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Research Article

Two Remarks on the Infinite Approximation of a Finite World in Economic Models

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While the assumption of infinity is prevalent in almost every area of economics, for two well-known frameworks in decision theory, we note some fundamental differences between the finite versions and their infinite counterpart. The first is on the usage of mixed strategies in finite games, while the second is on a characterization of the truth axiom in models of information and knowledge, where the properties of belief can be related to the properties of preference.

1. Introduction

It is well established that the infinity assumption is at the heart of contemporary economic methodology. Its usage in economic frameworks ranges from the most basic concepts like demand and supply often encountered by first year undergraduate students (I would like to emphasize that the moment we draw a demand or supply curve, we are assuming a continuum of possible prices and quantities.) to results on the existence of equilibrium in theory courses taken by graduate students. Infinite and continuum-like assumptions are welcomed by economists as it is well known that they offer the foundation for the usage of mathematical theories in economic modelling. Although it is hard to imagine the contemporary economic methodology without the infinite assumption, I will argue in this study that while the assumption does a tremendous work in establishing a rich and robust economic methodology, it does not come without loss of generality or content. In particular, I will show two instances in which the infinite generalization of concepts has fundamental differences with the original finite versions. In similar veins as Crespo and Tohmé [1], this study contributes to the debate on the traditional foundations of the mathematical tools in economics.

The first problem I consider is on the existence of equilibrium points in noncooperative finite games which was solved by Nash [2, 3] who showed that the mixed extension of the game has a Kakutani fixed point and thus an equilibrium point. While this result is one of the most celebrated results in game theory, it is well known that it does not guarantee the existence of a fixed point in pure strategies. In particular, Nash theorem for finite games only guarantees that an equilibrium (pure or mixed) exists, and therefore, the problem of an existence of equilibrium points in pure strategies for finite games remains open. Although many subsequent studies have provided sufficient conditions for the existence of pure strategies equilibrium (for example, see Rosenthal [4], Monderer and Shapley [5], Topkis [6], and Limura and Wanatabe [7]), to my knowledge, there are no known characterizations of pure strategy NE for finite games. There are also several generalizations of the Nash theorem that do not work if the strategy space is restricted to be a finite set [8, 9]. In this study, I will show some obvious difficulties that one may encounter while pursuing this line of research and will argue that the existence of pure strategies in the class of games studied by Nash remains an open, relevant, and a forgotten problem in game theory.

The second problem we consider relates to Morris [10] who developed a framework, based on Savage's approach [11], to decision theory to show that for a general class of models of information and knowledge, the properties of belief can be related to the properties of preference. Morris showed that one of the most fundamental axioms called the "knowledge" axiom relates to a property called nontriviality, which requires that preferences at a particular state are sensitive to what happens in that state. We will go one step further and demonstrate that in a finite framework, the knowledge axiom is equivalent to a property that can only be verified on complements of singletons. We then argue that when Morris framework is generalized to an infinite model following the Epstein and Wang' [12] methodology, the knowledge axiom is no longer verifiable as it becomes vacuously true. We then give some suggestions on some possible formulations of an infinite version of the knowledge axiom.

The rest of this study is organized as follows. In the next section, we demonstrate the gap between the existence of Nash equilibrium in mixed (infinite) strategies and in pure (finite) strategies in finite games. In Section 3, we discuss the knowledge axiom of the Morris framework, while Section 4 concludes the study.

2. Existence of Nash Equilibria in Finite Games

We consider a finite noncooperative game with n players, denoted by $\Gamma = (I, \{S_i\}_{i\in I}, \{u_i\}_{i\in I})$, where the set of players is denoted by $I = \{1, \ldots, n\}$, S_i the finite strategy space available to player i, and we also denote the set of all strategy profiles by S, so that each $s \in S$ is an n-tuple, where $S = \prod_{i=1}^{n} S_i$. Moreover, payoffs are defined by the vector-valued function $u\colon S \longrightarrow \mathbb{R}^n$. At this stage, it is important to note that it is the finiteness assumption of S_i that makes the game finite and that we do not consider games where S_i is infinite (countable or uncountable) in this study. The following definitions will be useful. We let $\Sigma_i \equiv \Delta S_i$ denote the set of mixed strategies available to player i. We denote a mixed strategy profile by $\sigma \in \prod_{i=1}^{n} \Sigma_i \equiv \Sigma$. For each player i, we define its best response map with respect to some strategy profile σ given by

$$\mathrm{BR}_i(\sigma) = \left\{ \sigma_i^* \in \sum_i : \forall \sigma_i \in \sum_i, u_i(\sigma_i^*, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i}) \right\} \tag{1}$$

.

We then define a Kakutani best response correspondence as follows.

$$\mathrm{BR}(\sigma) = \prod_{i=1}^{n} \mathrm{BR}_i(\sigma). \tag{2}$$

It is well known that the above correspondence has a fixed point, known as the Nash equilibrium of the game. However, it is also well known that the existence of the Nash equilibrium occurs in the set Σ, not necessarily in S. Therefore, the celebrated Nash theorem proves the existence of an equilibrium that cannot distinguish a pure strategy from a mixed strategy. To emphasize this point, we can consider the following three well known games: the Prisoners Dilemma, the Battle of Sexes, and the game of Matching Pennies. Although from the Nash theorem we know that all three games have at least one mixed strategy equilibrium, it cannot tell which of the above games have a pure strategy equilibrium. In the current formulation, a pure strategy is special case of a mixed strategy.

It is therefore desirable to have a theorem that can make such distinction. Although many subsequent studies have generalized the existence problem, to our knowledge, there are no known characterizations of pure strategy NE for finite games. An obvious difficulty that one may encounter while pursuing this line of research is that there is a lack of topological structure when the strategy space (without mixed extension) is a finite set. The fixed points theorems used in the game theory literature rely heavily on the machinery of the topological structure of either the payoff function and/or strategy space. On the other hand, existing results that can be applied to finite strategy sets assume other mathematical structures on the strategy space (e.g., supermodularity [6, 13], potential games [5], or symmetry and integer sets [7]). Unfortunately, these results do not help distinguish the type (pure of mixed) of equilibria in the Prisoners Dilemma, the Battle of Sexes, and the game of Matching Pennies. In order to accomplish this, we need a theorem that gives both a necessary and sufficient condition for the existence of a pure strategy Nash equilibrium in finite games.

One rare fixed-point theorem that considers finite sets is Abian's [14] fixed point theorem for functions on finite sets. Abian's theorem states that "for a given a function from a finite set into itself, the set cannot be partitioned into three sets so that the intersection of each of these sets with their image is empty if and only if the mapping has a fixed point." Luckraz [13] showed that two sets could suffice if the periodicity of the mapping is even. While Abian's theorem is useful in many applications in discrete mathematics, the result cannot be used in cases where the mapping is multivalued. For example, in the context of the problem at the hand, it is well known that the Nash best response mapping is not always single-valued. Nevertheless, in special cases where the best response is a function, the Abian theorem can successfully demarcate pure strategies from mixed strategies.

As for correspondences, we show that Abian's partitioning condition does not generalize to multivalued maps as follows.

Let X be a nonempty finite set and let $F\colon X \longrightarrow P(X)$ be a nonempty-valued correspondence. Often, we denote the correspondence as $F\colon X \twoheadrightarrow X$. For any $A \subseteq X$, let $F(A) = \bigcup_{x\in A} F(x)$. We first show that the following claim on the generalization of Abian's theorem is not true.

Claim 1. F has a fixed point iff X cannot be partitioned into three sets A, B, and C, such that $A \cap F(A) = B \cap F(B) = C \cap F(C) = \varnothing$.

The negation of this claim will state that the mapping has no fixed points iff such a partitioning exists. The following counter example shows that the claim is not true.

Example 1. Let $X = \{a, b, c, d\}$, $F(a) = \{b, c\}$, $F(b) = \{c\}$, $F(c) = \{b, d\}$, and $F(d) = \{a, b\}$. Then, clearly, F has no fixed points. Then, by the negation of Claim 1, we should be able to find a partition of X into sets A, B, and C, such that that $A \cap F(A) = B \cap F(B) = C \cap F(C) = \varnothing$. Suppose such three sets exist, then from the given correspondence F, a, b, and c need to be in three different sets. But d cannot be in any of these three sets which is a counter example to Claim 1.

The above example can be generalized $n \geq 3$ partitions. We first give the following definition.

Definition 1. Let X be a nonempty finite set, and let $F: X \longrightarrow P(X)$ be a nonempty-valued correspondence. Then collection $\mathcal{X} = \{X_i\}_{i\in\{1,..,k\}}$ is called a trivial partition of X iff $k = |X|$ and each X_i is a singleton.

Therefore, if the cardinality of X is n and the mapping has no fixed point, then there always exists a partition of X into n sets that satisfies Abian's condition. In the next example, we show there exists some mapping F that has no fixed points and such that only the trivial partition satisfies the Abian condition.

Example 2. Let X be a nonempty finite set, and let $\mathcal{F}$ denote the set of all correspondences from X to X. Then, there exist some $F \in \mathcal{F}$ that has no fixed points and that does not satisfy Abian's partitioning condition for any number of sets strictly less than $|X|$. Indeed, since X is finite, it can be enumerated as follows: $X = \{x_i\}_{i\in\{1,\ldots,|X|\}}$. For each pair i, j, let $x_i \in F(x_j)$ iff $i \neq j$. Then, clearly, X cannot be partitioned into n sets, such that $n < |X|$ and that satisfy Abian's condition.

We end this section on this negative note: while significant progress has been made on the existence of equilibrium in games with rich topological structures, very little progress has been made on the most fundamental problem, which is the existence of pure strategy equilibria in finite games.

3. The Logic of Beliefs

Morris [10] develops a framework to show that, for a general class of models of information and knowledge, the properties of belief can be related to the properties of preference. In that framework, Morris shows that the truth axiom is implied by a property called nontriviality. We will use a strengthening of this connection to show that when the Morris framework is generalized to infinity, using the Epstein and Wang [12] framework, the truth axiom becomes a property that holds true vacuously, in that it can only be violated on a set of measure zero.

Following Morris [10], we let Ω be any set of states, and we define the belief operator by mapping $B: 2^{\Omega} \longrightarrow 2^{\Omega}$ satisfying (i) $B(\varnothing) = \varnothing$, (ii) $B(\Omega) = \Omega$, (iii) $B(E \cap F) = B(E) \cap B(F)$, and (iv) $E \subseteq F \Rightarrow B(E) \subseteq B(F)$.

3.1. The Finite Case. For the finite case, we use the following definitions from Morris. For preference relation $\{\succsim_\omega\}_{\omega\in\Omega}$, the belief operator is defined as follows.

$$B(E) = \left\{\omega\colon\ (x_E, y_{-E}) \sim_\omega (x_E, z_{-E}) \text{ for all } x, y, z \in \mathbb{R}^{\Omega}\right\}. \quad (3)$$

3.2. Truth Axiom. Truth axiom (T): $B(A) \subseteq A$ for each $A \in 2^{\Omega}$.

Separation of singletons property (S): $\{\omega\} \cap B(-\{\omega\}) = \varnothing$ for each singleton $\{\omega\} \in 2^{\Omega}$.

For the finite case, the above is nothing but the nontriviality condition defined by Morris. Indeed, belief operator B is said to satisfy the nontriviality condition if

$$(x_\omega, z_{-\omega}) \succ_\omega (y_\omega, z_{-\omega}), \text{ for some } x, y, z \in \mathbb{R}^{\Omega}, \text{ for all } \omega \in \Omega \quad (4)$$

.

Lemma 1. *For any Ω and B satisfying (i)–(iv), if $\{\omega\} \cap B(-\{\omega\}) = \varnothing$ for each singleton $\{\omega\} \in 2^{\Omega}$, then $B(\{\omega\}) \subseteq \{\omega\}$.*

Proof. First, observe that for each $\omega \in \Omega$, $\{\omega\} \subseteq \{-\omega'\}$ for each $\omega' \neq \omega$. By (iv), $B(\{\omega\}) \subseteq B(\{-\omega'\}_{\omega' \neq \omega})$ for each $\omega \in \Omega$ and each $\omega' \neq \omega$. But by the hypothesis of the lemma, we have $\{\omega'\} \cap B(-\{\omega'\}) = \varnothing$ for each $\omega' \neq \omega$. Therefore,

$$\begin{aligned} B(\{\omega\}) &\subseteq \cap_{\omega' \neq \omega} B(\{-\omega'\}) \\ &\subseteq \{\omega\}. \end{aligned} \quad (5)$$

This completes the proof. □

Theorem 1. *For any Ω and B satisfying (i)–(iv), $S \Leftrightarrow T$.*

Proof (Necessity). By T, we have $B(-\{\omega\}) \subseteq -\{\omega\}$ for each singleton $\{\omega\} \in 2^{\Omega}$. Hence, $\{\omega\} \cap B(-\{\omega\}) = \varnothing$.

(Sufficiency). Let $X \subseteq \Omega$. By (i) and (ii), T is seen to be satisfied when $X = \varnothing$ or $X = \Omega$. Moreover, T is trivially satisfied for all singletons by S by Lemma 1. So, it suffices to consider the following two cases. (1) X is a nonempty strict subset of Ω satisfying $X = \Omega - \{\omega\}$ for some $\{\omega\} \in 2^{\Omega}$ and (2) X is a nonempty strict subset of Ω satisfying $X = \Omega - C$, where C is a nonempty, nonsingleton strict subset of Ω. We first see (1). By S, for each $\omega' \notin X$, we have

$$B(\Omega - \{\omega'\}) \subseteq \Omega - \{\omega'\} \quad (6)$$

.

We next see (2). By the definition of C, we have $X \subseteq \Omega - \{\omega'\}$ for some $\{\omega'\} \in 2^{\Omega}$. Then, it follows from (iv) that

$$\begin{aligned} B(X) &\subseteq B(\Omega - \{\omega'\}) \\ &\subseteq \Omega - \{\omega'\} \text{ by } (1). \end{aligned} \quad (7)$$

Since the above holds for all $\omega' \notin X$, we have

$$B(X) \subseteq \Omega - \{\omega'\}, \text{ for all } \omega' \notin X \quad (8)$$

.

The above implies that $B(X) \subseteq X$.

The following example illustrates Theorem 1. □

Example 3. Let $\Omega = \{a, b, c\}$ and B be given as follows. $B(\varnothing) = \varnothing$, $B(\{a\}) = \varnothing$, $B(\{b\}) = \{b\}$, $B(\{c\}) = \varnothing$, $B(\{a, b\}) = \{a, b\}$, $B(\{a, c\}) = \varnothing$, $B(\{b, c\}) = \{b, c\}$, and $B(\Omega) = \Omega$. It is easy to verify that the truth axiom T is satisfied in this example. The S property is also satisfied since

$$\begin{aligned} \{a\} \cap B(\{b, c\}) &= \varnothing, \\ \{b\} \cap B(\{a, c\}) &= \varnothing, \\ \{c\} \cap B(\{a, b\}) &= \varnothing. \end{aligned} \quad (9)$$

Now, consider the following property.

One state deviation property: $\omega \in B(-\{\omega\})$ for some $\omega \in \Omega$.

The above theorem implies the following. The one state deviation property is true if and only if the truth axiom is violated. It is precisely this property which will enable us to show that the truth (knowledge) axiom can only be violated on a set of measure zero in the infinite case. In order to do that, we need to define the infinite model formally.

3.3. The Infinite Case. For the infinite case, we follow Epstein and Wang [12]. In particular, they consider the case where Ω is a compact Hausdorff space, endowed with the Borel σ-algebra. An act is a Borel measurable function: $f: \Omega \longrightarrow [0,1]$. Let $\mathscr{F}(\Omega)$ denote the set of all acts, and let $\mathscr{P}(\Omega)$ denote the set of all preferences over $\mathscr{F}(\Omega)$. We restrict $\mathscr{P}(\Omega)$ to the set of utility functions $u \in \mathscr{P}(\Omega)$ and $u^{\omega} \in \mathscr{P}(\Omega)$ for each ω, satisfying the following.

Certainty equivalence is

$$u(r) = r, \text{for all}\, r \in [0,1] \quad (10)$$

Weak monotonicity is,

$$f' \in \mathscr{F}(\Omega) \quad (11)$$

Inner regularity is

$$u(f) = \sup\{u(g)\colon g \leq f, g \in \mathscr{F}^{u}(\Omega)\}, \text{for all}\, f \in \mathscr{F}(\Omega) \quad (12)$$

Outer regularity is

$$u(g) = \inf\left\{u(h)\colon h \geq g, h \in \mathscr{F}^{l}(\Omega)\right\}, \text{for all}\, g \in \mathscr{F}^{u}(\Omega). \quad (13)$$

where $\mathscr{F}^{u}(\Omega)$ and $\mathscr{F}^{l}(\Omega)$ denote the sets of all simple upper-semicontinuous and lower-semicontinuous acts, respectively.

For a closed event E, we let $\mathscr{P}(\Omega \mid E)$ denote the set of preferences for which the complement of E is Savage-null. Let $\mathscr{P}^{*}(\Omega \mid E)$ be the extension of $\mathscr{P}(\Omega \mid E)$ to all subsets of E of Ω. The belief operator is given as follows.

$$B(E) = \left\{\omega\colon u^{\omega} \in \mathscr{P}(\Omega \mid \bar{E})\, \text{for some close d set}\, \bar{E} \subseteq E\right\} \quad (14)$$

.

For the infinite case, the S property is equivalent to the following.

For each $\{\omega\} \in 2^{\Omega}$, there exist no closed set $\bar{E} \subseteq -\{\omega\}$, such that $u^{\omega} \in \mathscr{P}(\Omega \mid \bar{E})$.

Note that if $u^{\omega} \notin \mathscr{P}(\Omega \mid \bar{E})$ for any closed set $\bar{E} \subseteq -\{\omega\}$, then $u^{\omega} \notin \mathscr{P}^{*}(\Omega \mid -\{\omega\})$, where $\mathscr{P}^{*}(\Omega \mid .)$ is the extension of $\mathscr{P}(\Omega \mid .)$ to all subsets of Ω. Alternatively, one could approximate $-\{\omega\}$ in the following way. Since $\{\omega\}$ is Lebesgue measurable, so is $-\{\omega\}$. Therefore, its inner measure must be equal to its outer measure. Taking the inner measure to be the supremum over all compact subsets of $-\{\omega\}$, we obtain a compact and hence a closed set M that is equal to $-\{\omega\}$ almost everywhere. Then, $u^{\omega} \in \mathscr{P}(\Omega \mid M)$ iff $u^{\omega} \in \mathscr{P}^{*}(\Omega \mid -\{\omega\})$.

It follows from the above adaptation of the S property to the infinite model that the one state deviation property, which holds if and only if the truth axiom is violated, can only occur on a set of measure zero. As a result, the knowledge axiom can only be violated on a set of measure zero. This renders the concept of the truth axiom useless in the infinite formulation. We conclude by proposing a weaker notion of the truth axiom that can potentially resolve the above problem.

Uncountable states deviation property: there exists an uncountable set K, such that $K \subseteq B(-K)$.

Weak truth axiom: for every uncountable set K, $K \nsubseteq B(-K)$.

Remark 1. T implies weak truth axiom.

Proof. Suppose the truth axiom holds. Then, we need to show that for all uncountable set K, $K \nsubseteq B(-K)$. Now, each uncountable set K has some $\{\omega\}$ as its subset. By Theorem 1, $\{\omega\} \cap B(-\{\omega\}) = \varnothing$. Now, since $-K \subseteq -\{\omega\}$, by (iv), $B(-K) \subseteq B(-\{\omega\})$. Thus, if $\omega \in K$, we have $\omega \notin B(-K)$ since $\omega \notin B(-\{\omega\})$.

The above shows that using an infinite “allegedly” more general framework does not always come without loss of generality. A well-defined concept in a finite model does not necessarily carry over to an infinite model. □

4. Conclusion

While it is hard to imagine the world of economic methodology without the continuum or the assumption of infinity, the jump from a finite model to an infinite does come at the expense of some loss in generality in some cases. We gave two instances of such occurrences in some well-known frameworks used in Economics. In particular, we showed that the existence result in finite games does not come with a distinction between a pure strategy equilibrium and a purely mixed strategy equilibrium. We also showed that in well-established models of knowledge and beliefs, an important axiom known as the knowledge axiom does not fit well in an infinite framework. We would therefore like to draw the attention of researchers at the frontier of economic methodology that the infinite assumption is not completely free.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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