T}(\varrho)+\delta+\kappa>\kappa^*$

(2) $\varrho_\delta\in\vee(\kappa^*,q_\kappa)\mathscr{T}$ if $\varrho_\delta\in\mathscr{T}$ or $\varrho_\delta(\kappa^*,q_\kappa)\mathscr{T}$

(3) $\varrho_\delta\overline{\alpha}\mathscr{T}$ if $\varrho_\delta\alpha\mathscr{T}$ does not hold for $\alpha\in\{(\kappa^*,q_\kappa),\in\vee(\kappa^*,q_\kappa)\}$

Definition 4. A *FS* $\mathscr{T}$ of $\tilde{\mathscr{A}}$ is called an $(\in,\in\vee(\kappa^*,q_\kappa))$-fuzzy q-ideal (briefly, $(\in,\in\vee(\kappa^*,q_\kappa))$-FQI) of $\tilde{\mathscr{A}}$ if

(1) $\varrho_\delta\in\mathscr{T}$ implies $0_\delta\in\vee(\kappa^*,q_\kappa)\mathscr{T}$

(2) $(\varpi*(\Theta*\varrho))_\delta\in\mathscr{T}$ and $\Theta_\epsilon\in\mathscr{T}$ imply $(\varpi*\varrho)_{\delta\wedge\varepsilon}\in\vee(\kappa^*,q_\kappa)\mathscr{T}\,\forall\varpi,\Theta,\varrho\in\tilde{\mathscr{A}}$ and $\delta,\varepsilon\in(0,1]$.

Example 1. Consider $\tilde{\mathscr{A}}=\{0,1,j,\varrho,\Theta\}$ as a *BCI*-algebra under the operation $(*)$ which is defined in Table 1.

Define a *FS* $\mathscr{T}$ on $\tilde{\mathscr{A}}$ as

$$\mathscr{T}(\varpi)=\begin{cases}0.6 & \text{if }\varpi=0,\\ 0.1 & \text{if }\varpi\in\{1,\Theta\},\\ 0.3 & \text{if }\varpi=j,\\ 0.2 & \text{if }\varpi=\varrho.\end{cases} \tag{3}$$

Choose $\kappa^*=0.8$ and $\kappa=0.4$. It is straightforward to show that $\mathscr{T}$ is an $(\in,\in\vee(\kappa^*,q_\kappa))$-FQI of $\tilde{\mathscr{A}}$.

Definition 5. A *FS* $\mathscr{T}$ of $\tilde{\mathscr{A}}$ is called an $(\in,\in\vee q)$-fuzzy q-ideal (briefly, $(\in,\in\vee q)$-FQI) of $\tilde{\mathscr{A}}$ if

(1) $\varrho_\delta\in\mathscr{T}$ implies $0_\delta\in\vee q\mathscr{T}$

(2) $(\varpi*(\Theta*\varrho))_\delta\in\mathscr{T}$ and $\Theta_\epsilon\in\mathscr{T}$ imply $(\varpi*\varrho)_{\delta\wedge\varepsilon}\in\vee q\mathscr{T},\forall\varpi,\Theta,\varrho\in\tilde{\mathscr{A}}$ and $\delta,\varepsilon\in(0,1]$.

Example 2. Take a *BCI*-algebra $\tilde{\mathscr{A}}=\{0,1,j,\varpi\}$ with operation $(*)$ which is described in Table 2.

Define a FS $\mathscr{T}$ on $\tilde{\mathscr{A}}$ as

$$\mathscr{T}(\vartheta)=\begin{cases}0.6 & \text{if }\vartheta=0,\\ 0.1 & \text{if }\vartheta\in\{1,j\}.\end{cases} \tag{4}$$

It is straightforward to check that $\mathscr{T}$ is an $(\in,\in\vee q)$-FQI of $\tilde{\mathscr{A}}$.

Lemma 4. *In $\tilde{\mathscr{A}}$, every $(\in,\in\vee q)$-FQI is $(\in,\in\vee(\kappa^*,q_\kappa))$-FQI.*

Proof. Let $\mathscr{T}$ be an $(\in,\in\vee q)$-FQI of $\tilde{\mathscr{A}}$. Take any $\varrho_\delta\in\mathscr{T}$ for $\varrho\in\tilde{\mathscr{A}}$ and $\delta\in(0,1]$. Then, by hypothesis, $0_\delta\in\vee q\mathscr{T}$. It implies that $\mathscr{T}(0)\geq\delta$ or $\mathscr{T}(0)+\delta\geq1$, and thus, $\mathscr{T}(0)\geq\delta$ or $\mathscr{T}(0)+\kappa+\delta>\kappa^*$. Therefore, $0_\delta\in\vee(\kappa^*,q_\kappa)\mathscr{T}$. Next, take any $(\varpi*(\Theta*\varrho))_\delta\in\mathscr{T}$ and $\Theta_\epsilon\in\mathscr{T}$. So, $(\varpi*\varrho)_{\delta\wedge\epsilon}\in\vee q\mathscr{T}$ implies $\mathscr{T}(\varpi*\varrho)\geq\delta\wedge\varepsilon$ or $\mathscr{T}(\varpi*\varrho)+\delta\wedge\varepsilon>1$. Therefore, $\mathscr{T}(\varpi*\varrho)\geq\delta\wedge\varepsilon$ or $\mathscr{T}(\varpi*\varrho)+\kappa+\delta\wedge\varepsilon>\kappa^*$. Thus, $(\varpi*\varrho)_{\delta\wedge\epsilon}\in\vee(\kappa^*,q_\kappa)\mathscr{T}$. Hence, $\mathscr{T}$ is an $(\in,\in\vee(\kappa^*,q_\kappa))$-FQI of $\tilde{\mathscr{A}}$. □

In general, the converse of Lemma 4 is not valid, as illustrated by the following example.

Table 1: Cayley table of the binary operation *.

$*$	0	1	j	ϱ	Θ
0	0	0	0	ϱ	ϱ
1	1	0	1	Θ	ϱ
j	j	j	0	ϱ	ϱ
ϱ	ϱ	ϱ	ϱ	0	0
Θ	Θ	ϱ	Θ	1	0

Table 2: Cayley table of the binary operation *.

$*$	0	1	j
0	0	0	j
1	1	0	j
j	j	j	0

Example 3. Take a *BCI*-algebra $\widetilde{\mathscr{A}} = \{0, 1, j, \varrho\}$ with the operation $(*)$ described in Table 3.

Define a *FS* $\mathscr{T}$ on $\widetilde{\mathscr{A}}$ as

$$\mathscr{T}(\varpi) = \begin{cases} 0.7 & \text{if } \varpi = 0, \\ 0.6 & \text{if } \varpi = 1, \\ 0.4 & \text{if } \varpi \in \{j, \varrho\}. \end{cases} \quad (5)$$

Choose $\kappa^* = 0.82$ and $\kappa = 0.02$. It is straightforward to show that $\mathscr{T}$ is an $(\in, \in\vee(\kappa^*, q_\kappa))$-FQI of $\widetilde{\mathscr{A}}$ but not an $(\in, \in\vee q)$-FQI because $(\varrho * (0 * 1))_{\delta=0.5} \in \mathscr{T}$ and $0_{\varepsilon=0.5} \in \mathscr{T}$ but $(\varrho * 1)_{\delta\wedge\varepsilon=0.5}\overline{\in\vee q}\mathscr{T}$.

Definition 6. A *FS* $\mathscr{T}$ of $\widetilde{\mathscr{A}}$ is called an $(\in\vee(\kappa^*, q_\kappa), \in\vee(\kappa^*, q_\kappa))$-fuzzy q-ideal (briefly, $(\in\vee(\kappa^*, q_\kappa), \in\vee(\kappa^*, q_\kappa))$-*FQI*) of $\widetilde{\mathscr{A}}$ if

(1) $\varrho_\delta \in\vee(\kappa^*, q_\kappa)\mathscr{T}$ implies $0_\delta \in\vee(\kappa^*, q_\kappa)\mathscr{T}$

(2) $(\varpi * (\Theta * \varrho))_\delta \in\vee(\kappa^*, q_\kappa)\mathscr{T}$ and $\Theta_\epsilon \in\vee(\kappa^*, q_\kappa)\mathscr{T}$ imply $(\varpi * \varrho)_{\delta\wedge\varepsilon} \in\vee(\kappa^*, q_\kappa)\mathscr{T}$ $\forall \varpi, \Theta, \varrho \in \widetilde{\mathscr{A}}$ and $\delta, \varepsilon \in (0, 1]$

Example 4. Take a *BCI*-algebra $\widetilde{\mathscr{A}} = \{0, 1, j, \varrho\}$ with operation $(*)$ which is described in Table 4.

Define $\mathscr{T}\colon \widetilde{\mathscr{A}} \longrightarrow [0, 1]$ by

$$\mathscr{T}(\varpi) = \begin{cases} 0.9, & \text{if } \varpi = 0, \\ 0.3, & \text{if } \varpi \in \{1, j, \varrho\}. \end{cases} \quad (6)$$

Consider $\kappa^* = 0.2$ and $\kappa = 0.1$. It is easy to check that $\mathscr{T}$ is an $(\in, \in\vee(\kappa^*, q_\kappa))$-FI of $\widetilde{\mathscr{A}}$.

Lemma 5. *In $\widetilde{\mathscr{A}}$, every $(\in\vee(\kappa^*, q_\kappa), \in\vee(\kappa^*, q_\kappa))$-FQI is $(\in, \in\vee(\kappa^*, q_\kappa))$-FQI.*

Proof. Let $\mathscr{T}$ be any $(\in\vee(\kappa^*, q_\kappa), \in\vee(\kappa^*, q_\kappa))$-FQI of $\widetilde{\mathscr{A}}$. Take any $\varrho_\delta \in \mathscr{T}$ for $\varrho \in \widetilde{\mathscr{A}}$ and $\delta \in (0, 1]$. Then, $\varrho_\delta \in\vee(\kappa^*, q_\kappa)\mathscr{T}$. So, by hypothesis, $0_\delta \in\vee(\kappa^*, q_\kappa)\mathscr{T}$. Suppose that $(\varpi * (\Theta * \varrho))_\delta \in \mathscr{T}$ and $\Theta_\epsilon \in \mathscr{T}$. Then, $(\varpi * (\Theta * \varrho))_\delta \in\vee(\kappa^*, q_\kappa)\mathscr{T}$ and $\Theta_\epsilon \in\vee(\kappa^*, q_\kappa)\mathscr{T}$. Therefore, by hypothesis, $(\varpi * \varrho)_{\delta\wedge\varepsilon} \in\vee(\kappa^*, q_\kappa)\mathscr{T}$. Hence, $\mathscr{T}$ is an $(\in, \in\vee(\kappa^*, q_\kappa))$-FQI of $\widetilde{\mathscr{A}}$. □

Table 3: Cayley table of the binary operation *.

$*$	0	1	j	ϱ
0	0	ϱ	j	1
1	1	0	ϱ	j
j	j	1	0	ϱ
ϱ	ϱ	j	1	0

Table 4: Cayley table of the binary operation *.

$*$	0	1	j	ϱ
0	0	0	0	ϱ
1	1	0	0	ϱ
j	j	j	0	ϱ
ϱ	ϱ	ϱ	ϱ	0

In general, the converse of Lemma 5 is not valid, as illustrated in the following example.

Example 5. Take a *BCI*-algebra $\widetilde{\mathscr{A}} = \{0, 1, j, \varrho, \Theta\}$ with the operation $(*)$ described in Table 5.

Define $\mathscr{T}\colon \widetilde{\mathscr{A}} \longrightarrow [0, 1]$ by

$$\mathscr{T}(\varpi) = \begin{cases} 0.4, & \text{if } \varpi = 0, \\ 0.6, & \text{if } \varpi \in \{1, \varrho\}, \\ 0.1, & \text{if } \varpi = \{j, \Theta\}. \end{cases} \quad (7)$$

Consider $\kappa = 0$ and $\kappa^* = 0.7$. Then, $\mathscr{T}$ is an $(\in, \in\vee(\kappa^*, q_\kappa))$-FQI of $\widetilde{\mathscr{A}}$, but it is not an $(\in\vee(\kappa^*, q_\kappa), \in\vee(\kappa^*, q_\kappa))$-FQI of $\widetilde{\mathscr{A}}$ as $j_{\delta=0.95} = (j * (0 * 1))_{\delta=0.95} \in\vee(\kappa^*, q_\kappa)\mathscr{T}$ and $0_{\varepsilon=0.5} \in\vee(\kappa^*, q_\kappa)\mathscr{T}$, but $j = (j * 1)_{\delta\wedge\varepsilon=0.5}\overline{\in\vee(\kappa^*, q_\kappa)}\mathscr{T}$.

Lemma 6. *Let $\mathscr{T}$ be a FS of $\widetilde{\mathscr{A}}$. Then, $(\varpi * (\Theta * \varrho))_\delta \in \mathscr{T}$ and $\Theta_\epsilon \in \mathscr{T}$ imply $(\varpi * \varrho)_{\delta\wedge\varepsilon} \in\vee(\kappa^*, q_\kappa)\mathscr{T} \Leftrightarrow \mathscr{T}(\varpi * \varrho) \geq \mathscr{T}(\varpi * (\Theta * \varrho)) \wedge \mathscr{T}(\Theta) \wedge (\kappa^* - \kappa/2)$.*

Proof. $(\Rightarrow)$ On the contrary, suppose that $\mathscr{T}(\varrho) < \mathscr{T}(\varpi * (\Theta * \varrho)) \wedge \mathscr{T}(\Theta) \wedge (\kappa^* - \kappa/2)$ for some $\varpi, \Theta, \varrho \in \widetilde{\mathscr{A}}$. Choose $\delta \in (0, (\kappa^* - \kappa/2)]$ such that $\mathscr{T}(\varrho) < \delta \leq \mathscr{T}(\varpi * (\Theta * \varrho)) \wedge \mathscr{T}(\Theta) \wedge (\kappa^* - \kappa/2)$. Then, $(\varpi * (\Theta * \varrho))_\delta \in \mathscr{T}$ and $\Theta_\delta \in \mathscr{T}$, but $\varrho_\delta \overline{\in\vee(\kappa^*, q_\kappa)}\mathscr{T}$, which is not possible. Thus, $\mathscr{T}(\varpi * \varrho) \geq \mathscr{T}(\varpi * (\Theta * \varrho)) \wedge \mathscr{T}(\Theta) \wedge (\kappa^* - \kappa/2)$.

$(\Leftarrow)$ Let $(\varpi * (\Theta * \varrho))_\delta \in \mathscr{T}$ and $\Theta_\varepsilon \in \mathscr{T}$, $\forall \delta, \varepsilon \in (0, 1]$. Then, $\mathscr{T}(\varpi * (\Theta * \varrho)) \geq \delta$ and $\mathscr{T}(\Theta) \geq \varepsilon$. Thus,

$$\begin{aligned} \mathscr{T}(\varpi * \varrho) &\geq \mathscr{T}(\varpi * (\Theta * \varrho)) \wedge \mathscr{T}(\Theta) \wedge \frac{\kappa^* - \kappa}{2} \\ &\geq \delta \wedge \varepsilon \wedge \frac{\kappa^* - \kappa}{2}. \end{aligned} \quad (8)$$

Now, if $\delta\wedge\varepsilon \leq (\kappa^* - \kappa/2)$, then $\mathscr{T}(\varpi * \varrho) \geq \delta\wedge\varepsilon$; hence, $(\varpi * \varrho)_{\delta\wedge\varepsilon} \in \mathscr{T}$; otherwise, i.e., when $\delta\wedge\varepsilon > (\kappa^* - \kappa/2)$, $\mathscr{T}(\varpi * \varrho) \geq (\kappa^* - \kappa/2)$. So, we have

Table 5: Cayley table of the binary operation *.

$*$	0	1	j	ϱ	Θ
0	0	0	0	0	0
1	1	0	1	0	1
j	j	j	0	j	0
ϱ	ϱ	1	ϱ	0	ϱ
Θ	Θ	Θ	j	Θ	0

$$\mathcal{T}(\varpi * \varrho) + \delta\wedge\varepsilon > \frac{\kappa^* - \kappa}{2} + \frac{\kappa^* - \kappa}{2} = \kappa^* - \kappa. \tag{9}$$

This implies that $(\varpi * \varrho)_{\delta\wedge\varepsilon}(\kappa^*, q_\kappa)\mathcal{T}$. Hence, $(\varpi * \varrho)_{\delta\wedge\varepsilon} \in\vee(\kappa^*, q_\kappa)\mathcal{T}$, as required. □

By combining Lemma 1 and Lemma 6, we get the following theorem.

Theorem 1. *A FS $\mathcal{T}$ of $\tilde{\mathcal{A}}$ is an $(\in, \in\vee(\kappa^*, q_\kappa))$-FQI of $\tilde{\mathcal{A}} \Leftrightarrow$*

(1) $\mathcal{T}(0) \geq \mathcal{T}(\varrho)\wedge(\kappa^* - \kappa/2)$

(2) $\mathcal{T}(\varpi * \varrho) \geq \mathcal{T}(\varpi * (\Theta * \varrho))\wedge\mathcal{T}(\Theta)\wedge(\kappa^* - \kappa/2)$, $\forall\varpi, \Theta, \varrho \in \tilde{\mathcal{A}}$

Theorem 2. *Every $(\in, \in\vee(\kappa^*, q_\kappa))$-FQI of $\tilde{\mathcal{A}}$ is an $(\in, \in\vee(\kappa^*, q_\kappa))$-FI.*

Proof. Let $\mathcal{T}$ be an $(\in, \in\vee(\kappa^*, q_\kappa))$-FQI of $\tilde{\mathcal{A}}$. Then, $\forall\,\varpi, \Theta, \varrho \in \tilde{\mathcal{A}}$, we have

$$\mathcal{T}(\varpi * \varrho) \geq \mathcal{T}(\varpi * (\Theta * \varrho))\wedge\mathcal{T}(\Theta)\wedge\frac{\kappa^* - \kappa}{2}. \tag{10}$$

Substitute ϱ by 0, to obtain

$$\mathcal{T}(\varpi * 0) \geq \mathcal{T}(\varpi * (\Theta * 0))\wedge\mathcal{T}(\Theta)\wedge\frac{\kappa^* - \kappa}{2}. \tag{11}$$

Thus, $\mathcal{T}(\varpi) \geq \mathcal{T}(\varpi * \Theta)\wedge\mathcal{T}(\Theta)\wedge(\kappa^* - \kappa/2)$, as required. □

In general, the converse of Theorem 2 is not valid, as illustrated in the following example.

Example 6. Take a *BCI*-algebra $\tilde{\mathcal{A}} = \{0, 1, j, \varrho, \Theta\}$ with the operation $(*)$ described in Table 6.

Define a *FS* $\mathcal{T}$ on $\tilde{\mathcal{A}}$ as

$$\mathcal{T}(\varpi) = \begin{cases} 0.4 & \text{if } \varpi = 0, \\ 0.2 & \text{if } \varpi = 1, \\ 0 & \text{if } \varpi \in \{j, \varrho, \Theta\}. \end{cases} \tag{12}$$

Choose $\kappa = 0.1$ and $\kappa^* = 0.9$. It is straightforward to check that $\mathcal{T}$ is an $(\in, \in\vee(\kappa^*, q_\kappa))$-FI of $\tilde{\mathcal{A}}$, but it is not an $(\in, \in\vee(\kappa^*, q_\kappa))$-FQI because $0 = \mathcal{T}(\Theta * j) \ngeq \mathcal{T}(\Theta * (0 * j))\wedge\mathcal{T}(0)\wedge 0.4 = \mathcal{T}(0) = 0.4$.

Theorem 3. *If $\mathcal{T}$ is an $(\in, \in\vee(\kappa^*, q_\kappa))$-FI of $\tilde{\mathcal{A}}$, then the following statements are equivalent:*

Table 6: Cayley table of the binary operation *.

$*$	0	1	j	ϱ	Θ
0	0	0	Θ	ϱ	j
1	1	0	Θ	ϱ	j
j	j	j	0	Θ	ϱ
ϱ	ϱ	ϱ	j	0	Θ
Θ	Θ	Θ	ϱ	j	0

(1) $\mathcal{T}$ is an $(\in, \in\vee(\kappa^*, q_\kappa))$-FQI of $\tilde{\mathcal{A}}$

(2) $\mathcal{T}(\varpi * \Theta) \geq \mathcal{T}(\varpi * (0 * \Theta))\wedge(\kappa^* - \kappa/2)$, $\forall\varpi, \Theta \in \tilde{\mathcal{A}}$

(3) $\mathcal{T}((\varpi * \Theta) * \varrho) \geq \mathcal{T}(\varpi * (\Theta * \varrho))\wedge(\kappa^* - \kappa/2)$, $\forall\varpi, \Theta, \varrho \in \tilde{\mathcal{A}}$

Proof. (1)⇒(2). Let $\mathcal{T}$ be an $(\in, \in\vee(\kappa^*, q_\kappa))$-FQI of $\tilde{\mathcal{A}}$. Then, $\forall\,\varpi, \Theta, \varrho \in \tilde{\mathcal{A}}$, we have

$$\mathcal{T}(\varpi * \varrho) \geq \mathcal{T}(\varpi * (\Theta * \varrho))\wedge\mathcal{T}(\Theta)\wedge(\kappa^* - \kappa/2). \tag{13}$$

Replacing ϱ by Θ and Θ by ϱ, we get

$$\mathcal{T}(\varpi * \Theta) \geq \mathcal{T}(\varpi * (\varrho * \Theta))\wedge\mathcal{T}(\varrho)\wedge\frac{\kappa^* - \kappa}{2}. \tag{14}$$

Substitute ϱ by 0, to obtain

$$\mathcal{T}(\varpi * \Theta) \geq \mathcal{T}(\varpi * (0 * \Theta))\wedge\mathcal{T}(0) = \mathcal{T}(\varpi * (0 * \Theta))\wedge\frac{\kappa^* - \kappa}{2}. \tag{15}$$

(2)⇒(3). Let $\varpi, \Theta, \varrho \in \tilde{\mathcal{A}}$. Then,

$$\mathcal{T}((\varpi * \Theta) * \varrho) \geq \mathcal{T}((\varpi * \Theta) * (0 * \varrho))\wedge\frac{\kappa^* - \kappa}{2}. \tag{16}$$

Now, we have

$$\begin{aligned} &((\varpi * \Theta) * (0 * \varrho)) * (\varpi * (\Theta * \varrho)) \\ &= ((\varpi * \Theta) * (\varpi * (\Theta * \varrho))) * (0 * \varrho) \\ &\leq ((\Theta * \varrho) * \Theta) * (0 * \varrho) \\ &= ((\Theta * \Theta) * \varrho) * (0 * \varrho) \\ &= (0 * \varrho) * (0 * \varrho) \\ &= 0. \end{aligned} \tag{17}$$

By Lemma 3, we have

$$\mathcal{T}((\varpi * \Theta) * (0 * \varrho)) \geq \mathcal{T}(\varpi * (\Theta * \varrho))\wedge\frac{\kappa^* - \kappa}{2}. \tag{18}$$

From (16) and (18), we obtain $\mathcal{T}((\varpi * \Theta) * \varrho) \geq \mathcal{T}(\varpi * (\Theta * \varrho))\wedge(\kappa^* - \kappa/2)$.

(3)⇒(1). As $\mathcal{T}$ is an $(\in, \in\vee(\kappa^*, q_\kappa))$-FI of $\tilde{\mathcal{A}}$, $\forall\,\varpi, \Theta, \varrho \in \tilde{\mathcal{A}}$, we have

$$\begin{aligned} \mathcal{T}(\varpi * \varrho) &\geq \mathcal{T}((\varpi * \varrho) * \Theta)\wedge\mathcal{T}(\Theta)\wedge\frac{\kappa^* - \kappa}{2} \\ &\geq \mathcal{T}(\varpi * (\Theta * \varrho))\wedge\mathcal{T}(\Theta)\wedge\frac{\kappa^* - \kappa}{2}. \end{aligned} \tag{19}$$

□

Lemma 7. *Let $\mathscr{T}$ be an $(\in, \in\vee(\kappa^*, q_\kappa))$-FQI of $\widetilde{\mathscr{A}}$. Then, $\mathscr{T}(0*\varrho) \geq \mathscr{T}(\varrho)\wedge(\kappa^*-\kappa/2)\forall\varrho \in \widetilde{\mathscr{A}}$.*

Proof. Assume that $\mathscr{T}$ is an $(\in, \in\vee(\kappa^*, q_\kappa))$-FQI of $\widetilde{\mathscr{A}}$. Then, $\forall\, \varpi, \Theta, \varrho \in \widetilde{\mathscr{A}}$, we have

$$\mathscr{T}(\varpi*\varrho) \geq \mathscr{T}(\varpi*(\Theta*\varrho))\wedge\mathscr{T}(\Theta)\wedge\frac{\kappa^*-\kappa}{2}. \quad (20)$$

Substitute ϖ by 0 and Θ by ϱ, and we have

$$\mathscr{T}(0*\varrho) \geq \mathscr{T}(0*(\varrho*\varrho))\wedge\mathscr{T}(\varrho) = \mathscr{T}(\varrho)\wedge\frac{\kappa^*-\kappa}{2}. \quad (21)$$

□

Definition 7. Let $\mathscr{T}$ be a FS of $\widetilde{\mathscr{A}}$. The set

$$\mathscr{T}_\delta = \left\{\varrho \in \widetilde{\mathscr{A}} \mid \mathscr{T}(\varrho) \geq \delta\right\}, \text{ where } \delta \in (0,1], \quad (22)$$

is called the *level subset* of $\mathscr{T}$.

Theorem 4. *Let $\mathscr{T}$ be a FS of $\widetilde{\mathscr{A}}$. Then, $\mathscr{T}$ is an $(\in, \in\vee(\kappa^*, q_\kappa))$-FQI of $\widetilde{\mathscr{A}} \Leftrightarrow$ the set $\mathscr{T}_\delta\,(\neq\emptyset)$ is a q-ideal of $\widetilde{\mathscr{A}}$, $\forall\delta \in (0, (\kappa^*-\kappa/2)]$.*

Proof. $(\Rightarrow)$ Let $\delta \in (0, (\kappa^*-\kappa/2)]$ with $\mathscr{T}_\delta \neq \emptyset$. From Theorem 1, we have

$$\mathscr{T}(0) \geq \mathscr{T}(\varrho)\wedge\frac{\kappa^*-\kappa}{2}, \quad (23)$$

with $\varrho \in \mathscr{T}_\delta$. It implies that $\mathscr{T}(0) \geq \delta\wedge(\kappa^*-\kappa/2) = \delta$. Therefore, $0 \in \mathscr{T}_\delta$.

Next, assume that $(\varpi*(\Theta*\varrho)) \in \mathscr{T}_\delta$ and $\Theta \in \mathscr{T}_\delta$. Then, $\mathscr{T}(\varpi*(\Theta*\varrho)) \geq \delta$ and $\mathscr{T}(\Theta) \geq \delta$. Again, by Theorem 1, we have

$$\begin{aligned}\mathscr{T}(\varpi*\varrho) &\geq \mathscr{T}(\varpi*(\Theta*\varrho))\wedge\mathscr{T}(\Theta)\wedge\frac{\kappa^*-\kappa}{2}\\ &\geq \delta\wedge\frac{\kappa^*-\kappa}{2} = \delta.\end{aligned} \quad (24)$$

Therefore, $\varpi*\varrho \in \mathscr{T}_\delta$. Hence, $\mathscr{T}_\delta$ is a q-ideal of $\widetilde{\mathscr{A}}$.

$(\Leftarrow)$ Assume that $\mathscr{T}_\delta$ is q-ideal of $\widetilde{\mathscr{A}}$, $\forall\delta \in (0, (\kappa^*-k/2)]$. If $\mathscr{T}(0) < \mathscr{T}(\varrho)\wedge(\kappa^*-\kappa/2)$ for some $\varrho \in \widetilde{\mathscr{A}}$, then $\exists\delta \in (0, (\kappa^*-k/2)]$ such that $\mathscr{T}(0) < \delta \leq \mathscr{T}(\varrho)\wedge(\kappa^*-\kappa/2)$. It follows that $\varrho \in \mathscr{T}_\delta$ but $0 \notin \mathscr{T}_\delta$, a contradiction. Therefore, $\mathscr{T}(0) \geq \mathscr{T}(\varrho)\wedge(\kappa^*-\kappa/2)$. Also, if $\mathscr{T}(\varpi*\varrho) < \mathscr{T}(\varpi*(\Theta*\varrho))\wedge\mathscr{T}(\Theta)\wedge(\kappa^*-\kappa/2)$ for some $\varpi, \Theta, \varrho \in \widetilde{\mathscr{A}}$, then $\exists\delta \in (0, (\kappa^*-k/2)]$ such that

$$\mathscr{T}(\varpi*\varrho) < \delta \leq \mathscr{T}(\varpi*(\Theta*\varrho))\wedge\mathscr{T}(\Theta)\wedge\frac{\kappa^*-\kappa}{2}. \quad (25)$$

It follows that $(\varpi*(\Theta*\varrho)) \in \mathscr{T}_\delta$ and $\Theta \in \mathscr{T}_\delta$ but $\varpi*\varrho \notin \mathscr{T}_\delta$, which is again a contradiction. Therefore, $\mathscr{T}(\varpi*\varrho) \geq \mathscr{T}(\varpi*(\Theta*\varrho))\wedge\mathscr{T}(\Theta)\wedge(\kappa^*-\kappa/2)$. Hence, $\mathscr{T}$ is an $(\in, \in\vee(\kappa^*, q_\kappa))$-FQI of $\widetilde{\mathscr{A}}$. □

Definition 8. Let $\mathscr{T}$ be a FS of $\widetilde{\mathscr{A}}$. The set

$$\widetilde{[\mathscr{T}]}_\delta = \left\{\varrho \in \widetilde{\mathscr{A}} \mid \varrho_\delta \in\vee(\kappa^*, q_\kappa)\mathscr{T}\right\}, \text{ where } \delta \in (0,1], \quad (26)$$

is called an $(\in\vee(\kappa^*, q_\kappa))$*-level subset* of $\mathscr{T}$.

Theorem 5. *Let $\mathscr{T}$ be a FS of $\widetilde{\mathscr{A}}$. Then, $\mathscr{T}$ is an $(\in, \in\vee(\kappa^*, q_\kappa))$-FQI of $\widetilde{\mathscr{A}} \Leftrightarrow$ the $(\in\vee(\kappa^*, q_\kappa))$-level subset $\widetilde{[\mathscr{T}]}_\delta$ of $\mathscr{T}$ is a q-ideal of $\widetilde{\mathscr{A}}$, $\forall\delta \in (0,1]$.*

Proof. $(\Rightarrow)$ Suppose $\mathscr{T}$ is an $(\in, \in\vee(\kappa^*, q_\kappa))$-FQI of $\widetilde{\mathscr{A}}$. Take any $\varrho \in \widetilde{[\mathscr{T}]}_\delta$. Then, $\varrho_\delta \in\vee(\kappa^*, q_\kappa)\mathscr{T}$. So, $\mathscr{T}(\varrho) \geq \delta$ or $\mathscr{T}(\varrho) + \delta > \kappa^*-\kappa$. Now, by Theorem 1, we have $\mathscr{T}(0) \geq \mathscr{T}(\varrho)\wedge(\kappa^*-\kappa/2)$. Thus, $\mathscr{T}(0) \geq \delta\wedge(\kappa^*-\kappa/2)$ when $\mathscr{T}(\varrho) \geq \delta$. If $\delta > (\kappa^*-\kappa/2)$, then $\mathscr{T}(0) \geq (\kappa^*-\kappa/2)$ implies $0 \in \widetilde{[\mathscr{T}]}_\delta$. Also, if $\delta \leq (\kappa^*-\kappa/2)$, then $\mathscr{T}(0) \geq \delta$ implies $0 \in \widetilde{[\mathscr{T}]}_\delta$. Similarly, $0 \in \widetilde{[\mathscr{T}]}_\delta$ when $\mathscr{T}(\varrho) + \delta > \kappa^*-\kappa$.

Next, take any $(\varpi*(\Theta*\varrho)) \in \widetilde{[\mathscr{T}]}_\delta$ and $\Theta \in \widetilde{[\mathscr{T}]}_\delta$. Then, $(\varpi*(\Theta*\varrho)) \in\vee(\kappa^*, q_\kappa)\mathscr{T}$ and $\Theta \in\vee(\kappa^*, q_\kappa)\mathscr{T}$, i.e., either $\mathscr{T}(\varpi*(\Theta*\varrho)) \geq \delta$ or $\mathscr{T}(\varpi*(\Theta*\varrho)) + \delta > \kappa^*-\kappa$ and either $\mathscr{T}(\Theta) \geq \delta$ or $\mathscr{T}(\Theta) + \delta > \kappa^*-\kappa$. By assumption, $\mathscr{T}(\varrho) \geq \mathscr{T}(\varpi*(\Theta*\varrho))\wedge\mathscr{T}(\Theta)\wedge(\kappa^*-\kappa/2)$. Thus, the following cases arise. □

Case 1. Let $\mathscr{T}(\varpi*(\Theta*\varrho)) \geq \delta$ and $\mathscr{T}(\Theta) \geq \delta$. If $\delta > (\kappa^*-\kappa/2)$, then

$$\begin{aligned}\mathscr{T}(\varpi*\varrho) &\geq \mathscr{T}(\varpi*(\Theta*\varrho))\wedge\mathscr{T}(\Theta)\wedge\frac{\kappa^*-\kappa}{2}\\ &\geq \delta\wedge\frac{\kappa^*-\kappa}{2}\\ &= \frac{\kappa^*-\kappa}{2},\end{aligned} \quad (27)$$

and so, $(\varpi*\varrho)_\delta \in (\kappa^*, q_\kappa)\mathscr{T}$. If $\delta \leq (\kappa^*-\kappa/2)$, then

$$\begin{aligned}\mathscr{T}(\varpi*\varrho) &\geq \mathscr{T}(\varpi*(\Theta*\varrho))\wedge\mathscr{T}(\Theta)\wedge\frac{\kappa^*-\kappa}{2}\\ &\geq \delta\wedge\frac{\kappa^*-\kappa}{2}\\ &= \delta.\end{aligned} \quad (28)$$

So, $(\varpi*\varrho)_\delta \in \mathscr{T}$. Hence, $\varrho_\delta \in\vee(\kappa^*, q_\kappa)\mathscr{T}$.

Case 2. Let $\mathscr{T}(\varpi*(\Theta*\varrho)) \geq \delta$ and $\mathscr{T}(\Theta) + \delta \geq \kappa^*-\kappa$. If $\delta > (\kappa^*-\kappa/2)$, then

$$\begin{aligned}\mathscr{T}(\varpi*\varrho) &\geq \mathscr{T}(\varpi*(\Theta*\varrho))\wedge\mathscr{T}(\Theta), \frac{\kappa^*-\kappa}{2}\\ &\geq \delta\wedge\kappa^*-\kappa-\delta\wedge\frac{\kappa^*-\kappa}{2}\\ &= \kappa^*-\kappa-\delta,\end{aligned} \quad (29)$$

i.e., $\mathscr{T}(\varpi*\varrho) + \delta > \kappa^*-\kappa$, and thus, $(\varpi*\varrho)_\delta \in (\kappa^*, q_\kappa)\mathscr{T}$. If $\delta \leq (\kappa^*-\kappa/2)$, then

$$\mathcal{T}(\varpi * \varrho) \geq \mathcal{T}(\varpi * (\Theta * \varrho)) \wedge \mathcal{T}(\Theta) \wedge \frac{\kappa^* - \kappa}{2}$$
$$\geq \delta \wedge \kappa^* - \kappa - \delta \wedge \frac{\kappa^* - \kappa}{2} = \delta, \tag{30}$$

and so, $(\varpi * \varrho)_\delta \in \mathcal{T}$. Hence, $(\varpi * \varrho)_\delta \in\vee (\kappa^*, q_\kappa)\mathcal{T}$.

Similarly, for other cases, i.e., when $\mathcal{T}(\varpi * (\Theta * \varrho)) + \delta > \kappa^* - \kappa$, $\mathcal{T}(\Theta) \geq \delta$, $\mathcal{T}(\varpi * (\Theta * \varrho)) + \delta > \kappa^* - \kappa$, and $\mathcal{T}(\Theta) + \delta > \kappa^* - \kappa$, we have $(\varpi * \varrho)_\delta \in\vee (\kappa^*, q_\kappa)\mathcal{T}$. Hence, for each case, $(\varpi * \varrho)_\delta \in\vee (\kappa^*, q_\kappa)\mathcal{T}$, and thus, $\varpi * \varrho \in \widetilde{[\mathcal{T}]}_\delta$.

$(\Leftarrow)$ Let $\widetilde{[\mathcal{T}]}_\delta$ be a q-ideal of $\widetilde{\mathscr{A}}$, $\forall \delta \in (0, 1]$. On the contrary, let

$$\mathcal{T}(0) < \mathcal{T}(\varrho) \wedge \frac{\kappa^* - \kappa}{2}, \tag{31}$$

for some $\varrho \in \widetilde{\mathscr{A}}$. Then, $\exists \delta \in (0, 1]$ such that $\mathcal{T}(0) < \delta \leq \mathcal{T}(\varrho) \wedge (\kappa^* - \kappa/2)$. It follows that $\varrho \in \widetilde{[\mathcal{T}]}_\delta$, but $0 \notin \widetilde{[\mathcal{T}]}_\delta$, which is not possible. Therefore,

$$\mathcal{T}(0) \geq \mathcal{T}(\varrho) \wedge \frac{\kappa^* - \kappa}{2}. \tag{32}$$

Also, if $\mathcal{T}(\varpi * \varrho) < \mathcal{T}(\varpi * (\Theta * \varrho)) \wedge \mathcal{T}(\Theta) \wedge (\kappa^* - \kappa/2)$ for some $\varrho, \varpi \in \widetilde{\mathscr{A}}$, then $\exists \delta \in (0, 1]$ such that

$$\mathcal{T}(\varpi * \varrho) < \delta \leq \mathcal{T}(\varpi * (\Theta * \varrho)) \wedge \mathcal{T}(\Theta) \wedge \frac{\kappa^* - \kappa}{2}. \tag{33}$$

Thus, $(\varpi * (\Theta * \varrho)) \in \widetilde{[\mathcal{T}]}_\delta$ and $\Theta \in \widetilde{[\mathcal{T}]}_\delta$, but $\varpi * \varrho \notin \widetilde{[\mathcal{T}]}_\delta$, which is again a contradiction. Therefore, $\mathcal{T}(\varpi * \varrho) \geq \mathcal{T}(\varpi * (\Theta * \varrho)) \wedge \mathcal{T}(\Theta) \wedge (\kappa^* - \kappa/2)$. Hence, $\mathcal{T}$ is an $(\in, \in\vee (\kappa^*, q_\kappa))$-FQI of $\widetilde{\mathscr{A}}$.

4. Conclusion

The main aim of the present paper is to introduce the concept of $(\in, \in\vee (\kappa^*, q_\kappa))$-FQI in *BCI*-algebras. We provided some equivalent conditions and different characterizations of the $(\in, \in\vee (\kappa^*, q_\kappa))$-FQI in terms of level subsets and $(\in\vee (\kappa^*, q_\kappa))$-level subsets of *BCI*-algebras. It has been shown that in any *BCI*-algebras, the $(\in, \in\vee (\kappa^*, q_\kappa))$-FQI is $(\in, \in\vee (\kappa^*, q_\kappa))$-FI but the converse does not hold and an example is provided to support this. Furthermore, relation between $(\in, \in\vee (\kappa^*, q_\kappa))$-FQI and q-ideal of *BCI*-algebras has been considered. In this investigation, we get to the following conclusions:

(1) If we choose $\kappa^* = 1$ and $k = 0$, then $(\in, \in\vee (\kappa^*, q_\kappa))$-FQI reduces to the notion $(\in, \in\vee q)$-FQI of X as established in [16]

(2) If we choose $\kappa^* = 1$ and $k = k$, then $(\in, \in\vee (\kappa^*, q_\kappa))$-fuzzy subalgebras and $(\in, \in\vee (\kappa^*, q_\kappa))$-FQI reduce to the concepts $(\in, \in\vee q_\kappa)$-fuzzy subalgebras and $(\in, \in\vee q_\kappa)$-FQI of X as introduced in [28]

Consequently, the notions introduced in this paper are more general than the existing notions. In future study, these ideas may be extended to other algebraic structures such as rings, hemirings, *LA*-semigroups, semi-hypergroups, semi-hyper-rings, *BL*-algebras, *MTL*-algebras, *R0*-algebras, *EQ*-algebras, *MV*-algebras, and lattice implication algebras.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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