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Research Article

A New Result for ψ-Hilfer Fractional Pantograph-Type Langevin Equation and Inclusions

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In this paper, we deal with the existence and uniqueness of solution for ψ-Hilfer Langevin fractional pantograph differential equation and inclusion; these types of pantograph equations are a special class of delay differential equations. The existence and uniqueness results are obtained by making use of the Krasnoselskii fixed-point theorem and Banach contraction principle, and for the inclusion version, we use the Martelli fixed-point theorem to get the existence result. In the end, we are giving an example to illustrate our results.

1. Introduction

Over the past few years, fractional differential equations have attracted the interest of many mathematicians due to their ability to describe several complex problems in different scientific and engineering fields such as physics, biology, chemistry, and control theory (for more details, see [1–14]).

With the recent outstanding development in fractional differential equations, the Langevin equation has been considered a part of fractional calculus, and its form is $m(d^2x)/dt^2 = \lambda(dx/dt) + \eta(t)$, introduced by Paul Langevin in 1908. The Langevin equation is an effective tool that can describe processes as regards time evolution of the velocity of the Brownian motion [15–19] and also describe the evolution of physical phenomena in fluctuating environments [8] (for more details see, for example, [2, 20, 21]).

There are diverse definitions of fractional integrals and derivatives, the famous definitions are the Riemann-Liouville and the Caputo fractional derivatives. Hilfer [3] introduced the generalization of these derivatives under the name of Hilfer fractional derivative of order α and parameter $\beta \in [0, 1]$.

The authors in [14] have investigated the existence and uniqueness of solution for an initial value problem of Langevin equation involving two fractional orders, as follows:

$$\begin{cases} D^{\beta}(D^{\alpha} + \lambda)x(t) = f(t, x(t)), 0 \le t \le 1, 0 < \alpha \le 1, 1 \le \beta \le 2, \\ x(0) = x(1) = 0, D^{2\alpha}x(1) + \lambda D^{\alpha}x(1) = 0. \end{cases} \tag{1}$$

where D^{α} is is the Caputo fractional derivative of order α, $f : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function, and λ is a real number.

Motivated by the mentioned work and the new research going on in this direction, we study a new and a challenging case of fractional derivative called the ψ-Hilfer derivative [22]; this brand of fractional derivative generalizes the well-known fractional derivatives (Riemman-Liouville, Caputo, ψ-Riemman-Liouville, Hilfer-Hadamard, and Katugampola derivative), for different values of function ψ and parameter β; these values are described in what follows.

In this paper, we study the existence and uniqueness results of the solutions for the following problem:

$$\begin{cases} {}^{H}D^{\alpha_1,\beta_1;\psi}\left({}^{H}D^{\alpha_2,\beta_2;\psi} + \mu\right)x(t) = f(t, x(t), x(\lambda t)), \quad a \le t \le b \\ x(a) = 0, \quad x(b) = \sum_{i=1}^{n} \omega_i(I^{\sigma_i;\psi}(x))(\eta_i). \end{cases} \tag{2}$$

where ${}^{H}D^{\alpha_j,\beta_j;\psi}$, $j=1,2$ are the ψ-Hilfer fractional derivative of order α_j, $0<\alpha_j<1$ and parameter β_j, $0\le\beta_j\le 1$, $j=1,2$, $\mu\in\mathbb{R}^*$, $0<\lambda<1$, $a\ge 0$, $I^{\sigma_i;\psi}$, are the ψ-Riemann-Liouville fractional integral of order $\sigma_i>0$, $\omega_i\in\mathbb{R}^*$, $i=1,\cdots,n$, $a<\eta_1\ldots\cdots<\eta_n<b$, and $f:[a,b]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ is a continuous function.

We also will study the multivalued version of the problem (2) by considering the following problem:

$$\begin{cases} {}^{H}D^{\alpha_1,\beta_1;\psi}\left({}^{H}D^{\alpha_2,\beta_2;\psi}+\mu\right)x(t)\in F(t,x(t),x(\lambda t)), & a\le t\le b \\ x(a)=0, \quad x(b)=\sum_{i=1}^{n}\omega_i(I^{\sigma_i;\psi}(x))(\eta_i). \end{cases} \tag{3}$$

where $F:[a,b]\times\mathbb{R}^2\longrightarrow\mathscr{P}(\mathbb{R})$ is a multivalued map ($\mathscr{P}(\mathbb{R})$ is the family of all nonempty subjects of $\mathbb{R}$).

The novelty of this paper and its challenges are that it collects and generalizes the types of fractional derivative, for different values of function ψ and parameter β_i, $i=1,2$ such as follows:

(i) If $\psi(x)=x$ and $\beta_i=1$, then the problems (2) and (3) reduce to the Caputo-type

(ii) If $\psi(x)=x$ and $\beta_i=0$, then the problems (2) and (3) reduce to the Riemman-Liouville-type

(iii) If $\beta_i=0$, then the problems (2) and (3) reduce to the ψ-Riemman-Liouville-type

(iv) If $\psi(x)=x$, then the problems (2) and (3) reduce to the Hilfer-type

(v) If $\psi(x)=\log(x)$, then the problems (2) and (3) reduce to the Hilfer-Hadamard-type

(vi) If $\psi(x)=x^\rho$, then the problems (2) and (3) reduce to Katugampola-type

This paper is structured as follows: In the second section, we will present some auxiliary lemmas, some basic definitions, and theorems which are needed throughout this paper. In the third section, we discuss the existence and uniqueness results for the first problem, by using Krasnoselskii's fixed-point theorem and Banach's contraction principle. In the fourth section, we deal with the existence results for the inclusion version, by making use of Martelli's fixed-point theorem, which is applicable to completely continuous operators. Finally, in the last part, we give an example to support our study.

2. Preliminaries and Notations

2.1. Fractional Calculus. In this section, we introduce some definitions, lemmas, and useful notations that will be used throughout this paper.

Let $\mathscr{C}=C([a,b],\mathbb{R})$ denote the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm defined by $\|f\|=sup_{t\in[a,b]}\{|f(t)|\}$. We denote by $AC^n([a,b],\mathbb{R})$ the n-times absolutely continuous functions given by

$$AC^n([a,b],\mathbb{R})=\left\{f:[a,b]\longrightarrow\mathbb{R}\,;f^{(n-1)}\in AC([a,b],\mathbb{R})\right\}. \tag{4}$$

Definition 2.1 [23]. Let (a,b), $-\infty\le a<b\le+\infty$, be a finite or infinite interval of the half-axis $(0,+\infty)$ and $\alpha>0$. In addition, let $\psi(t)$ be a positive increasing function on $(a,b]$, which has a continuous derivative $\psi'(t)$ on (a,b). The ψ-Riemann-Liouville fractional integral of a function f with respect to another function ψ on $[a,b]$ is defined by

$$I_{a^+}^{\alpha;\psi}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\psi'(t)(\psi(t)-\psi(s))^{\alpha-1}f(s)ds, \tag{5}$$

where $\Gamma(.)$ represents the Gamma function.

Definition 2.2 [23]. Let $\psi'(t)\ne 0$ and $\alpha>0$, $n\in\mathbb{N}$. The Riemann-Liouville derivative of a function f with respect to another function ψ of order α, correspondent to the Riemann-Liouville is defined by

$$\begin{aligned} D_{a^+}^{\alpha;\psi}f(t)&=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha;\psi}f(t),\\ &=\frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\\ &\quad\cdot\int_a^t\psi'(t)(\psi(t)-\psi(s))^{n-\alpha-1}f(s)ds, \end{aligned} \tag{6}$$

where $n-1<\alpha<n, n=[\alpha]+1$, and $[\alpha]$ denote the integer part of the real number α.

Definition 2.3 [23]. Let $n-1<\alpha<n$ with $n\in\mathbb{N}$, $[a,b]$ is the interval such that $-\infty\le a<b\le+\infty$ and $f,\psi\in C^n([a,b],\mathbb{R})$ are two functions such that ψ is increasing and $\psi'(t)\ne 0$ for all $t\in[a,b]$. The ψ-Hilfer fractional derivative of a function f of order α and type $0\le\beta\le 1$ is defined by

$$\begin{aligned} {}^{H}D_{a^+}^{\alpha,\beta;\psi}f(t)&=I_{a^+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{(1-\beta)(n-\alpha);\psi}f(t)\\ &=I_{a^+}^{\gamma-\alpha;\psi}D_{a^+}^{\gamma;\psi}f(t), \end{aligned} \tag{7}$$

where $n-1<\alpha<n, n=[\alpha]+1$, and $[\alpha]$ denote the integer part of the real number α, with $\gamma=\alpha+\beta(n-\alpha)$.

Lemma 2.4 [23]. *Let $\alpha,\beta>0$. Then we have the following semigroup property given by*

$$I_{a^+}^{\alpha;\psi}I_{a^+}^{\beta;\psi}f(t)=I_{a^+}^{\alpha+\beta;\psi}f(t), t>a. \tag{8}$$

Proposition 2.5 [23, 24]. *Let $a \geq 0$, $\upsilon > 0$, and $t > a$. Then, ψ-fractional integral and derivative of a power function are given by*

(i) $I_{a^+}^{\alpha;\psi}(\psi(s)-\psi(a))^{\upsilon-1}(t) = (\Gamma(\upsilon)/\Gamma(\upsilon+\alpha))(\psi(s)-\psi(a))^{\upsilon+\alpha-1}(t)$

(ii) ${}^H D_{a^+}^{\alpha;\psi}(\psi(s)-\psi(a))^{\upsilon-1}(t) = (\Gamma(\upsilon)/\Gamma(\upsilon+\alpha))(\psi(s)-\psi(a))^{\upsilon-\alpha-1}(t)$, $n-1<\alpha<n, \upsilon > n$

Lemma 2.6 [23]. *If $f \in C^n([a,b],\mathbb{R})$, $n-1<\alpha<n$, $0 \leq \beta \leq 1$, and $\gamma = \alpha + \beta(n-\alpha)$, then*

$$I_{a^+}^{\alpha;\psi}\left({}^H D_{a^+}^{\alpha,\beta;\psi} f\right)(t) = f(t) - \sum_{k=1}^{n} \frac{(\psi(t)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\psi}^{[n-k]} I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(a), \tag{9}$$

for all $t \in [a,b]$, where $f_{\psi}^{[n-k]} f(t) := ((1/\psi'(t))(d/dt))^n f(t)$.

Lemma 2.7. *Let $a \geq 0$, $0<\alpha_i<1$, $i=1,2$, and $h \in C([a,b],\mathbb{R})$. The function x is a solution of the problem:*

$$\begin{cases} {}^H D^{\alpha_1,\beta_1;\psi}\left({}^H D^{\alpha_2,\beta_2;\psi} + \mu\right)x(t) = h(t), \quad a \leq t \leq b, \\ x(a) = 0, \quad x(b) = \sum_{i=1}^{n} \omega_i (I^{\sigma_i;\psi}(x))(\eta_i), \quad a < \eta_i < b, i = 1, \cdots, n, \end{cases} \tag{10}$$

if and only if

$$x(t) = I^{\alpha_1+\alpha_2;\psi}h(t) - \mu I^{\alpha_2;\psi}x(t) + \left((\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}\right)/\Delta\Gamma(\gamma_1+\alpha_2) \cdot \left[I^{\alpha_1+\alpha_2;\psi}h(b) - \mu I^{\alpha_2;\psi}x(b) - \sum_{i=1}^{n}\omega_i I^{\alpha_1+\alpha_2+\sigma_i;\psi}h(\eta_i) + \mu\sum_{i=1}^{n}\omega_i I^{\alpha_2+\sigma_i;\psi}x(\eta_i)\right], \tag{11}$$

where

$$\Delta = \sum_{i=1}^{n}\omega_i \frac{(\psi(\eta_i)-\psi(a))^{\gamma_1+\alpha_2+\sigma_i-1}}{\Gamma(\gamma_1+\alpha_2+\sigma_i)} - \frac{(\psi(b)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Gamma(\gamma_1+\alpha_2)} \neq 0. \tag{12}$$

Proof. The problem (10) can be written as

$$I^{\gamma_1-\alpha_1;\psi H}D^{\gamma_1;\psi}\left({}^H D^{\alpha_2,\beta_2;\psi} + \mu\right)x(t) = h(t). \tag{13}$$

Applying the ψ-Riemann-Liouville fractional integral of order α_1 to both sides, we obtain the following by using Lemma 2.6

$${}^H D^{\alpha_2,\beta_2;\psi}x(t) + \mu x(t) = I^{\alpha_1;\psi}h(t) + \frac{d_0}{\Gamma(\gamma_1)}((\psi(t)-\psi(a))^{\gamma_1-1}, \tag{14}$$

where d_0 is constant and $\gamma_1 = \alpha_1 + \beta_1 - \alpha_1\beta_1$. Next, applying the ψ-Riemann-Liouville fractional integral of order α_2 to both sides of (14), we get by using Lemma 2.6:

$$\begin{aligned} x(t) = {} & I^{\alpha_1+\alpha_2;\psi}h(t) - \mu I^{\alpha_2;\psi}x(t) \\ & + \frac{d_0}{\Gamma(\gamma_1+\alpha_2)}((\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1} \\ & + \frac{d_1}{\Gamma(\gamma_2)}((\psi(t)-\psi(a))^{\gamma_2-1}. \end{aligned} \tag{15}$$

From using the boundary condition $x(a)=0$ in (15), we obtain that $d_1 = 0$. We get

$$x(t) = I^{\alpha_1+\alpha_2;\psi}h(t) - \mu I^{\alpha_2;\psi}x(t) + \frac{d_0}{\Gamma(\gamma_1+\alpha_2)}((\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}. \tag{16}$$

From using the boundary condition $x(b) = \sum_{i=1}^{n}\omega_i(I^{\sigma_i;\psi}(x))(\eta_i)$, in (15), we find

$$\begin{aligned} d_0 = \frac{1}{\Delta}\Bigg[& I^{\alpha_1+\alpha_2;\psi}h(b) - \mu I^{\alpha_2;\psi}x(b) - \sum_{i=1}^{n}\omega_i I^{\alpha_1+\alpha_2+\sigma_i;\psi}h(\eta_i) \\ & + \mu\sum_{i=1}^{n}\omega_i I^{\alpha_2+\sigma_i;\psi}x(\eta_i)\Bigg]. \end{aligned} \tag{17}$$

Substituting the value of (d_0) in (16), we obtain the solution (11). The converse follows by direct computation. □

2.2. Multivalued Analysis. For a normed space $(X, \|.\|)$, we define

$$\begin{aligned} \mathscr{P}(X) &= \{Y \subset X : Y \neq \varnothing\}, \\ \mathscr{P}_{b,c,cl}(X) &= \{Y \subset X : Y \text{ is bounded, convex and closed}\}. \end{aligned} \tag{18}$$

For more details of multivalued analysis, see ([2, 3]).

Definition 2.8. A multivalued map $F : [a,b] \times \mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longrightarrow F(t,x)$ is measurable for each $x \in \mathbb{R}$

(ii) $x \longrightarrow F(t,x)$ is upper semicontinuous for almost all $t \in [a,b]$

Furthermore, a Carathéodory function F is called $\mathbb{L}^1$-Carathéodory if:

(iii) For each $\rho > 0$, there exists $\varphi_\rho \in \mathbb{L}^1([a,b];\mathbb{R})$ such that

$$\|F(t,x)\| = \sup\{|v|\colon v \in F(t,x)\} \leq \varphi_\rho(t), \tag{19}$$

for all $x \in \mathbb{R}$ with $\|x\| \leq \rho$ and for a.e. $t \in [a,b]$.

Theorem 2.9 (Krasnoselskii fixed-point theorem [25]). *Let $\mathcal{M}$ be a closed, bounded, convex, and nonempty subset of a Banach space. Let $\mathcal{A}, \mathcal{B}$ be the operators such that*

(i) $\mathcal{A}x + \mathcal{B}y \in \mathcal{M}$ whenever $x, y \in \mathcal{M}$

(ii) $\mathcal{A}$ is compact and continuous

(iii) $\mathcal{B}$ is contraction mapping

Then there exists $z \in \mathcal{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$.

Theorem 2.10 (Martelli fixed-point theorem [26]). *Let X be a Banach space and $T : X \longrightarrow \mathcal{P}_{b,c,cl}(X)$ be a completely continuous multivalued map. If the set $\Lambda = \{x \in X : \theta x \in T(x), \theta > 1\}$ is bounded, then T has a fixed point.*

3. Existence and Uniqueness Results for Problem (2)

In this section, we investigate the existence and uniqueness results for the problem (2), for this to simplify the computations, we use the following notations:

$$\Phi_1 = \frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(\psi(b)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_1+\alpha_2)} \cdot \left[\frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \sum_{i=1}^{n}|\omega_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha_1+\alpha_2+\sigma_i}}{\Gamma(\alpha_1+\alpha_2+\sigma_i+1)}\right], \tag{20}$$

$$\Phi_2 = |\mu|\left\{\frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(\psi(b)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_1+\alpha_2)} \cdot \left[\frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{n}|\omega_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha_2+\sigma_i}}{\Gamma(\alpha_2+\sigma_i+1)}\right]\right\}. \tag{21}$$

In view of Lemma 2.7, we define the operator $\mathcal{K} : \mathcal{C} \longrightarrow \mathcal{C}$ by

$$\begin{aligned}(\mathcal{K}x)(t) = {} & I^{\alpha_1+\alpha_2;\psi}f(t,x(t),x(\lambda t)) - \mu I^{\alpha_2;\psi}x(t) + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1+\alpha_2)} \\ & \cdot \Bigg[I^{\alpha_1+\alpha_2;\psi}f(b,x(b),x(\lambda b)) - \mu I^{\alpha_2;\psi}x(b) \\ & - \sum_{i=1}^{n}\omega_i I^{\alpha_1+\alpha_2+\sigma_i;\psi}f(\eta_i,x(\eta_i),x(\lambda\eta_i)) \\ & + \mu\sum_{i=1}^{n}\omega_i I^{\alpha_2+\sigma_i;\psi}x(\eta_i)\Bigg],\end{aligned} \tag{22}$$

where $\mathcal{C} = C([a,b],\mathbb{R})$ denotes the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm $\|x\| \coloneqq \sup\{|x(t)|\,;\, t \in [a,b]\}$.

3.1. Existence Result via Krasnoselskii's Fixed-Point Theorem

Theorem 3.1. *Assume that*

(H1)$f : [a,b]\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function such that $|f(t,x,y)| \leq \phi(t)$, $\forall (t,x,y) \in [a,b]\times\mathbb{R}^2$, with $\phi \in C([a,b]\,;\,\mathbb{R}^+)$

(H2)$\Phi_2 < 1$, where Φ_2 is given by (21)

Then there exists at least one solution for the problem (2) on $[a,b]$.

Proof. Let $\sup_{t\in[a,b]}|\phi(t)| = \|\phi\|$ and $\mathcal{B}_r = \{x \in \mathcal{C}\,;\, \|x\| \leq r\}$, where $r \geq (\|\phi\|\Phi_1)/(1-\Phi_2)$, we will show that the operator $\mathcal{K}$ defined by (22) satisfies the conditions of Krasnoselskii's fixed-point theorem, for that we split the operator $\mathcal{K}$ into the sum of two operators $\mathcal{K}_1$ and $\mathcal{K}_2$ defined, on the closed ball, by

$$\begin{aligned}(\mathcal{K}_1x)(t) &= I^{\alpha_1+\alpha_2}f(t,x(t),x(\lambda t)) + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1+\alpha_2)} \cdot \left[I^{\alpha_1+\alpha_2}f(b,x(b),x(\lambda b)) - \sum_{i=1}^{n}\omega_i I^{\alpha_1+\alpha_2+\sigma_i}f(\eta_i,x(\eta_i),x(\lambda\eta_i))\right], \\ (\mathcal{K}_2x)(t) &= -\mu I^{\alpha_2}x(t) + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1+\alpha_2)} \cdot \left[-\mu I^{\alpha_2}x(b) + \mu\sum_{i=1}^{n}\omega_i I^{\alpha_2+\sigma_i}x(\eta_i)\right].\end{aligned} \tag{23}$$

For every $x, y \in \mathcal{B}_r$, we have

$$\begin{aligned}|(\mathcal{K}_1x)(t) + (\mathcal{K}_2y)(t)| &\leq I^{\alpha_1+\alpha_2}|f(t,x(t),x(\lambda t))| + |\mu|I^{\alpha_2}|y(t)| \\ &\quad + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_1+\alpha_2)} \times \Bigg[I^{\alpha_1+\alpha_2}|f(b,x(b),x(\lambda b))| \\ &\quad + \sum_{i=1}^{n}|\omega_i|I^{\alpha_1+\alpha_2+\sigma_i}|f(\eta_i,x(\eta_i),x(\lambda\eta_i))| + |\mu|I^{\alpha_2}|y(b)| \\ &\quad + |\mu|\sum_{i=1}^{n}|\omega_i|I^{\alpha_2+\sigma_i}|y(\eta_i)|\Bigg], \\ &\leq \|\phi\|\left\{\frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(\psi(b)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_1+\alpha_2)} \times \left[\frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \sum_{i=1}^{n}|\omega_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha_1+\alpha_2+\sigma_i}}{\Gamma(\alpha_1+\alpha_2+\sigma_i+1)}\right]\right\} \\ &\quad + \|y\||\mu|\left\{\frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(\psi(b)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_1+\alpha_2)} \times \left[\frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{n}|\omega_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha_2+\sigma_i}}{\Gamma(\alpha_2+\sigma_i+1)}\right]\right\}, \\ &\leq \|\phi\|\Phi_1 + r\Phi_2, \leq r,\end{aligned} \tag{24}$$

then $\|\mathcal{K}_1x + \mathcal{K}_2y\| \leq r$, which implies that $\mathcal{K}_1x + \mathcal{K}_2y \in \mathcal{B}_r$.

Since f is continuous, then the operator $\mathscr{K}_1$ is continuous, and it is uniformly bounded on $\mathscr{B}_r$ as

$$\|\mathscr{K}_1 x\| \leq \Phi_1 \|\phi\|. \tag{25}$$

Next, we prove that the operator $\mathscr{K}_1$ is compact, for that setting $\sup_{(t,x,y)\in[a,b]\times\mathscr{B}_r}|f(t,x,y)| = \bar{f} < \infty$, and let $\tau_1, \tau_2 \in [a,b]$, $\tau_1 < \tau_2$; we obtain

$$\begin{aligned}
&|(\mathscr{K}_1 x)(\tau_2) - (\mathscr{K}_1 x)(\tau_1)| \\
&= \Bigg| I^{\alpha_1+\alpha_2} f(\tau_2, x(\tau_2), x(\lambda\tau_2)) + \frac{(\psi(\tau_2)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1+\alpha_2)} \\
&\quad \times \left[I^{\alpha_1+\alpha_2} f(b, x(b), x(\lambda b)) - \sum_{i=1}^{n} \omega_i I^{\alpha_1+\alpha_2+\sigma_i} f(\eta_i, x(\eta_i), x(\lambda\eta_i)) \right] \\
&\quad - I^{\alpha_1+\alpha_2} f(\tau_1, x(\tau_1), x(\lambda\tau_1)) - \frac{(\psi(\tau_1)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1+\alpha_2)} \\
&\quad \cdot \left[I^{\alpha_1+\alpha_2} f(b, x(b), x(\lambda b)) - \sum_{i=1}^{n} \omega_i I^{\alpha_1+\alpha_2+\sigma_i} f(\eta_i, x(\eta_i), x(\lambda\eta_i)) \right] \Bigg| \\
&\leq \frac{\bar{f}}{\Gamma(\alpha_1+\alpha_2)} \Bigg| \int_a^{\tau_1} \psi'(s)\left((\psi(\tau_2)-\psi(s))^{\alpha_1+\alpha_2-1} - (\psi(\tau_1)-\psi(s))^{\alpha_1+\alpha_2-1} ds\right) \\
&\quad + \int_{\tau_1}^{\tau_2} \psi'(s)(\psi(\tau_2)-\psi(s))^{\alpha_1+\alpha_2-1} ds \Bigg| \\
&\quad + \frac{\left|(\psi(\tau_2)-\psi(a))^{\gamma_1+\alpha_2-1} - (\psi(\tau_1)-\psi(a))^{\gamma_1+\alpha_2-1}\right|}{|\Delta|\Gamma(\gamma_1+\alpha_2)} \\
&\quad \times \left[\bar{f} \frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \bar{f} \sum_{i=1}^{n} |\omega_i| \frac{(\psi(\eta_i)-\psi(a))^{\alpha_1+\alpha_2+\sigma_i}}{\Gamma(\alpha_1+\alpha_2+\sigma_i+1)} \right].
\end{aligned} \tag{26}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \longrightarrow 0$, independently of $x \in \mathscr{B}_r$. Then, $\mathscr{K}_1$ is equicontinuous; hence, $\mathscr{K}_1$ is relatively compact on $\mathscr{B}_r$. By the Arzelà-Ascoli theorem, it implies that $\mathscr{K}_1$ is compact on $\mathscr{B}_r$.

In the next step, we will show that $\mathscr{K}_2$ is a contraction mapping; for that, let $x, y \in \mathscr{C}$, and for $t \in [a,b]$, we have

$$\begin{aligned}
&|(\mathscr{K}_2 x)(t) + (\mathscr{K}_2 y)(t)| \\
&\leq |\mu| I^{\alpha_2}|x(t)-y(t)| + \frac{(\psi(b)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_1+\alpha_2)} \\
&\quad \times \left[|\mu| I^{\alpha_2}|x(b)-y(b)| + |\mu| \sum_{i=1}^{n} |\omega_i| I^{\alpha_2+\sigma_i}|x(\eta_i)-y(\eta_i)| \right], \\
&\leq \|x-y\| |\mu| \left\{ \frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(\psi(b)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_1+\alpha_2)}, \right. \\
&\quad \left. \times \left[\frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{n} |\omega_i| \frac{(\psi(\eta_i)-\psi(a))^{\alpha_2+\sigma_i}}{\Gamma(\alpha_2+\sigma_i+1)} \right] \right\}, \\
&\leq \Phi_2 \|x-y\|.
\end{aligned} \tag{27}$$

This implies that $\|\mathscr{K}_2 x + \mathscr{K}_2 y\| \leq \Phi_2 \|x-y\|$, by using $(H2)$; we deduce that $\mathscr{K}_2$ is a contraction mapping. It follows by using Krasnoselskii's fixed-point theorem, the problem (2) has at least one solution on $[a,b]$. □

3.2. Uniqueness Result via Banach's Fixed-Point Theorem. To deal with the uniqueness of solution for our problem (2), we use Banach's fixed-point theorem.

Theorem 3.2. *Assume that $|f(t,x,y) - f(t,z,w)| \leq L(|x-z| + |y-w|)$; $L > 0$, for each $t \in [a,b]$ and $x, y, z, w \in \mathbb{R}$.*

If $2L\Phi_1 + \Phi_2 < 1$, where Φ_1, Φ_2 are, respectively, given by (20) and (21); then the problem (2) has a unique solution on $[a,b]$.

Proof. By considering the operator $\mathscr{K}$ defined in (22), we transform the problem (2) into a fixed-point problem $x = \mathscr{K}x$. By using Banach contraction principle, we will show that $\mathscr{K}$ has a unique fixed point.

We set $\sup_{t\in[a,b]} = |f(t,0,0)| = M < \infty$ and choose $\rho > 0$ such that

$$\rho \geq \frac{M\Phi_1}{1 - 2L\Phi_1 - \Phi_2}. \tag{28}$$

$\mathscr{B}_\rho = \{x \in \mathscr{C}([a,b],\mathbb{R})\,;\, \|x\| \leq \rho)\}$, where Φ_1, Φ_2 are, respectively, given by (20) and (21).

Step 1: we show that $\mathscr{K}\mathscr{B}_\rho \subset \mathscr{B}_\rho$.

For any $x \in \mathscr{B}_\rho$ we have

$$\begin{aligned}
|f(t,x(t),x(\lambda t))| &\leq |f(t,x(t),x(\lambda t)) - f(t,0,0)| + |f(t,0,0)| \\
&\leq L(|x(t)| + |x(\lambda t)|) + M \leq 2L\|x\| + M.
\end{aligned} \tag{29}$$

Then we have

$$\begin{aligned}
|(\mathscr{K}x)(t)| &\leq I^{\alpha_1+\alpha_2}|f(t,x(t),x(\lambda t))| + |\mu| I^{\alpha_2}|x(t)| \\
&\quad + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_1+\alpha_2)} \times \Bigg[I^{\alpha_1+\alpha_2}|f(b,x(b),x(\lambda b))| \\
&\quad + \sum_{i=1}^{n} |\omega_i| I^{\alpha_1+\alpha_2+\sigma_i}|f(\eta_i,x(\eta_i),x(\lambda\eta_i))| + |\mu| I^{\alpha_2}|x(b)| \\
&\quad + |\mu| \sum_{i=1}^{n} |\omega_i| I^{\alpha_2+\sigma_i}|x(\eta_i)| \Bigg], \\
&\leq (2L\|x\| + M) \left\{ \frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(\psi(b)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_1+\alpha_2)} \right. \\
&\quad \left. \times \left[\frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \sum_{i=1}^{n} |\omega_i| \frac{(\psi(\eta_i)-\psi(a))^{\alpha_1+\alpha_2+\sigma_i}}{\Gamma(\alpha_1+\alpha_2+\sigma_i+1)} \right] \right\} \\
&\quad + \|x\| |\mu| \left\{ \frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(\psi(b)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_1+\alpha_2)} \right. \\
&\quad \left. \times \left[\frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{n} |\omega_i| \frac{(\psi(\eta_i)-\psi(a))^{\alpha_2+\sigma_i}}{\Gamma(\alpha_2+\sigma_i+1)} \right] \right\}, \\
&\leq (2L\|x\| + M)\Phi_1 + \|x\|\Phi_2 \leq (2L\rho + M)\Phi_1 + \rho\Phi_2, \leq \rho,
\end{aligned} \tag{30}$$

which implies that $\mathscr{K}\mathscr{B}_\rho \subset \mathscr{B}_\rho$.

Step 2: next we show that the operator $\mathscr{K}$ is a contraction.

For any $x, y \in \mathscr{C}$, and for $t \in [a, b]$, we have

$$\begin{aligned}
&|(\mathscr{K}x)(t) - (\mathscr{K}y)(t)| \\
&\leq \left\{ \frac{(\psi(b) - \psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1 + \alpha_2 + 1)} + \frac{(\psi(b) - \psi(a))^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_1 + \alpha_2)} \right. \\
&\quad \times \left. \left[\frac{(\psi(b) - \psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1 + \alpha_2 + 1)} + \sum_{i=1}^{n} |\omega_i| \frac{(\psi(\eta_i) - \psi(a))^{\alpha_1+\alpha_2+\sigma_i}}{\Gamma(\alpha_1 + \alpha_2 + \sigma_i + 1)} \right] \right\} 2L\|x - y\| + |\mu| \\
&\quad \cdot \left\{ \frac{(\psi(b) - \psi(a))^{\alpha_2}}{\Gamma(\alpha_2 + 1)} + \frac{(\psi(b) - \psi(a))^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_1 + \alpha_2)} \right. \\
&\quad \times \left. \left[\frac{(\psi(b) - \psi(a))^{\alpha_2}}{\Gamma(\alpha_2 + 1)} + \sum_{i=1}^{n} |\omega_i| \frac{(\psi(\eta_i) - \psi(a))^{\alpha_2+\sigma_i}}{\Gamma(\alpha_2 + \sigma_i + 1)} \right] \right\} \|x - y\| \\
&\leq (2L\Phi_1 + \Phi_2)\|x - y\|,
\end{aligned} \tag{31}$$

which implies $\|(\mathscr{K}x)(t) - (\mathscr{K}y)(t)\| \leq (2L\Phi_1 + \Phi_2)\|x - y\|$. As $2L\Phi_1 + \Phi_2 < 1$, then $\mathscr{K}$ is a contraction, and by applying Banach's fixed-point theorem, we get that the operator $\mathscr{K}$ has a unique fixed point which is the unique solution of our problem (2). □

4. Existence Results for the Inclusion Version

In this section, we will investigate the existence result for the inclusion version defined as problem (3).

Definition 4.1. A continuous function x is said to be a solution of problem (3) if $x(a) = 0\,; x(b) = \sum_{i=1}^{n} \omega_i (I^{\sigma_i;\psi}(x))(\eta_i)$, and there exists a function $v \in \mathbb{L}^1([a, b], \mathbb{R})$ with $v \in F(t, x(t), x(\lambda t))$, i.e., on $[a, b]$ such that

$$\begin{aligned}
x(t) = I^{\alpha_1+\alpha_2;\psi} v(t) - \mu I^{\alpha_2;\psi} x(t) + \frac{(\psi(t) - \psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1 + \alpha_2)} \\
\cdot \left[I^{\alpha_1+\alpha_2;\psi} v(b) - \mu I^{\alpha_2;\psi} x(b) \right. \\
\left. - \sum_{i=1}^{n} \omega_i I^{\alpha_1+\alpha_2+\sigma_i;\psi} v(\eta_i) + \mu \sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i;\psi} x(\eta_i) \right],
\end{aligned} \tag{32}$$

for each $x \in \mathscr{C}([a, b], \mathbb{R})$, define the set of selections of F by

$$\mathcal{S}_{F,x} := \left\{ v \in \mathbb{L}^1([a, b], \mathbb{R}) : v \in F(t, x(t), x(\lambda t)) \; on \; [a, b] \right\}. \tag{33}$$

Lemma 4.2 ([5]). *Let X be a Banach space and $F : [a, b] \times \mathbb{R}^2 \longrightarrow \mathscr{P}_{b,c,cl}$ be a $\mathbb{L}^1$-Carathéodory multivalued map. And let Ξ be a linear continuous mapping from $L^1([a, b], X)$ to $\mathscr{C}([a, b], X)$. Then the operator*

$$\begin{aligned}
\Xi \circ \mathcal{S}_F : \mathscr{C}([a, b], X) \longrightarrow \mathscr{P}_{b,c,cl}(\mathscr{C}([a, b], X))\,; x \\
\longrightarrow (\Xi \circ \mathcal{S}_F)(x) = \Xi(\mathcal{S}_{F,x})
\end{aligned} \tag{34}$$

is a closed graph operator in $\mathscr{C}([a, b], X) \times \mathscr{C}([a, b], X)$.

In what follows, we deal with the upper semicontinuous case, and for the existence results, we use Martelli's fixed-point theorem.

Theorem 4.3. *Suppose that (H2) holds and the following assumptions hold:*

(H3) $F : [a, b] \times \mathbb{R}^2 \longrightarrow \mathscr{P}_{b,c,cl}(\mathbb{R})$ is $\mathbb{L}^1$-Carathéodory and has nonempty convex values, and for each fixed $x \in \mathscr{C}([a, b], \mathbb{R})$, the set

$$\mathcal{S}_{F,x} = \left\{ v \in \mathbb{L}^1([a, b], X) : v(t) \in F(t, x(t), x(\lambda t))\,; t \in [a, b] \right\} \tag{35}$$

is nonempty and convex.

(H4) $|F(t, x, y)| := \sup\{|v| : v \in F(t, x, y)\} \leq p(t)\Psi(|x| + |y|)\}$ for all $t \in [a, b]$ and all $x, y \in \mathscr{C}([a, b], X)$, where $p \in \mathbb{L}^1([a, b], \mathbb{R}^+)$ and $\Psi : \mathbb{R}^+ \longrightarrow [0, +\infty)$ is continuous and nondecreasing function.

Then the problem (3) has at least one solution on $[a, b]$.

Proof. In order to transform the problem (3) into a fixed-point problem. Let $\mathscr{K} : \mathscr{C}([a, b], \mathbb{R}) \longrightarrow \mathscr{P}([a, b], \mathbb{R})$ be defined by

$$\begin{aligned}
\mathscr{K}(x) := \Bigg\{ k \in \mathscr{C}([a, b], \mathbb{R}) : k(t) \\
= \Bigg\{ I^{\alpha_1+\alpha_2;\psi} v(t) - \mu I^{\alpha_2;\psi} x(t) + \frac{(\psi(t) - \psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1 + \alpha_2)} \\
\times \left[I^{\alpha_1+\alpha_2;\psi} v(b) - \mu I^{\alpha_2} x(b) - \sum_{i=1}^{n} \omega_i I^{\alpha_1+\alpha_2+\sigma_i;\psi} v(\eta_i) \right. \\
\left. + \mu \sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i;\psi} x(\eta_i) \right]\,; t \in [a, b], v \in \mathcal{S}_{F,x} \Bigg\}.
\end{aligned} \tag{36}$$

We will show that $\mathscr{K}$ satisfies the conditions of the Theorem 2.10; for the poof, we give it in steps:

Step 1: $\mathscr{K}(x)$ is convex for each $x \in \mathscr{C}([a, b], \mathbb{R})$.

If k_1, k_2 belong to $\mathscr{K}(x)$, then there exist $v_1, v_2 \in \mathcal{S}_{F,x}$ such that for each $t \in [a, b]$ we have:

$$\begin{aligned}
k_j(t) = I^{\alpha_1+\alpha_2;\psi} v_j(t) - \mu I^{\alpha_2} x(t) + \frac{(\psi(t) - \psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1 + \alpha_2)} \\
\cdot \left[I^{\alpha_1+\alpha_2;\psi} v_j(b) - \mu I^{\alpha_2 \psi} x(b) - \sum_{i=1}^{n} \omega_i I^{\alpha_1+\alpha_2+\sigma_i;\psi} v_j(\eta_i) \right. \\
\left. + \mu \sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i;\psi} x(\eta_i) \right],
\end{aligned} \tag{37}$$

for $j = 1, 2$. Let $0 \leq \xi \leq 1$; then, for each $t \in [a, b]$, we have

$$\begin{aligned}\xi k_1(t)+(1-\xi)k_2(t)=&I^{\alpha_1+\alpha_2;\psi}[\xi v_1(s)+(1-\xi)v_2(s)]\\&-\mu I^{\alpha_2;\psi}x(t)+\frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1+\alpha_2)}\\&\times\Bigg[I^{\alpha_1+\alpha_2;\psi}[\xi v_1(s)+(1-\xi)v_2(s)]-\mu I^{\alpha_2;\psi}x(b)\\&-\sum_{i=1}^n\omega_i I^{\alpha_1+\alpha_2+\sigma_i;\psi}[\xi v_1(\eta_i)+(1-\xi)v_2(\eta_i)]\\&+\mu\sum_{i=1}^n\omega_i I^{\alpha_2+\sigma_i;\psi}x(\eta_i)\Bigg],\end{aligned}\tag{38}$$

Thus, $\xi v_1+(1-\xi)v_2\in\mathscr{K}(x)$ (because $\mathscr{S}_{F,x}$ is convex); then $\mathscr{K}(x)$ is convex for each $x\in\mathscr{C}([a,b],\mathbb{R})$

Step 2: $\mathscr{K}$ is bounded.

For a positive number ρ, let $\mathscr{B}_\rho=\{x\in\mathscr{C}([a,b],\mathbb{R})\colon \|x\|\le\rho\}$ be bounded ball in $\mathscr{C}([a,b],\mathbb{R}$; then for each $k\in\mathscr{K}(x)$ and $x\in\mathscr{B}_\rho$, there exists $v\in\mathscr{S}_{F,x}$, such that

$$\begin{aligned}k(t)=&I^{\alpha_1+\alpha_2;\psi}v(t)-\mu I^{\alpha_2;\psi}x(t)+\frac{(\psi(t)-(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1+\alpha_2)}\\&\cdot\Bigg[I^{\alpha_1+\alpha_2;\psi}v(b)-\mu I^{\alpha_2;\psi}x(b)\\&-\sum_{i=1}^n\omega_i I^{\alpha_1+\alpha_2+\sigma_i;\psi}v(\eta_i)+\mu\sum_{i=1}^n\omega_i I^{\alpha_2+\sigma_i;\psi}x(\eta_i)\Bigg],\end{aligned}\tag{39}$$

then for every $t\in[a,b]$, we have

$$\begin{aligned}|k(t)|\le&\sup_{t\in[a,b]}\Bigg\{I^{\alpha_1+\alpha_2;\psi}|v(t)|+|\mu|I^{\alpha_2;\psi}|x(t)|+\frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_1+\alpha_2)}\\&\times\Bigg[I^{\alpha_1+\alpha_2;\psi}|v(b)|+\sum_{i=1}^n|\omega_i|I^{\alpha_1+\alpha_2+\sigma_i;\psi}|v(\eta_i)|\\&+|\mu|I^{\alpha_2;\psi}|x(b)|+|\mu|\sum_{i=1}^n|\omega_i|I^{\alpha_2+\sigma_i;\psi}|x(\eta_i)|\Bigg]\Bigg\}\\\le&\|p\|\Psi(2\|x\|)\Bigg\{\frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{(\psi(b)-(a))^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_1+\alpha_2)}\\&\times\Bigg[\frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}+\sum_{i=1}^n|\omega_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha_1+\alpha_2+\sigma_i}}{\Gamma(\alpha_1+\alpha_2+\sigma_i+1)}\Bigg]\Bigg\}\\&+\|x\||\mu|\Bigg\{\frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{(\psi(b)-\psi(a))^{\gamma_1+\alpha_2-1}}{|\Delta|\Gamma(\gamma_1+\alpha_2)}\\&\times\Bigg[\frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)}+\sum_{i=1}^n|\omega_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha_2+\sigma_i}}{\Gamma(\alpha_2+\sigma_i+1)}\Bigg]\Bigg\}\\\le&\|p\|\Psi(2\|x\|)\Phi_1+\|x\|\Phi_2\le\|p\|\Psi(2\rho)\Phi_1+\rho\Phi_2.\end{aligned}\tag{40}$$

Then

$$\|\mathscr{K}(x)\|\le\|p\|\Psi(2\rho)\Phi_1+\rho\Phi_2\coloneqq l,\tag{41}$$

where Φ_1,Φ_2 are, respectively, given by (20) and (21).

Step 3: $\mathscr{K}$ is equicontinuous.

Let $t_1,t_2\in[a,b]$; $t_1<t_2$, and $x\in\mathscr{B}_\rho$ where $\mathscr{B}_\rho$, as above then for each $x\in\mathscr{B}_\rho$ and $k\in\mathscr{K}(x)$; there exist $v\in\mathscr{S}_{F,x}$; then we obtain

$$\begin{aligned}|k(t_2)-k(t_1)|\le&\frac{1}{\Gamma(\alpha_1+\alpha_2)}\Bigg|\int_a^{t_1}\psi'(s)\big((\psi(t_2)-\psi(s))^{\alpha_1+\alpha_2-1}\\&-(\psi(t_1)-\psi(s))^{\alpha_1+\alpha_2-1}\big)v(s)ds\\&+\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha_1+\alpha_2-1}v(s)ds\Bigg|\\&+\frac{|\mu|}{\Gamma(\alpha_2)}\Bigg|\int_a^{t_1}\psi'(s)\big((\psi(t_2)-\psi(s))^{\alpha_2-1}\\&-(\psi(t_1)-\psi(s))^{\alpha_2-1}\big)x(s)ds\\&+\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha_2-1}x(s)ds\Bigg|\\&+\frac{|(\psi(t_2)-\psi(a))^{\gamma_1+\alpha_2-1}-(\psi(t_1)-\psi(a))^{\gamma_1+\alpha_2-1}|}{|\Delta|\Gamma(\gamma_1+\alpha_2)},\\&\times\Bigg[\|v(s)\|\frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}\\&+\|v(s)\|\sum_{i=1}^n|\omega_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha_1+\alpha_2+\sigma_i}}{\Gamma(\alpha_1+\alpha_2+\sigma_i+1)}\\&+\|x(b)\||\mu|\frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)}\\&+\|x(\eta_i)\||\mu|\sum_{i=1}^n|\omega_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha_2+\sigma_i}}{\Gamma(\alpha_2+\sigma_i+1)}\Bigg],\\\le&\frac{\|p\|\Psi(2\rho)}{\Gamma(\alpha_1+\alpha_2)}\Bigg|\int_a^{t_1}\psi'(s)\big((\psi(t_2)-\psi(s))^{\alpha_1+\alpha_2-1}\\&-(\psi(t_1)-\psi(s))^{\alpha_1+\alpha_2-1}\big)ds+\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha_1+\alpha_2-1}ds\Bigg|\\&+\frac{\rho|\mu|}{\Gamma(\alpha_2)}\Bigg|\int_a^{t_1}\psi'(s)\big((\psi(t_2)-\psi(s))^{\alpha_2-1}-(\psi(t_1)-\psi(s))^{\alpha_2-1}\big)ds\\&+\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha_2-1}ds\Bigg|\\&+\frac{|(\psi(t_2)-\psi(a))^{\gamma_1+\alpha_2-1}-(\psi(t_1)-\psi(a))^{\gamma_1+\alpha_2-1}|}{|\Delta|\Gamma(\gamma_1+\alpha_2)}\\&\times\Bigg[\|p\|\Psi(2\rho)\frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}\\&+\|p\|\Psi(2\rho)\sum_{i=1}^n|\omega_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha_1+\alpha_2+\sigma_i}}{\Gamma(\alpha_1+\alpha_2+\sigma_i+1)}\\&+\rho|\mu|\frac{(\psi(b)-\psi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)}+\rho|\mu|\sum_{i=1}^n|\omega_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha_2+\sigma_i}}{\Gamma(\alpha_2+\sigma_i+1)}\Bigg].\end{aligned}\tag{42}$$

As $t_2\longrightarrow t_1$, the right-hand side of the above inequality tends to zero, implying that $\mathscr{K}(x)$ is equicontinuous. By using the Arzelà-Ascoli theorem, we get that $\mathscr{K}$ is relatively compact; then $\mathscr{K}$ is completely continuous.

To prove that the operator $\mathscr{K}$ is upper semicontinuous, it is enough to show that $\mathscr{K}$ has a closed graph.

Step 4: $\mathscr{K}$ has a closed graph.

Let $x_n\longrightarrow x_*$, $k_n\in\mathscr{K}(x_n)$ and $k_n\longrightarrow k_*$; we will prove that $k_*\in\mathscr{K}(x_*)$.

For $k_n \in \mathscr{K}(x_n)$, then there exists $v_n \in \mathcal{S}_{F,x_n}$ such that for each $t \in [a, b]$:

$$k_n(t) = I^{\alpha_1+\alpha_2;\psi} v_n(t) - \mu I^{\alpha_2;\psi} x_n(t) + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1+\alpha_2)} \cdot \left[I^{\alpha_1+\alpha_2;\psi} v_n(b) - \mu I^{\alpha_2;\psi} x_n(b) - \sum_{i=1}^{n} \omega_i I^{\alpha_1+\alpha_2+\sigma_i;\psi} v_n(\eta_i) + \mu \sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i;\psi} x_n(\eta_i) \right]. \tag{43}$$

We should prove that $v_* \in \mathcal{S}_{F,x_*}$ such that for each $t \in [a, b]$

$$k_*(t) = I^{\alpha_1+\alpha_2;\psi} v_*(t) - \mu I^{\alpha_2;\psi} x_*(t) + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1+\alpha_2)} \cdot \left[I^{\alpha_1+\alpha_2;\psi} v_*(b) - \mu I^{\alpha_2;\psi} x_*(b) - \sum_{i=1}^{n} \omega_i I^{\alpha_1+\alpha_2+\sigma_i;\psi} v_*(\eta_i) + \mu \sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i;\psi} x_*(\eta_i) \right]. \tag{44}$$

We have that

$$\left\| \left(k_n(t) + \lambda I^{\alpha_2;\psi} x_n(t) - \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1+\alpha_2)} \cdot \left[-\mu I^{\alpha_2;\psi} x_n(b) + \mu \sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i;\psi} x_n(\eta_i) \right] \right) - \left(k_*(t) + \mu I^{\alpha_2;\psi} x_*(t) - \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1+\alpha_2)} \cdot \left[-\lambda I^{\alpha_2;\psi} x_*(b) + \mu \sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i;\psi} x_*(\eta_i) \right] \right) \right\| \longrightarrow 0, \tag{45}$$

as $n \longrightarrow \infty$.

Consider the operator defined by

$$\begin{aligned} \Xi : \mathbb{L}^1([a, b], \mathbb{R}) &\longrightarrow \mathscr{C}([a, b], \mathbb{R}), \\ v &\longrightarrow \Xi(v)(t). \end{aligned} \tag{46}$$

with

$$\Xi(v)(t) = I^{\alpha_1+\alpha_2;\psi} v(t) + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1+\alpha_2)} \cdot \left[I^{\alpha_1+\alpha_2;\psi} v(b) - \sum_{i=1}^{n} \omega_i I^{\alpha_1+\alpha_2+\sigma_i;\psi} v(\eta_i) \right], \tag{47}$$

By using Lemma 4.2, $\Xi \circ \mathcal{S}_F$ is a closed graph operator; then we get

$$\left(k_n(t) + \mu I^{\alpha_2;\psi} x_n(t) - \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1+\alpha_2)} \cdot \left[-\mu I^{\alpha_2;\psi} x_n(b) + \mu \sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i;\psi} x_n(\eta_i) \right] \right) \in Y\left(\mathcal{S}_{F,x_n}\right). \tag{48}$$

Since $x_n \longrightarrow x_*$, and $k_n \longrightarrow k_*$, then

$$\left(k_*(t) + \mu I^{\alpha_2;\psi} x_*(t) - \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1+\alpha_2)} \cdot \left[-\mu I^{\alpha_2;\psi} x_*(b) + \mu \sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i;\psi} x_*(\eta_i) \right] \right) = \Xi(v_*) \in \Xi\left(\mathcal{S}_{F,x_*}\right). \tag{49}$$

It follows that $v_* \in \mathcal{S}_{F,x_*}$ such that

$$k_*(t) = I^{\alpha_1+\alpha_2;\psi} v_*(t) - \mu I^{\alpha_2;\psi} x_*(t) + \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1+\alpha_2)} \cdot \left[I^{\alpha_1+\alpha_2;\psi} v_*(b) - \mu I^{\alpha_2;\psi} x_*(b) - \sum_{i=1}^{n} \omega_i I^{\alpha_1+\alpha_2+\sigma_i;\psi} v_*(\eta_i) + \mu \sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i;\psi} x_*(\eta_i) \right]. \tag{50}$$

We deduce that $\mathscr{K}$ is an upper semicontinuous multivalued map, with convex closed values.

Step 5: $\Lambda = \{x \in X : \theta x \in \mathscr{K}(x), \theta > 1\}$ is bounded.

Let $x \in \Lambda$, and then $\theta x \in \mathscr{K}(x)$ for some $\theta > 1$; thus, there exists a function $v \in \mathcal{S}_{F,x}$ such that

$$x(t) = \frac{1}{\theta} I^{\alpha_1+\alpha_2;\psi} v(t) - \frac{1}{\theta} \mu I^{\alpha_2;\psi} x(t) + \frac{1}{\theta} \frac{(\psi(t)-\psi(a))^{\gamma_1+\alpha_2-1}}{\Delta\Gamma(\gamma_1+\alpha_2)} \cdot \left[I^{\alpha_1+\alpha_2;\psi} v(b) - \mu I^{\alpha_2;\psi} x(b) - \sum_{i=1}^{n} \omega_i I^{\alpha_1+\alpha_2+\sigma_i;\psi} v(\eta_i) + \mu \sum_{i=1}^{n} \omega_i I^{\alpha_2+\sigma_i;\psi} x(\eta_i) \right]. \tag{51}$$

From Step 2, and for every $t \in [a, b]$, we have

$$\|x\| \leq \|p\| \Psi(2\|x\|) \Phi_1 + \|x\| \Phi_2. \tag{52}$$

Then, by using $(H2)$, we get

$$\|x\| \leq \frac{\|p\| \Psi(2\rho) \Phi_1}{(1-\Phi_2)}. \tag{53}$$

Finally the set Λ is bounded; then from Theorem 2.10, we deduce that the problem (3) has at least one solution. □

5. Example

Consider the following ψ-Hilfer fractional pantograph Langevin equation given by

$$\begin{cases} {}^{H}D^{2/5,4/5;e^t/3}\left({}^{H}D^{1/5,3/5;e^t/3} + \frac{1}{9}\right)x(t) = \frac{1}{5} + \frac{1}{7}x(t) + \frac{1}{7}\sin x\left(\frac{t}{3}\right), \quad 0 \le t \le 1 \\ x(0) = 0, \quad x(1) = \frac{5}{8}I^{3/2;e^t/3}x\left(\frac{1}{3}\right) + \frac{7}{8}I^{3/2;e^t/3}x\left(\frac{1}{2}\right). \end{cases} \tag{54}$$

where $\alpha_1 = 2/5$, $\alpha_2 = 1/5$, $\beta_1 = 4/5$, $\beta_2 = 3/5$, $\mu = 1/9$, $\lambda = 1/3$, $a = 0$, $b = 1$, $n = 2$, $\sigma_1 = 3/2$, $\sigma_2 = 5/2$, $\omega_1 = 5/8$, $\omega_2 = 7/8$, $\eta_1 = 1/3$, $\eta_2 = 1/2$, and $\psi(t) = e^t/3$. With this given data, we get $\gamma_1 = \alpha_1 + \beta_1 - \alpha_1\beta_1 = 22/25$, $|\Delta| \simeq 1,7825119$, $\Phi_1 = 2,1294613$, and $\Phi_2 = 0,2195946 < 1$.

Set $f(t,x,y) = 1/5 + 1/7x(t) + 1/7 \sin y(t/3)$, $x, y \in \mathbb{R}$, $t \in [0,1]$.

Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $t \in [0,1]$; then we get

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| 1/7|x_1 - x_2| + 1/7|y_1 - y_2| 1/7(|x_1 - x_2| + |y_1 - y_2|). \tag{55}$$

This implies that the assumption of Theorem 3.2 holds with $L = 1/7$.

It follows that $2L\Phi_1 + \Phi_2 \simeq 0.8280121 < 1$; then by applying Theorem 3.2, our problem has a unique solution on $[0,1]$.

6. Conclusion

The present paper examined the ψ-Hilfer fractional pantograph Langevin equation and inclusion. The challenges and the novelty of this work generalize the types of fractional derivatives. With the assistance of Krasnoselskii and Banach fixed-point theorems, we investigate the existence and uniqueness results for the single valued problem, and by making use of the Martelli fixed-point theorem, we study the existence result for the multivalued problem. In the end, we illustrate our result with an example.

Disclosure

No potential conflict of interest was reported by the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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