Research Article

On a Fixed Point Theorem for General Multivalued Mappings on Finite Sets with Applications in Game Theory

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Received 23 January 2022; Revised 25 March 2022; Accepted 1 April 2022; Published 2 June 2022

1. Introduction

The finite version of Abian’s [1] fixed point theorem states that given a function from a finite set into itself, the set cannot be partitioned into three sets so that the intersection of each of these sets with their image is empty if and only if the mapping has a fixed point. Although this result can be applied to all functions that have a finite domain, it can be shown that it does not generalize to multivalued mappings which are often referred to as correspondences (1 see [2] for a proof of this claim). As a result, Abian’s theorem cannot be applied to problems in game theory where the best response map is multivalued. In this paper, we solve this problem by providing a new fixed point theorem that can be defined over any abstract finite set and that works for multivalued maps. In its most general form, our theorem completely characterizes the fixed point problem for correspondences over finite sets.

The fixed point theorem we propose, when put in the context of the widely studied class of finite games, can help fill the gap between the existence of a completely mixed strategy equilibrium and the existence of a pure strategy equilibrium as it is well known that the existence theorem of Nash (1950, 1951) [3, 4] does not distinguish between the two (2 although from the Nash theorem, we know that the Prisoner’s Dilemma, the Battle of Sexes, and the game of Matching Pennies have equilibrium points, we cannot tell which of these games have pure strategy equilibria). In fact, although the Nash theorem is one of the most important results in game theory, it only guarantees that an equilibrium (pure or mixed) exists and therefore a part of the problem of existence of equilibrium points in pure strategies remains unsolved. While most of the subsequent research on the problem of existence of equilibrium points in games have generalized Nash’s theorem in more general and abstract infinite spaces, (for example, see [5–17, 19–26]), fewer studies have retained the finite strategy space assumption (3 for example, see [27–32]). (It is important to note that the generalization of the problem to infinite spaces with a richer topological structure does not solve the pure strategy equilibrium problem when the space is finite and mixed strategies are not allowed). Nevertheless, as observed by [2], the latter work do not give a complete characterization of equilibrium points in finite games as they only provide sufficient conditions and hence, there are no known characterizations of pure strategy equilibria for finite games.

It was argued by [2] that it is the lack of a topological structure when the strategy space (without mixed extension) is a finite set which renders the problem at hand difficult. [2] also observes that Abian’s partitioning method does not work for multivalued maps. In this paper, we solve this problem by using an equivalent fixed point result due to the

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1 Abian’s theorem

2 Nash’s theorem

3 Finite games

4 Generalized Nash’s theorem

5 Infinite spaces
2. Abian’s Theorem and Multivalued Maps

In this section, we introduce the fixed point theorem due to Abian and show that its generalization to multivalued mappings on finite sets fails.

2.1. Abian’s Theorem. Let \( X \) be a nonempty finite set and let \( f: X \rightarrow X \). Then, \( f \) has a fixed point if \( X \) cannot be partitioned into three sets \( A, B, \) and \( C \) such that \( A \cap F(A) = B \cap F(B) = C \cap F(C) = \emptyset \).

It was showed by [33] that when the periodicity of \( f \) is even, then two sets suffice in the statement of Abian’s theorem. We now consider the above statement when the mapping is generalized to a set-valued map. For a given nonempty finite set \( X \), let \( F: X \rightarrow P(X) \), where \( P(X) \) is the power set of \( X \), be a nonempty valued correspondence. At times, we denote the correspondence by \( F: X \rightarrow X \). [2] recently showed, using a counterexample, that it is not possible to characterize the fixed points of \( F \) using Abian’s three set partitioning method and as a result, an Abian-like theorem cannot be given for multivalued mappings on finite sets. The authors of [2] also went one step further to show that the theorem does not generalize even if the number of partitions is increased. In fact, it was shown that only the trivial partition (singletons) always works for multivalued maps.

The above negative result and the desirability of a fixed point for correspondences over finite domains that can distinguish a pure strategy equilibrium from the set of mixed strategy equilibria in Nash’s theorem are our main motivations for this paper.

3. Fixed Point Theorem for Multivalued Maps on Finite Sets

For finite set \( X \), let \( F: X \rightarrow P(X) \) be a nonempty valued correspondence as defined in the previous section. From \( F \), we define digraph \( G_F = (X, E) \), where \( E \subseteq X \times X \) is the set of the edges of \( G_F \), satisfying the following. (5 See [47] for more details on digraphs).

\[
\forall x, y \in X, \quad (x, y) \in E \text{ iff } y \in F(x).
\]

We assign weights to the edges of \( E \) through mapping \( \lambda \), so that \( \lambda: E \rightarrow [0, 1] \) is defined as follows.

\[
\forall e = (x, y) \in E, \quad \lambda((x, y)) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}
\]

Let \( T_{G_F}(X) \) denote the set of all infinite walks of \( G_F \). For a given finite walk \( w = \langle w_j \rangle_{j=1}^k \) in \( G_F \), we denote by \( l(w) \) the total weight of edges it encounters, with the convention that if an edge is repeated, then the total weight of the number of its occurrences contributes to \( l(w) \). Formally, for each finite walk \( w = \langle w_j \rangle_{j=1}^k \), the length of finite walk \( w \), denoted by \( l(w) \), is defined as follows.
For any vertices \( x, y \in X \), let \( W(x, y) \) be the set of all finite walks connecting \( x \) and \( y \). Then, we can define the distance between \( x \) and \( y \) as follows.

\[
d(x, y) = \begin{cases} 
\inf_{w \in W(x,y)} d(w) & \text{if } W(x, y) \neq \emptyset, \\
+\infty & \text{otherwise}. 
\end{cases}
\] (5)

We say that an infinite walk \( s = \langle s_j \rangle_{j=1}^{\infty} \in T_{G_F}(X) \) is Cauchy if for every \( \varepsilon > 0 \), there exist some natural number \( N \) such that for all \( m, n > N \), we have \( d(s_m, s_n) < \varepsilon \). We say that a graph \( G_F \) is closed if every sequence \( s \in T_{G_F}(X) \) is Cauchy.

**Theorem 1.** If \( G_F \) is a closed graph, then \( F \) has a fixed point.

**Proof.** We prove by contradiction. Suppose that \( G_F \) is closed but \( F \) has no fixed points. Then, for all \( x \in X \), we have \( x \not\in F(x) \). As a result, there exist some \( s = \langle s_j \rangle_{j=1}^{\infty} \in T_{G_F}(X) \) such that for each \( j, F(s_j) \neq s_{j+1} \). Let \( \varepsilon = 1/2 \). Then, since \( G_F \) is a closed graph, for every natural \( N \), there exist \( m, n > N \), such that \( m = j \) and \( n = j + 1 \) for some \( j \) th term of \( s \). Now, since \( s_{j+1} \neq s_j \), we have \( d(s_j, s_{j+1}) = 1 > 1/2 = \varepsilon \). As a result, this would imply that \( s \) is not Cauchy, and hence, \( G_F \) is not closed, which is a contradiction.

Theorem 1 gives a sufficient condition for the existence of fixed points for multivalued mappings on finite sets. The closed graph condition can be seen as the counterpart of the upper-hemicontinuity assumption of Kakutani’s fixed point theorem (see [33]). When the relationship defined by the mapping is represented by a digraph, the closed graph condition guarantees the existence of fixed points. The next result makes this closed graph condition more transparent and relates it directly to the acyclicity of the graph. We first define cycles more formally as follows.

We define the graph \( G_F \) has a cycle if it has some vertex \( x' \) and finite walk \( w = \langle w_j \rangle_{j=1}^{\infty} \in W(X') \) satisfying the following: For all \( m, n \in \{1, \ldots, k\} \), where \( k > 2 \), \( m \neq n \) imply \( w_m \neq w_n \), except at \( m = 1 \) and \( n = k \), where \( w_1 = w_k = x' \). (6) Although the graph may have longer cycles with repeated vertices, the condition used in this definition is necessary for any kind of cycles to exist).

**Theorem 2.** If \( G_F \) is a closed graph, then \( F \) is acyclic.

**Proof.** We prove by contradiction. Suppose that \( G_F \) is closed but contains some cycle. Then, \( G_F \) has some vertex \( x' \) and finite walk \( w = \langle w_j \rangle_{j=1}^{\infty} \in W(X') \) satisfying for all \( m, n \in \{1, \ldots, k\} \), where \( k > 2 \), \( m \neq n \) imply \( w_m \neq w_n \), except at \( m = 1 \) and \( n = k \). Concatenating \( w \) with \( w \) ad infinitum, we obtain \( s = \langle w; w; \ldots \rangle \). Hence, \( s = \langle s_j \rangle_{j=1}^{\infty} \in T_{G_F}(X) \) such that for each \( j, F(s_j) \neq s_{j+1} \). Then, we can use an argument similar to that in the proof of Theorem 1 to show that \( s \) is not Cauchy, and therefore, \( G_F \) is not closed, which is a contradiction.

Theorem 2 shows that the closed graph condition relates the acyclicity of the mapping which in turn relates to Abian’s fixed point theorem ([33]). Since from the standpoint of directed graphs the acyclicity condition may be easier to verify, Theorem 2 can be useful in the development of graph theoretic fixed point search algorithms.

Since our aim is to achieve a result that is parallel to Abian’s theorem for multivalued maps, the next theorem gives a complete characterization of fixed points for multivalued maps. The following definitions will be useful in establishing this result. We first make use of the notion of distance between two vertices to define an index on \( G_F \). For any given walk \( w = \langle w_j \rangle_{j=1}^{\infty} \in G_F \), let \( m(w) \) be defined as follows. (7) Here, \( w \) can be finite or infinite.

\[
m(w) = \begin{cases} 
\inf_{i\neq j, n} d(w_i, w_j) & \text{if } \inf_{i\neq j, n} d(w_i, w_j) \neq 0, \\
+\infty & \text{otherwise}. 
\end{cases}
\]

We will use \( m \) to define an index on \( G_F \). We first index an element of \( T_{G_F}(X) \) by \( w^j \) for \( j \in K \), where \( K \) can be countable or uncountable, so that \( T_{G_F}(X) = \{ w^j \}_{j\in K} \). We can now define index \( M \) on \( G_F \) as follows:

\[
M(G_F) = \begin{cases} 
1 & \text{if } m(w^j) \neq +\infty, \\
+\infty & \text{otherwise}. 
\end{cases}
\]

**Theorem 3.** \( F \) has a fixed point if \( M(G_F) = +\infty \).

**Proof.** (Only if) Suppose \( F \) has a fixed point. Then, there exist some Cauchy sequence in \( T_{G_F}(X) \). An example of such a sequence is \( s = \langle s; x, x, \ldots \rangle \). For such \( s \), we have \( \inf_{i\neq j, n} d(w_i, w_j) = 0 \) and therefore, \( m(s) = +\infty \), which in turn implies that \( M(G_F) = +\infty \).

(If) We prove by contradiction. Suppose that \( M(G_F) = +\infty \) but \( F \) has no fixed points. Then, we claim that there are no Cauchy sequences in \( T_{G_F}(X) \). Indeed, suppose that \( s = \langle s_k \rangle_{k=1}^{\infty} \) were some Cauchy sequence in \( T_{G_F}(X) \), then for all \( e \in (0, 1) \), there would exist some \( N \) such that for all \( i, j > N, d(s_i, s_j) < e \). However, this would have implied that for some \( k^{th} \) term of the sequence, we would have \( s_k = s_{k+1} \), which would contradict the fact that \( F \) has no fixed points. Therefore, all sequences of \( T_{G_F}(X) \) are non-Cauchy.

Now, since \( M(G_F) = +\infty \), there must exist some \( s \in T_{G_F}(X) \) such that \( m(s) = +\infty \). Hence, for such an \( s \), \( \inf_{i\neq j, n} d(w_i, w_j) = 0 \), which implies that there exist some edge \( e = \langle x, y \rangle \in G_F \) satisfying \( x = y \). Then, we can construct Cauchy sequence \( \langle x, x, \ldots \rangle \in T_{G_F}(X) \), which leads to a contradiction.

\[
\langle s_k \rangle_{k=1}^{\infty}
\]
The above theorem reduces the fixed point problem to the construction of an index on the digraph induced by the mapping. The index we construct gives a “measure” to each walk in \( T_G (X) \) and it is shown that a fixed point exists if and only if this measure is infinite. Our result can also be seen as a finite interpretation of the well-known Lefschetz fixed point theorem [48] which uses an index that counts the loops of the underlying topological space. (8 Fixed point indices for mappings on topological spaces was invented by Lefschetz (1937).

We next show that a strengthening of Theorem 1 can also be given as a characterization of fixed points for multivalued maps.

**Theorem 4.** \( F \) has a fixed point if \( G_F \) has a Cauchy sequence. The proof is similar to the proof of Theorem 1.

Theorem 4 shows that as long as the digraph induced by the mapping has a Cauchy sequence, the map has a fixed point. We will show that Theorem 4 can be used in a straightforward manner to prove the existence of Nash equilibria in finite games.

**Remark 1.** Theorems 1–4 readily generalize to mappings with countable domains as the assumption of finiteness was not used in the proofs.

### 4. Characterizing Pure Strategy Nash Equilibria in Finite Games

In this section, we apply the theorems obtained in the previous section in the context of finite noncooperative games. We denote a finite noncooperative game by triple \( \Gamma = (I, \{ S_i \}_{i \in I}, \{ u_i \}_{i \in I}) \), where the set of players is denoted by \( I = \{1, \ldots, n\} \) and \( S_i \) the finite strategy space available to player \( i \). We also denote the set of all strategy profiles by \( S \) so that each \( s \in S \) is an \( n \)-tuple, where \( S = \prod_{i=1}^{n} S_i \). Finally, payoffs are defined by the vector-valued function \( u: S \rightarrow \mathbb{R}^{\ast} \). (Note that assuming strictly positive payoffs for each player does not lead to any loss of generality). Our aim will be to use the results obtained in the previous section to derive a necessary and sufficient condition for the existence of a pure strategy Nash equilibrium (NE) in any general finite game. As remarked by [2], the lack of the topological structure of space \( S \), which can be an abstract set, hinders the usage of the powerful toolbox of functional analysis for fixed point theorems. (10 For example, games like the Prisoner’s Dilemma, The Battle of Sexes, and the Matching Pennies have abstract strategy spaces). Moreover, since we want to be able to distinguish pure strategies NE from mixed strategies NE, we cannot rely on the topological structure of the mixed extension as is typically done in the literature.

#### 4.1. Fixed Points of the Unilateral Best Response Correspondence

The following definitions will be useful in establishing our results. We define the best response map of player \( i \) with respect to some strategy profile \( s \) by the following:

\[
BR_i(s) = \{ s_i^* \in S_i : \forall s_i \in S_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \}.
\]  

From \( BR_i(s) \), we define the best response correspondence of \( \Gamma \) by \( BR(s) = \prod_{i=1}^{n} BR_i(s) \). We can then define a unilateral best response correspondence \( UNBR: S \rightarrow S \) as follows:

\[
UNBR(s) = \begin{cases} 
\{ s' = (s_i', s_{-i}) \in S \mid s_i' \in BR_i(s) \text{ and } u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \text{ for some } i \}, \\
\{ s \} \text{ otherwise.}
\end{cases}
\]

While it is well known that the game has a Nash equilibrium if and only if the best response map \( BR \) has a fixed point, it is straightforward to show that \( UNBR \) has a fixed point if and only if the game has a Nash equilibrium. Indeed, suppose that \( s \) is a NE. Then, there exists no \( i \) such that \( u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \). Thus, \( UNBR(s) = \{ s \} \) is a fixed point. Next, suppose that \( UNBR(s) \) has a fixed point, say \( s \). Then, no \( (s_i', s_{-i}) \in S \) satisfying \( u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \) are in \( UNBR(s) \). Hence, \( s \in UNBR(s) \) is a NE. (11 Our notion of UNBR is in similar veins as the concepts of better (or best) improvement paths studied by [30, 34]).

For game \( \Gamma = (I, \{ S_i \}_{i \in I}, \{ u_i \}_{i \in I}) \), we define function \( u: S \rightarrow \mathbb{R} \) as follows:

\[
u(s) = \sum_{i=1}^{n} u_i(s).
\]

We then say that game \( \Gamma = (I, \{ S_i \}_{i \in I}, \{ u_i \}_{i \in I}) \) is a payoff sum separable if \( u \) is a bijection from \( S \) into \( \text{Range}(u) \).

Since \( UNBR(s) \) is a correspondence from a finite set into itself, we can define the directed graph \( G_{UNBR(s)} \) as in the previous section and we can let \( T_G (u) \) be the set of all infinite walks of \( G_{UNBR(s)} \). We then have the following definition.

We say that payoff sum function \( u \) converges weakly if for some sequence \( t = \langle s^k \rangle_{k=1}^{\infty} \) in \( T_G (u) \), sequence \( \langle u(s^k) \rangle_{k=1}^{\infty} \) converges to some point in \( \text{Range}(u) \).

The next theorem achieves our objective of completely characterizing pure strategy NE in finite games for the class of payoff-sum separable games.

**Theorem 6.** Suppose that \( \Gamma \) is a payoff-sum separable. Then, \( \Gamma \) has a pure strategy NE if \( u \) converges weakly.

**Proof.** (If) We show that if \( \langle u(s^k) \rangle_{k=1}^{\infty} \) converges to some \( u^* \) in the range of \( u \) for some sequence \( t = \langle s^k \rangle_{k=1}^{\infty} \in T_G (u) \) of graph \( G_{UNBR(s)} \), then \( UNBR(s) \) has a fixed point. Since
\[ \langle u(s^k) \rangle_{k=1} \text{ converges to } u^*, \text{ for every } \epsilon > 0, \text{ there exist integer } N \text{ such that for all } m > N \text{ we have the following:} \]
\[ |u(s^m) - u^*| < \epsilon. \quad (11) \]

Since \( u^* \) is in the range of \( u \), there exist some \( s^* \) satisfying \( u^* = u(s^*) \). Let
\[ \epsilon = \min_{s,t \in S, s \neq t} \left| \left[ u(s) - u(s') \right] \right|. \quad (12) \]

Since the game is payoff-sum separable and finite, \( \epsilon > 0 \). Thus, if \( |u(s^m) - u^*(s^*)| < \epsilon \) holds for all \( m > N \) along the sequence, we must have \( s^m = s^* \) for all \( m > N \) since
\[ \left| u(s^m) - u^*(s^*) \right| < \epsilon < |u(s^m) - u(s^*)| \quad \text{for all } s^* \neq s^m. \quad (13) \]

Therefore, \( t \) is a Cauchy sequence and by Theorem 4, \( UNBR(s) \) has a fixed point. Hence, \( \Gamma \) has a NE.

(Only If) Suppose that \( \Gamma \) has a NE, say \( s^* \). Then, some Cauchy sequence \( t = \langle s^k \rangle_{k=0}^\infty = \langle s^*, s^*, \ldots \rangle \in T_{\Gamma, \text{NE}, \text{NE}} \) exists. Along this sequence, the payoff sum function sequence \( \langle u(s^k) \rangle_{k=1} \) converges to \( u(s^*) \).

While Theorem 6 gives us a characterization of \( \Gamma \), it is restricted to the class of payoff sum separable games. We next show that the result can be generalized to all finite games. For that, we need the following definitions.

We say that \( \Gamma = (I, \{S_i\}_{i \in I}, \{u_i\}_{i \in I}) \) and \( \Gamma' = (I, \{S'_i\}_{i \in I}, \{u'_i\}_{i \in I}) \) are argmax payoff equivalent if the following condition holds:

For every \( i \) and \( s_i \in BR_i(s_{-i}) \) in \( \Gamma \Leftrightarrow s_i \in BR_i(s_{-i}) \) in \( \Gamma' \). \quad (14)

It is to be noted that the only difference between \( \Gamma \) and \( \Gamma' \) is the payoff function. An important implication of this definition is that finding a NE in \( \Gamma \) is equivalent to finding a NE in \( \Gamma' \). The next theorem shows that every finite game has an argmax payoff equivalent game that is payoff separable.

**Theorem 7.** Let \( \Gamma = (I, \{S_i\}_{i \in I}, \{u_i\}_{i \in I}) \) be a finite game. Then, \( \Gamma \) has an argmax payoff equivalent game \( \Gamma' = (I, \{S'_i\}_{i \in I}, \{u'_i\}_{i \in I}) \) that is payoff separable.

**Proof.** For a given \( \Gamma \), we construct an argmax payoff equivalent sum separable game \( \Gamma' = (I, \{S'_i\}_{i \in I}, \{u'_i\}_{i \in I}) \) by modifying the payoff of some player in \( \Gamma \) each time a pair of profiles yield the same payoff sum. The construction is as follows.

(i) Step 1: Fix \( S_i \) for player 1 and enumerate \( S_{-i} \) by \( \{s_{-i}^k\}_{k=1}^M \). For each \( p \), consider set \( S_i \times \{s_{-i}^k\} \). (Note that (Tex translation failed)). For each \( p \) and every pair \( s_j, s'_j \in BR_j(s_{-j}^p) \), if \( u(s_j, s_{-j}^p) = u(s'_j, s_{-j}^p) \), \( s_j, s'_{-j}^p \) can be made payoff sum separable by modifying the payoffs of player 1 so that \( u(s_j, s_{-j}^p) = u(s_j, s_{-j}^p) \). Since pair \( s_j, s'_{-j}^p \notin BR_j(s_{-j}^p) \), \( \Gamma \) remains equivalent to \( \Gamma' \). Since pair \( s_j, s'_{-j}^p \notin BR_j(s_{-j}^p) \), \( \Gamma \) remains equivalent to \( \Gamma' \).

(ii) Step 2: For \( p, q \in \{1, \ldots, M\} \) and \( s_j \in \Gamma \), if \( u(s_j, s_{-j}^p) = u(s_j, s_{-j}^p) \), \( s_j \Leftrightarrow \Gamma \) satisfies \( (\text{i}) \) payoff sum separability and \( (\text{ii}) \) it converges weakly.
5. Applications

In this section, we apply the results obtained in the previous sections to some widely studied classes of games like potential games (Monderer and Shapley (1996) [31]), supermodular games (Topkis (1998) [40] and Vives (1990) [41]) and satisfaction form games (Debreu (1952) [42]). We will show that the existence of equilibrium results of potential games and supermodular games are special cases of Theorem 8.

5.1. Potential Games. Monderer and Shapley (1996) [31] showed that if there exist some function $Q: S \rightarrow \mathbb{R}$ such that

$$\forall i, s_{-i} \in S_{-i},$$

$$\forall s_i, s'_i \in S_i, u_i(s'_i, s_{-i}) - u_i(s_i, s_{-i}) > 0,$$

$$\Leftrightarrow Q(s'_i, s_{-i}) - Q(s_i, s_{-i}) > 0.$$  \hfill (15)  \hfill (16)

Then, the game is a potential game and has a pure strategy NE. Note that the above condition implies that if $u_i(s'_i, s_{-i}) - u_i(s_i, s_{-i}) > 0$ for some $i$, then the inequality holds for all $i$. We now show that if a game is payoff sum separable and has potential, then the hypothesis of Theorem 6 is satisfied.

Theorem 9. Let $\Gamma = (I, \{X, Y\}, [u_i]_{i \in I})$ be a payoff sum separable potential game. Then, it has a Nash equilibrium.

Proof. Suppose that the game is potential, that is, inequalities (15) and (16) imply that $u$ is always increasing along any sequence of $UNBR(s)$. By the finiteness of the strategy space and payoff sum separability, $u$ converges weakly and therefore by Theorem 6, the game has a Nash equilibrium. \hfill □

While the above result is given in terms of a payoff separable game, a theorem analogous to Theorem 8 can be given to show that if any game satisfies the potential game condition, then there exist some argmax payoff equivalent payoff sum separable potential game $\Gamma$ such that inequalities (15) and (16) are preserved and such that $u'$ converges weakly. \hfill □

5.2. Supermodular Games. It is well known that supermodular games (see (Topkis (1998) [40] and Vives (1990) [41]) have the NE property due to the results by Topkis and Tarski. We will construct a simple proof of this theorem for finite games using Theorems 4. For the sake of illustration, we will restrict the analysis to the case of two players and thus, we assume that $\Gamma = (I, \{X, Y\}, [u_i]_{i \in I})$ is a supermodular game, where $I = \{1, 2\}$, $X (Y)$ is an ordered lattice strategy space of player 1 (2) and such that the following increasing differences condition is satisfied by $u_i$ for each $i$. For all $x' \geq x$ and $y' \geq y$, we have the following:

$$u_i(x', y') - u_i(x, y') \geq u_i(x', y) - u_i(x, y).$$  \hfill (17)

We will also make use of the following result due to Topkis.

If $u_i$ is supermodular in $(x, y)$, and $X$ and $Y$ are lattices, then $x^* (y) \equiv \text{Argmax}_{x \in X} u_i(x, y)$ is increasing in $y$ and $y^* (x) = \text{Argmax}_{y \in Y} u_i(x, y)$ is increasing in $x$.

Theorem 10. Let $\Gamma = (I, \{X, Y\}, [u_i]_{i \in I})$ be a payoff sum separable supermodular game. Then, $\Gamma$ has a Nash equilibrium.

Proof. We prove by contradiction by supposing that the game is payoff sum separable and supermodular but does not have a NE. Consider some profile $(x_0, y_0)$ so that both $x_0$ and $y_0$ are the minimum (in the lattice ordering) of $X$ and $Y$ respectively. Since $(x_0, y_0)$ is not NE, without any loss of generality, we can assume that $u_1(x_1, y_0) - u_1(x_0, y_0) > 0$ for some $x_1 \in BR_1((x_0, y_0))$, satisfying $x_1 > x_0$. Now, either $(x_1, y_0)$ is a NE, or by Topkis theorem, there exist some $y_1 > y_0$ such that $y_1 \in BR_2((x_1, y_0))$ satisfying $u_2(x_1, y_1) - u_2(x_1, y_0) > 0$. (Note that at $(x_1, y_0)$, there is no profitable deviation from player 1 and hence, the only deviation can come from player 2.) But since $(x_1, y_1) \gg (x_0, y_0)$, either $(x_1, y_1)$ is a NE or there exist some unilateral deviation that leads to some profile $x'_1, y'_1 > x_1, y_1$.
(x_2, y_2) \gg (x_1, y_1). By repeating the latter argument, one can construct sequence \( \langle (x_i, y_i) \rangle_{i=1}^\infty \) satisfying \( (x_{i+1}, y_{i+1}) \gg (x_i, y_i) \). By the ordering of the lattice, since either \( x_i \) or \( y_i \) is strictly increasing along \( \langle (x_i, y_i) \rangle_{i=1}^\infty \), one can construct some \( s \in T_{\text{GUNBR}} \) that does not have any finite cyclic subsequence. Then, on the one hand, \( s \) must be eventually Cauchy as it does not have any finite cyclic subsequences and on the other hand by payoff sum separability and since the game has no NE, Theorems 6 and 4 imply that \( G_{\text{UNBR}(s)} \) has no Cauchy sequences, a contradiction.

While the above result is given in terms of a payoff separable game, a theorem analogous to Theorem 8 can be given to show that if any game satisfies the supermodularity condition, then there exists some argmax payoff equivalent supermodular game which preserves inequality (17) and satisfies the condition of Theorem 4 so that it has a NE. \( \square \)

5.3. Games in a Satisfaction Form. Our fixed point theorem readily applies to games in satisfaction form as it can establish the existence of a satisfaction equilibrium (SE) when the set of actions of each player is finite. The concept of SE was first introduced by Debreu (1952 [42]) and developed further recently by [42–44, 46] to study the learning behavior of players in games where players can only observe their own payoffs. An SE is an equilibrium in the sense that a player who is satisfied with her payoff has no incentives to deviate from her current action. It is well-known that in the context of electrical engineering (15 for the analysis of quality of service (QoS) in wireless ad hoc networks), SE has proved to be particularly useful. For example, the “players” of the game are often described by radio devices (network components) which can choose among different possible operating configurations with the objective of satisfying some targeted QoS, (16 more recently, SE was used in the fifth generation of cellular communications (5G) for tackling the problem of energy efficiency, spectrum sharing, and transmitting power control (see [44, 45]).

More formally, a game in satisfaction form is given by the triple \( \Lambda = (I, \{S_i\}_{i \in \Lambda}, \{f_{i}\}_{i \in \Lambda}) \), where \( I \) is a finite set of players, each player \( i \) has a finite set of actions denoted by \( S_i \) and preference mapping \( f_i \) given by the following correspondence.

\[
f_i: S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots S_n \longrightarrow 2^{S_i}. \quad (18)
\]

As usual, we denote by \( S = \prod_{i=1}^n S_i \) the set of strategy profiles of the game. We say that \( s^* \in S \) is a satisfaction equilibrium (SE) if \( s^*_i \in f_i \) for all \( i \). We show that Theorems 1, 3, and 4 can be used to prove the existence of an SE in \( \Lambda \). For that, we define mapping \( g_i: S \longrightarrow 2^{S_i} \) which extends \( f_i \) to \( S \) as follows. For each \( s \in S \), let \( g_i(s, s_i) = f_i(s_i) \) and let \( g: S \longrightarrow 2^S \) be a correspondence satisfying \( g(s) = \prod_{i=1}^n g_i(s) \). Then, it is clear that correspondence \( g \) has a fixed point if \( \Lambda = (I, \{S_i\}_{i \in \Lambda}, \{f_{i}\}_{i \in \Lambda}) \) has an SE. (17) Indeed, suppose that \( s^* \) is an SE of \( \Lambda \). Then, for each \( i \), \( s_i^* \in f_i(s_i^*) = g_i(s_i^*) \). Thus, \( s^* \in \prod_{i=1}^n g_i(s^*) \) and hence, it is a fixed point of \( g \). Conversely, suppose that \( s^* \) is a fixed point of \( g \). Then, for each \( i \), we have \( s_i^* \in g_i(s^*) = f_i(s^*) \) and hence, it is an SE. The following results follow from Theorems 1, 3, and 4.

(i) If \( G_g \) is a closed graph, then \( \Lambda \) has an SE.

(ii) \( \Lambda \) has an SE if \( \mathcal{M}(G_g) = +\infty \).

(iii) \( \Lambda \) has an SE if \( G_g \) has a Cauchy sequence.

6. Conclusion

We have shown that the original problem of existence of an equilibrium in finite games can be fully characterized without the need to extend the strategy space to mixed strategies. We proceeded by generalizing Abian’s theorem for correspondences and applying it in the context of finite games. Our result sharpens the celebrated Nash theorem as it can filter out the exact class of games that have pure strategy NE from the class of finite games (for which a NE in mixed strategies always exists). We also show that the existence of equilibrium points problem studied in supermodular games, potential games, and games in satisfaction form follow as special cases of our theorem for the finite case.

Data Availability

This manuscript does not use any data in any form.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The author would like to thank Wong Lee Chin for her support.

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