Hindawi
Journal of Mathematics
Volume 2022, Article ID 2510204, 12 pages
https://doi.org/10.1155/2022/2510204

Research Article

A Convolution-Based Computational Technique for Subdivision Depth of Doo-Sabin Subdivision Surface

Faheem Khan[1] **Bushra Shakoor**[2] **Ghulam Mustafa**[1] **and Sidra Razaq**[1]

[1]*Department of Mathematics, The Islamia University of Bahawalpur, Pakistan*
[2]*Department of Mathematics, University of Sargodha, Pakistan*

Correspondence should be addressed to Ghulam Mustafa; ghulam.mustafa@iub.edu.pk

Received 28 June 2022; Revised 7 August 2022; Accepted 20 August 2022; Published 24 September 2022

Academic Editor: R. U. Gobithaasan

Subdivision surface schemes are used to produce smooth shapes, which are applied for modelling in computer-aided geometric design. In this paper, a new and efficient numerical technique is presented to estimate the error bound and subdivision depth of the uniform Doo-Sabin subdivision scheme. In this technique, first, a result for computing bounds between P^k (a polygon at kth level) and P^∞ (limit surface) of the Doo-Sabin scheme is obtained. After this, subdivision depth (the number of iterations) is computed by using the user-defined error tolerance. In addition, the results of the proposed technique are verified by taking distinct valence numbers of the Doo-Sabin surface scheme.

1. Introduction

The subdivision scheme is a powerful tool to design high-quality smooth curves, surfaces, and other geometrical objects. The subdivision scheme gives the rules to create a new polygon by using the vertices as a linear combination of old vertices for each step. Then, repeating the rule leads to a limit curve or surfaces. Recent work has been done on construction and properties of subdivision curves and surfaces by [1, 2]. Many subdivisions can be regarded as schemes for arbitrary topologies, such as $\sqrt{3}$ subdivision [3], which twice causes a uniform refinement with trisection of every original edge, hence, it is named $\sqrt{3}$ subdivision scheme.

The stencil for the subdivision rule has a minimum size and maximum symmetry. Similarly, much research work for estimating error bounds/subdivision depth has been done, such as Doo-Sabin [4], Catmull-Clark subdivision [5], Loop subdivision [6], 4-8 subdivision [7], quad/triangle subdivision [8, 9], and Butterfly scheme [10, 11]. By given user-defined error tolerance, how many subdivision steps are needed to subdivide the control polygon is called subdivision depth. This error control technique is called subdivision depth computation. It is required in all tessellation-based applications such as subdivision surface trimming, finite element polygon generation, Boolean operators, and surface tessellation for rendering.

Some novel numerical techniques are presented to estimate the error bounds and subdivision depths only for binary subdivision curves and surfaces [12–14]. Still, there is space to extend this work to find the error bound/subdivision depth for well-known subdivision schemes for arbitrary topology.

It is valuable for precomputing the error bounds and subdivision depth of subdivision curves/surfaces in advance in many engineering applications such as curve/surface intersection, mesh generation, NC machining, and surface rendering [15]. In this paper, we answer the question: how well does the Do-Sabin scheme approximate the initial polygon/polyhedron to the limiting curve/surface? In another way, given an error tolerance, how many levels of subdivision should be performed on the initial polygon/polyhedron so that the error between the resulting control polygon/polyhedron and the limiting curve/surface would be less than the error tolerance? So, an efficient and optimal approach is presented to estimate error bounds and subdivision depth for a uniform Doo-Sabin surface scheme. For a good understanding of the present work, a brief introduction of the Doo-Sabin scheme is presented. The main feature of the paper is to demonstrate the proposed results with the help of some experimental tests on distinct valence numbers of Doo-Sabin surfaces. Graphical representations are also given to illustrate the efficiency of the proposed work. In the end, the conclusion of the work is presented.

2. Analysis of Uniform Doo-Sabin Scheme

The uniform biquadratic B-spline is used to construct Doo-Sabin subdivision scheme on arbitrary control grids. The rules for the scheme are an iterative process that takes arbitrary mesh as an input and applies the rules to get a new refined mesh. In this scheme, each vertex of each face at the k-th stage generates a new vertex point of the $(k+1)$ -th stage as a linear combination of vertices of the same face. For each face, connect the new refined points that have been generated for each vertex of the face to form a new refined n face. Connect the newly obtained points that were generated for the faces adjacent to each vertex to form a valence of n. Finally, connect the new points to obtain the new edges that have been generated for the new faces. The scheme of computation for Doo-Sabin surfaces is shown in Figure 1.

In the control mesh (or polygon, polyhedron), the number of vertices directly adjacent to a vertex is called the vertex's valence. If it has a vertex of valence four, it is called a regular; otherwise, it is called an irregular. Similarly, any face with four sides (face valence is four) is regular, otherwise, irregular. In the Doo-Sabin scheme, at each subdivision step, new vertices are generated and then connected to form new faces. After each subdivision, every vertex of a refined polygon will be regular, and the number of irregular faces will remain constant.

The subdivision rule of Doo-Sabin scheme can be formalized in a specific way. In this a valence n face in level k with vertices P_i^k, $i=0,1,2,\cdots,n-1$, after subdivision, a new valence n face with vertices P_i^{k+1}, $i=0,1,2,\cdots,n-1$ are computed. Each vertex P_i^{k+1} is a linear combination of the vertex P_i^k as follows;

$$P_i^{k+1}=\sum_{j=0}^{n-1}\omega_{i,j}^u P_j^k. \quad i=0,1,2,\cdots,n-1, \tag{1}$$

where the weights $\omega_{i,j}^u$ are

$$\omega_{i,j}^u=\begin{cases}\dfrac{n+5}{4n}, & \text{if} \quad i=j;\\[2mm] \dfrac{3+2\cos(2(j-i)\pi/n)}{4n}, & \text{else}.\end{cases} \tag{2}$$

The geometrical computation of the scheme (1) is shown in Figure 2, which will help out to find our proposed results.

Now, some new iterative points are presented to see the applicability of the scheme (1).

From (1) and (2)

$$\begin{aligned}P_0^{k+1}&=\sum_{j=0}^{n-1}\omega_{0,j}^u P_j^k=\omega_{0,0}^u P_0^k+\omega_{0,1}^u P_1^k+\omega_{0,2}^u P_2^k+\cdots+\omega_{0,n-1}^u P_{n-1}^k\\&=\left(\frac{1}{4}+\frac{5}{4n}\right)P_0^k+\left(\frac{3+2\cos(2\pi/n)}{4n}\right)P_1\\&\quad+\left(\frac{3+2\cos(4\pi/n)}{4n}\right)P_2^k+\cdots\\&\quad+\left(\frac{3+2\cos(2(n-1)\pi/n)}{4n}\right)P_{n-1}=\left(\frac{1}{4}+\frac{5}{4n}\right)P_0^k\\&\quad+\sum_{j=1}^{n-1}\left(\frac{3+2\cos(2j\pi/n)}{4n}\right)P_j^k.\end{aligned} \tag{3}$$

Again using (1) and (2)

$$\begin{aligned}P_1^{k+1}&=\sum_{j=0}^{n-1}\omega_{1,j}^u P_j^k=\omega_{1,0}^u P_0^k+\omega_{1,1}^u P_1^k+\omega_{1,2}^u P_2^k+\cdots+\omega_{1,n-1}^u P_{n-1}^k\\&=\left(\frac{3+2\cos(2\pi/n)}{4n}\right)P_0^k+\left(\frac{1}{4}+\frac{5}{4n}\right)P_1^k\\&\quad+\left(\frac{3+2\cos(2\pi/n)}{4n}\right)P_2^k+\cdots\\&\quad+\left(\frac{3+2\cos(2(n-2)\pi/n)}{4n}\right)P_{n-1}^k\\&=\left(\frac{3+2\cos(2\pi/n)}{4n}\right)P_0^k+\left(\frac{1}{4}+\frac{5}{4n}\right)P_1^k\\&\quad+\sum_{j=1}^{n-2}\left(\frac{3+2\cos(2j\pi/n)}{4n}\right)P_{j+1}^k.\end{aligned} \tag{4}$$

Similarly

$$\begin{aligned}P_2^{k+1}&=\sum_{j=0}^{n-1}\omega_{2,j}^u P_j^k=\omega_{2,0}^u P_0^k+\omega_{2,1}^u P_1^k+\omega_{2,2}^u P_2^k+\cdots+\omega_{2,n-1}^u P_{n-1}^k\\&=\left(\frac{3+2\cos(4\pi/n)}{4n}\right)P_0^k+\left(\frac{3+2\cos(2\pi/n)}{4n}\right)P_1^k\\&\quad+\left(\frac{1}{4}+\frac{5}{4n}\right)P_2^k+\sum_{j=1}^{n-3}\left(\frac{3+2\cos(2j\pi/n)}{4n}\right)P_{j+2}^k.\end{aligned} \tag{5}$$

Continuing in the same way, we have

$$\begin{aligned}P_{n-1}^{k+1}&=\sum_{j=0}^{n-1}\omega_{n-1,j}^u P_j^k=\sum_{j=1}^{n-1}\left(\frac{3+2\cos(2j\pi/n)}{4n}\right)P_{j-1}^k\\&\quad+\left(\frac{1}{4}+\frac{5}{4n}\right)P_{n-1}^k.\end{aligned} \tag{6}$$

2.1. Reformulation of Successive Convolutions. With the use of analysis developed in [13], there are some useful results, which will be used to construct the new technique for Doo-Sabin surfaces scheme in upcoming sections. Now, we move forward and construct associated constants for the

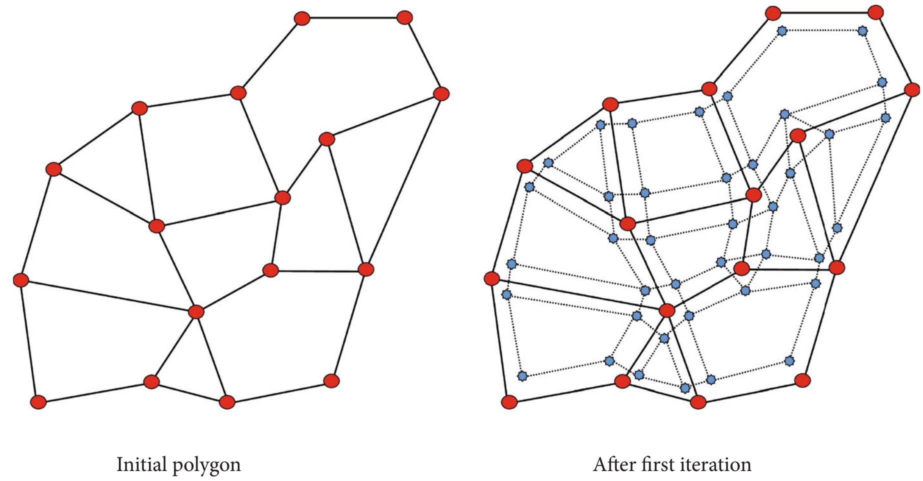


Figure 1: Scheme of computation of new face points by using Doo-Sabin subdivision scheme.

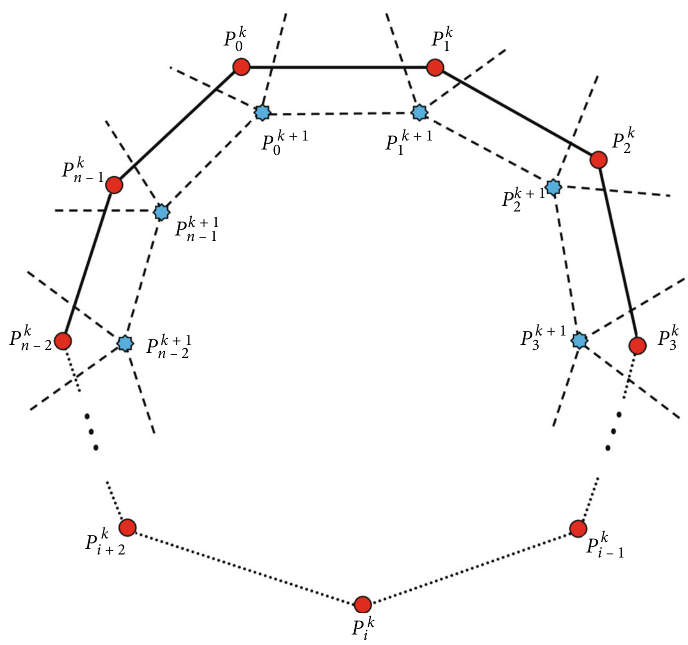


Figure 2: Geometric rules for Doo-Sabin surface for each face.

Doo-Sabin surface scheme by using successive convolutions for a one-dimensional array of finite length vectors.

Lemma 1. *Let $u = \{u_i\}$ be the vector of finite length and $g = \{g_i\}_{i=0}^{n-2}$, $h = \{h_i\}_{i=0}^{n-2}$ with $g_i = h_i = 0$ for $i \geq n-1$. The convolution product of $u = \{u_i\}$, $g = g_i, i \geq 0$, and $h = h_j, j \geq 0$ for Doo-Sabin subdivision surface scheme (1) along with (2) is given by*

$$u^{\varsigma_0}_{i,j} = \left(u^{\varsigma_0-1;0} * gh\right)_{i,j} = \sum_{m=0}^{\lfloor i/2 \rfloor} \sum_{l=0}^{\lfloor j/2 \rfloor} u^{\varsigma_0-1}_{m,l} g_{i-2m} h_{j-2l}. \tag{7}$$

Similarly, we get the following reformulation of generalized $\varsigma_0 - th$ convolution

$$\begin{aligned} u^{\varsigma_0}_{i,j} &= \left(\cdots\left(\left(\left(u^{\varsigma_0-1;0} * gh\right) * gh\right) * \cdots * gh\right) * gh\right)_{i,j} \\ &= \left\{ \sum_{m=0}^{\lfloor i/2 \rfloor} \left\{ \sum_{l=0}^{\lfloor j/2 \rfloor} u^0_{m,l} C^{\varsigma_0,g}_{m,i} C^{\varsigma_0,h}_{l,j} \right\} \right\}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} C^{\varsigma_0,g}_{m,i} &= \sum_{p=2m}^{\lfloor i/2^{\varsigma_0}-1 \rfloor} C^{\varsigma_0-1,g}_{m,p} C^{\varsigma_0-1,g}_{p,i}, \\ C^{\varsigma_0,h}_{l,j} &= \sum_{r=2l}^{\lfloor j/2^{\varsigma_0}-1 \rfloor} C^{\varsigma_0-1,h}_{l,s} C^{\varsigma_0-1,h}_{l,j}. \end{aligned} \tag{9}$$

Proof. See in [13]. □

Corollary 2. *The associated constants of a ς_0-th convolution with vector $g = \{g_0, g_1, \cdots, g_{n-2}\}$ and $h = \{h_0, h_1, \cdots, h_{n-2}\}$ for subdivision surface scheme are defined as*

$$G_{\varsigma_0} = \max_{i\in\Sigma(\varsigma_0,n)} \left\{ \sum_{m=0}^{\lfloor i/2^{\varsigma_0} \rfloor} \left| C_{m,i}^{\varsigma_0} \right| \right\}, \tag{10}$$

$$H_{\varsigma_0} = \max_{j\in\Sigma(\varsigma_0,n)} \left\{ \sum_{l=0}^{\lfloor j/2^{\varsigma_0} \rfloor} \left| C_{l,j}^{\varsigma_0} \right| \right\}. \tag{11}$$

Here

$$j \in \Sigma(\varsigma_0, n) = \{\Omega(\varsigma_0, n) - 2^{\varsigma_0} + 1, \Omega(\varsigma_0, n) - 2^{\varsigma_0} + 2, \cdots, \Omega(\varsigma_0, n)\}, \tag{12}$$

where

$$\Omega(\varsigma_0, n) = (2^{\varsigma_0} - 1)(2n - 1). \tag{13}$$

Proof. See in [13]. □

2.2. Recursion Formula for Uniform Doo-Sabin Subdivision Scheme. In this section, using the subdivision scheme (1) with weights given in (2), the recursion formula For Doo-Sabin subdivision scheme for $(k+1)^{\text{th}}$ level is constructed, which will be used in the main results.

Theorem 3. *Given initial control polygon $P^0 = \{p_j^0 ; j \in \mathbb{Z}\}$ and the polygon $P^{k+1} = \{p_j^{k+1} ; j \in \mathbb{Z}, k \geq 1\}$ be defined recursively by uniform Doo-Sabin subdivision process (1) together with (2). Then, the following inequalities hold for consecutive levels k and $k+1$*

$$\left\| \Delta^{k+1} \right\| \leq \sum_{j=1}^{n-2} \left| g_j^u \right| \left| \Delta^k \right\|, \tag{14}$$

and

$$\left\| \Delta^{k+1} \right\| \leq \sum_{j=1}^{n-2} \left| h_j^u \right| \left| \Delta^k \right\|. \tag{15}$$

Where

$$g_j^u = \sum_{l=0}^{j} (\omega_{i,l}^u - \omega_{i+1,l}^u), \tag{16}$$

$$h_j^j = \sum_{l=0}^{j} (\omega_{n-1,l}^u - \omega_{0,l}^u), \tag{17}$$

and

$$\Delta^k = P_{j+1}^k - P_j^k. \tag{18}$$

Proof. To obtain recursion formula of Uniform Doo-Sabin subdivision with the help from Figure 2. First, consider

(i) For $n = 3$

The first difference is

$$\begin{aligned} P_1^{k+1} - P_0^{k+1} &= \sum_{j=0}^{2} \left(\omega_{1,j}^u - \omega_{0,j}^u \right) P_j^k = \left(\omega_{1,0}^u - \omega_{0,0}^u \right) P_0^k \\ &\quad + \left(\omega_{1,1}^u - \omega_{0,1}^u \right) P_1^k + \left(\omega_{1,2}^u - \omega_{0,2}^u \right) P_2^k, \end{aligned} \tag{19}$$

By the scheme defined in (1) and (2)

$$\begin{aligned} P_1^{k+1} - P_0^{k+1} &= \left(\omega_{0,0}^u - \omega_{1,0}^u \right) \left(P_1^k - P_0^k \right) \\ &\quad + \left(\left(\omega_{0,0}^u - \omega_{1,0}^u \right) + \left(\omega_{0,1}^u - \omega_{1,1}^u \right) \right) \left(P_2^k - P_1^k \right), \end{aligned} \tag{20}$$

this implies

$$P_1^{k+1} - P_0^{k+1} = \sum_{j=0}^{1} \left(\sum_{l=0}^{j} \left(\omega_{0,l}^u - \omega_{1,l}^u \right) \right) \left(P_{j+1}^k - P_j^k \right). \tag{21}$$

Now, the second difference is

$$\begin{aligned} P_2^{k+1} - P_1^{k+1} &= \sum_{j=0}^{2} \left(\omega_{2,j}^u - \omega_{1,j}^u \right) P_j^k = \left(\omega_{2,0}^u - \omega_{1,0}^u \right) P_0^k \\ &\quad + \left(\omega_{2,1}^u - \omega_{1,1}^u \right) P_1^k + \left(\omega_{2,2}^u - \omega_{1,2}^u \right) P_2^k \\ &= \left(\omega_{1,0}^u - \omega_{2,0}^u \right) \left(P_1^k - P_0^k \right) \\ &\quad + \left(\left(\omega_{1,0}^u - \omega_{2,0}^u \right) + \left(\omega_{1,1}^u - \omega_{2,1}^u \right) \right) \left(P_2^k - P_1^k \right). \end{aligned} \tag{22}$$

This implies

$$P_2^{k+1} - P_1^{k+1} = \sum_{j=0}^{1} \left(\sum_{l=0}^{j} \left(\omega_{1,l}^u - \omega_{2,l}^u \right) \right) \left(P_{j+1}^k - P_j^k \right). \tag{23}$$

By the same procedure, the end point difference is

$$P_0^{k+1} - P_2^{k+1} = \sum_{j=0}^{1} \left(\sum_{l=0}^{j} \left(\omega_{0,l}^u - \omega_{2,l}^u \right) \right) \left(P_{j+1}^k - P_j^k \right). \tag{24}$$

(ii) For $n = 4$

The difference between P_1 and P_0 at $(k+1)^{\text{th}}$ level is

$$\begin{aligned} P_1^{k+1} - P_0^{k+1} &= \sum_{j=0}^{3} \left(\omega_{1,j}^u - \omega_{0,j}^u \right) P_j^k = \left(\omega_{1,0}^u - \omega_{0,0}^u \right) P_0^k \\ &\quad + \left(\omega_{1,1}^u - \omega_{0,1}^u \right) P_1^k + \left(\omega_{1,2}^u - \omega_{0,2}^u \right) P_2^k \\ &\quad + \left(\omega_{1,3}^u - \omega_{0,3}^u \right) P_3^k, \end{aligned} \tag{25}$$

this implies

$$P_1^{k+1} - P_0^{k+1} = \sum_{j=0}^{2}\left(\sum_{l=0}^{j}\left(\omega_{0,l}^u - \omega_{1,l}^u\right)\right)\left(P_{j+1}^k - P_j^k\right). \tag{26}$$

Now, the second difference is

$$\begin{aligned} P_2^{k+1} - P_1^{k+1} = \sum_{j=0}^{3}\left(\omega_{2,j}^u - \omega_{1,j}^u\right)P_j^k &= \left(\omega_{2,0}^u - \omega_{1,0}^u\right)P_0^k \\ &+ \left(\omega_{2,1}^u - \omega_{1,1}^u\right)P_1^k + \left(\omega_{2,2}^u - \omega_{1,2}^u\right)P_2^k \\ &+ \left(\omega_{2,3}^u - \omega_{1,3}^u\right)P_3^k, \end{aligned} \tag{27}$$

this implies

$$P_2^{k+1} - P_1^{k+1} = \sum_{j=0}^{2}\left(\sum_{l=0}^{j}\left(\omega_{1,l}^u - \omega_{2,l}^u\right)\right)\left(P_{j+1}^k - P_j^k\right). \tag{28}$$

Similarly,

$$\begin{aligned} P_3^{k+1} - P_2^{k+1} = \sum_{j=0}^{3}\left(\omega_{3,j}^u - \omega_{2,j}^u\right)P_j^k &= \left(\omega_{3,0}^u - \omega_{2,0}^u\right)P_0^k \\ &+ \left(\omega_{3,1}^u - \omega_{2,1}^u\right)P_1^k + \left(\omega_{3,2}^u - \omega_{2,2}^u\right)P_2^k \\ &+ \left(\omega_{3,3}^u - \omega_{2,3}^u\right)P_3^k, \end{aligned} \tag{29}$$

this implies

$$P_3^{k+1} - P_2^{k+1} = \sum_{j=0}^{2}\left(\sum_{l=0}^{j}\left(\omega_{1,l}^u - \omega_{2,l}^u\right)\right)\left(P_{j+1}^k - P_j^k\right). \tag{30}$$

By the same procedure the end point difference is

$$\begin{aligned} P_0^{k+1} - P_3^{k+1} = \sum_{j=0}^{3}\left(\omega_{0,j}^u - \omega_{3,j}^u\right)P_j^k &= \left(\omega_{0,0}^u - \omega_{3,0}^u\right)P_0^k \\ &+ \left(\omega_{0,1}^u - \omega_{3,1}^u\right)P_1^k + \left(\omega_{0,2}^u - \omega_{3,2}^u\right)P_2^k \\ &+ \left(\omega_{0,3}^u - \omega_{3,3}^u\right)P_3^k, \end{aligned} \tag{31}$$

this implies

$$P_0^{k+1} - P_3^{k+1} = \sum_{j=0}^{2}\left(\sum_{l=0}^{j}\left(\omega_{3,l}^u - \omega_{0,l}^u\right)\right)\left(P_{j+1}^k - P_j^k\right). \tag{32}$$

In general, it can be expressed as

$$P_{i+1}^{k+1} - P_i^{k+1} = \sum_{j=0}^{n-2}\left(\sum_{l=0}^{j}\left(\omega_{i,l}^u - \omega_{i+1,l}^u\right)\right)\left(P_{j+1}^k - P_j^k\right), \tag{33}$$

for $i = 0, 1, 2, \cdots, n-2$and

$$P_0^{k+1} - P_{n-1}^{k+1} = \sum_{j=0}^{n-2}\left(\sum_{l=0}^{j}\left(\omega_{n-1,l}^u - \omega_{0,l}^u\right)\right)\left(P_{j+1}^k - P_j^k\right). \tag{34}$$

Taking norm on both sides of (33), the following inequality will be obtained

$$\left\|P_{i+1}^{k+1} - P_i^{k+1}\right\| \leq \sum_{j=0}^{n-2}\left|\sum_{l=0}^{j}\left(\omega_{i,l}^u - \omega_{i+1,l}^u\right)\right|\left\|P_{j+1}^k - P_j^k\right\|. \tag{35}$$

Consider $\|P_{i+1}^k - P_i^k\| = \|\Delta^k\|$, which implies

$$\left\|\Delta^{k+1}\right\| \leq \sum_{j=0}^{n-2}\left|g_j^u\right|\left\|\Delta^k\right\|, \tag{36}$$

where $g_j^u = \sum_{l=0}^{j}(\omega_{i,l}^u - \omega_{i+1,l}^u)$.

Taking norm on both sides of (34), we have

$$\left\|P_0^{k+1} - P_{n-1}^{k+1}\right\| \leq \sum_{j=0}^{n-2}\left|\sum_{l=0}^{j}\left(\omega_{n-1,l}^u - \omega_{0,l}^u\right)\right|\left\|P_{j+1}^k - P_j^k\right\|, \tag{37}$$

which implies

$$\left\|\Delta^{k+1}\right\| \leq \sum_{j=0}^{n-2}\left|h_j^u\right|\left\|\Delta^k\right\|, \tag{38}$$

where $h_j^u = \sum_{l=0}^{j}(\omega_{n-1,l}^u - \omega_{0,l}^u)$. □

Now using Lemma 1 and Corollary 2 with (36) and (38), we have the associated constants $G_{\varsigma_0}^u$ and $H_{\varsigma_0}^u$, where ς_0 is the order of convolution. These constants can be used in above result as follows

$$\left\|\Delta^{k+1}\right\| \leq G_{\varsigma_0}^u H_{\varsigma_0}^u \left\|\Delta^k\right\|. \tag{39}$$

So recursively, the following main result is obtained

$$\left\|\Delta^k\right\| \leq \left(G_{\varsigma_0}^u H_{\varsigma_0}^u\right)^k \|\Delta^0\|. \tag{40}$$

3. The Subdivision Depth for Uniform Doo-Sabin Subdivision Scheme

In this section, first error bounds for the uniform Doo-Sabin subdivision scheme for distinct valence numbers and order of convolution are computed. Then, a numerical technique to find the subdivision depth within a given error tolerance is presented. Before going to the main work, consider two sequences, g_q^u and h_q^u, such that

$$\begin{cases} h_{2q}^u = \sum_{l=0}^{2q} \left(\omega_{n-1,l}^u - \omega_{0,l}^u\right), \\ h_{2q+1}^u = \sum_{l=0}^{2q+1} \left(\omega_{n-1,l}^u - \omega_{0,l}^u\right), \\ g_{2q}^u = \sum_{l=0}^{2q} \left(\omega_{i,l}^u - \omega_{i+1,l}^u\right), \\ g_{2q+1}^u = \sum_{l=0}^{2q+1} \left(\omega_{i,l}^u - \omega_{i+1,l}^u\right), \quad q = 0, 1, 2, \cdots, \left\lfloor \frac{n-2}{2} \right\rfloor. \end{cases} \tag{41}$$

Theorem 4. *Consider the initial control polygon $P^0 = \{p_\nu^0; \nu \in \mathbb{Z}\}$ and the polygon $P^k = \{p_\mu^k; \mu \in \mathbb{Z}, k \geq 1\}$, recursively at $k-th$ level defined by uniform Doo-Sabin scheme (1) together with (2). The error bound between two consecutive levels k and $k+1$ with associated constants $G_{\varsigma_0}^u$ and $H_{\varsigma_0}^u$ is*

$$\left\| P^{k+1} - P^k \right\|_\infty \leq b^u \left(G_{\varsigma_0}^u H_{\varsigma_0}^u \right)^k \beta, \tag{42}$$

with

$$\beta = \Delta^0 = \left\| P_{i+1}^0 - P_i^0 \right\|, \tag{43}$$

$$b^u = \max\left\{ b_0^u, b_1^u, b_2^u, \cdots, b_{n-1}^u \right\}, \tag{44}$$

where

$$b_j^u = \begin{cases} \sum_{i=1}^{n-1} \left| \frac{3 + 2\cos(2i\pi/n)}{4n} \right| i, & \text{if } j = 0, n-1; \\ \sum_{i=1}^{n-j-1} \left| \frac{3 + 2\cos(2i\pi/n)}{4n} \right| i + \sum_{i=1}^{j} \left| \frac{3 + 2\cos(2i\pi/n)}{4} \right| i, & \text{if } j = 1, 2, \cdots, n-2. \end{cases} \tag{45}$$

Proof. Let $\|.\|_\infty$ denote the uniform norm. Consider the first difference between $(k+1)^{\text{th}}$ and k^{th} level is

$$P_0^{k+1} - P_0^k = \left(\frac{1}{4} + \frac{5}{4n} \right) P_0^k + \sum_{j=1}^{n-1} \left(\frac{3 + 2\cos(2j\pi/n)}{4n} \right) P_j^k - P_0^k. \tag{46}$$

Using (1), the above equation can be rewritten as follows

$$\begin{aligned} P_0^{k+1} - P_0^k &= \left(\frac{1}{4} + \frac{5}{4n} \right) P_0^k + \sum_{i=1}^{n-1} \left(\frac{3 + 2\cos(2i\pi/n)}{4n} \right) P_i^k \\ &\quad - \left(\left(\frac{1}{4} + \frac{5}{4n} \right) + \sum_{i=1}^{n-1} \left(\frac{3 + 2\cos(2i\pi/n)}{4n} \right) \right) P_0^k \\ &= \sum_{i=1}^{n-1} \left(\frac{3 + 2\cos(2i\pi/n)}{4n} \right) \left(P_i^k - P_0^k \right), \end{aligned} \tag{47}$$

By taking $P_i^k = \sum_{j=1}^{i-1} (P_{j+1}^k - P_j^k) + P_1^k$, we have

$$\begin{aligned} P_0^{k+1} - P_0^k &= \sum_{i=1}^{n-1} \left(\frac{3 + 2\cos(2i\pi/n)}{4n} \right) \left(P_1^k - P_0^k \right) \\ &\quad + \sum_{i=2}^{n-1} \left(\frac{3 + 2\cos(2i\pi/n)}{4n} \right) \times \left(\sum_{j=1}^{i-1} \left(P_{j+1}^k - P_j^k \right) \right). \end{aligned} \tag{48}$$

Taking norm $\|.\|$, we have

$$\begin{aligned} \left\| P_0^{k+1} - P_0^k \right\| &\leq \sum_{i=1}^{n-1} \left| \frac{3 + 2\cos(2i\pi/n)}{4n} \right| \left\| P_1^k - P_0^k \right\| \\ &\quad + \sum_{i=2}^{n-1} \left| \frac{3 + 2\cos(2i\pi/n)}{4n} \right| \left(\sum_{j=1}^{i-1} \left\| P_{j+1}^k - P_j^k \right\| \right). \end{aligned} \tag{49}$$

Consider $\|P_{i+1}^{k+1} - P_i^k\| = \|\Delta^k\|$; then, the above expression leads

$$\begin{aligned} \left\| P_0^{k+1} - P_0^k \right\| &\leq \sum_{i=1}^{n-1} \left| \frac{3 + 2\cos(2i\pi/n)}{4n} \right| \left\| \Delta^k \right\| \\ &\quad + \sum_{i=2}^{n-1} \left| \frac{3 + 2\cos(2i\pi/n)}{4n} \right| \times \left(\sum_{j=1}^{i-1} \left\| \Delta^k \right\| \right) \\ &= \sum_{i=1}^{n-1} \left| \frac{3 + 2\cos(2i\pi/n)}{4n} \right| i \left\| \Delta^k \right\|, \end{aligned} \tag{50}$$

which implies

$$\left\| P_0^{k+1} - P_0^k \right\| \leq b_0^u \left\| \Delta^k \right\|, \tag{51}$$

where

$$b_0^u = \sum_{i=1}^{n-1} \left| \frac{3 + 2\cos(2i\pi/n)}{4n} \right| i. \tag{52}$$

Consider

$$P_1^{k+1} - P_1^k = \left(\frac{3 + 2\cos(2\pi/n)}{4n}\right) P_0^k + \left(\frac{1}{4} + \frac{5}{4n}\right) P_1^k + \sum_{i=1}^{n-2} \left(\frac{3 + 2\cos(2i\pi/n)}{4n}\right)\left(P_{i+1}^k - P_1^k\right), \tag{53}$$

Using (2), we have

$$\begin{aligned} P_1^{k+1} - P_1^k &= \left(\frac{3 + 2\cos(2\pi/n)}{4n}\right) P_0^k + \left(\frac{1}{4} + \frac{5}{4n}\right) P_1^k \\ &\quad + \sum_{i=1}^{n-2}\left(\frac{3 + 2\cos(2i\pi/n)}{4n}\right) P_{i+1}^k \\ &\quad - \left(\left(\frac{1}{4} + \frac{5}{4n}\right) + \sum_{i=1}^{n-1}\left(\frac{3 + 2\cos(2i\pi/n)}{4n}\right)\right) P_1^k \\ &= \left(\frac{3 + 2\cos(2\pi/n)}{4n}\right)\left(P_2^k - P_1^k\right) \\ &\quad + \sum_{i=1}^{n-2}\left(\frac{3 + 2\cos(2i\pi/n)}{4n}\right)\left(P_{i+1}^k - P_1^k\right) \\ &\quad - \left(\frac{3 + 2\cos(2\pi/n)}{4n}\right)\left(P_1^k - P_0^k\right). \end{aligned} \tag{54}$$

By taking $P_{i+1}^k = \sum_{j=1}^{i-1}(P_{j+2}^k - P_{j+1}^k) + P_2^k$, the above expression can be written as

$$\begin{aligned} P_1^{k+1} - P_1^k &= \sum_{i=1}^{n-2}\left(\frac{3 + 2\cos(2i\pi/n)}{4n}\right)\left(P_2^k - P_1^k\right) \\ &\quad + \sum_{i=2}^{n-2}\left(\frac{3 + 2\cos(2i\pi/n)}{4n}\right)\left(\sum_{j=1}^{i-1}\left(P_{j+2}^k - P_{j+1}^k\right)\right) \\ &\quad - \left(\frac{3 + 2\cos(2\pi/n)}{4n}\right)\left(P_1^k - P_0^k\right) \end{aligned} \tag{55}$$

Taking norm $\|.\|$, we have

$$\begin{aligned} \left\|P_1^{k+1} - P_1^k\right\| &\le \sum_{i=1}^{n-2}\left|\frac{3 + 2\cos(2i\pi/n)}{4n}\right|\left\|P_2^k - P_1^k\right\| \\ &\quad + \sum_{i=2}^{n-2}\left|\frac{3 + 2\cos(2i\pi/n)}{4n}\right|\left(\sum_{j=1}^{i-1}\left\|P_{j+2}^k - P_{j+1}^k\right\|\right) \\ &\quad + \left|\frac{3 + 2\cos(2\pi/n)}{4n}\right|\left\|P_1^k - P_0^k\right\|, \end{aligned} \tag{56}$$

which implies

$$\left\|P_1^{k+1} - P_1^k\right\| \le b_1^u\left\|\Delta^k\right\|, \tag{57}$$

where

$$b_1^u = \sum_{i=1}^{n-2}\left|\frac{3 + 2\cos(2i\pi/n)}{4n}\right| + \left|\frac{3 + 2\cos(2\pi/n)}{4n}\right|. \tag{58}$$

By the same procedure, we have

$$\left\|P_2^{k+1} - P_2^k\right\| \le b_2^u\left\|\Delta^k\right\|, \tag{59}$$

where

$$b_2^u = \sum_{i=1}^{n-3}\left|\frac{3 + 2\cos(2i\pi/n)}{4n}\right| i + \left|\frac{3 + 2\cos(2\pi/n)}{4n}\right| + 2\left|\frac{3 + 2\cos(4\pi/n)}{4n}\right|. \tag{60}$$

Similarly

$$\left\|P_3^{k+1} - P_3^k\right\| \le b_3^u\left\|\Delta^k\right\|, \tag{61}$$

where

$$b_3^u = \sum_{i=1}^{n-4}\left|\frac{3 + 2\cos(2i\pi/n)}{4n}\right| i + \left|\frac{3 + 2\cos(2\pi/n)}{4n}\right| + \left|\frac{3 + 2\cos(4\pi/n)}{4n}\right| + \left|\frac{3 + 2\cos(6\pi/n)}{4n}\right|. \tag{62}$$

In the same way, we get

$$\left\|P_{n-1}^{k+1} - P_{n-1}^k\right\| \le b_{n-1}^u\left\|\Delta^k\right\|, \tag{63}$$

where

$$b_{n-1}^u = \sum_{i=1}^{n-1}\left|\frac{3 + 2\cos(2i\pi/n)}{4n}\right| i. \tag{64}$$

From (51), (57), (59), (63), and (64), the following general form is obtained

$$\left\|P^{k+1} - P^k\right\| \le b^u\left\|\Delta^k\right\|, \tag{65}$$

with

$$b^u = \max\left\{b_0^u, b_1^u, b_2^u, \cdots, b_{n-1}^u\right\}, \tag{66}$$

where

$$b_j^u = \begin{cases} \displaystyle\sum_{i=1}^{n-1}\left|\frac{3 + 2\cos(2i\pi/n)}{4n}\right| i, & \text{if } \quad j = 0, n-1; \\ \displaystyle\sum_{i=1}^{n-j-1}\left|\frac{3 + 2\cos(2i\pi/n)}{4n}\right| i + \sum_{i=1}^{j}\left|\frac{3 + 2\cos(2i\pi/n)}{4}\right| i, & \text{if } \quad j = 1, 2, \cdots, n-2. \end{cases} \tag{67}$$

From (40), the following inequality holds

$$\left\|P^{k+1}-P^{k}\right\| \leq b^{u}\left(G_{\varsigma_0}^{u} H_{\varsigma_0}^{u}\right)^{k}\left\|\Delta^{0}\right\|. \tag{68}$$

If we take $\|\Delta^{0}\|=\beta$, then

$$\left\|P^{k+1}-P^{k}\right\| \leq b^{u}\left(G_{\varsigma_0}^{u} H_{\varsigma_0}^{u}\right)^{k}\beta. \tag{69}$$

This completes the proof. □

Theorem 5. *Let P^{∞} be the limit surface generated from uniform Doo-Sabin scheme; then, under the same conditions used in Theorem 4, the following inequality holds*

$$\left\|P^{\infty}-P^{k}\right\|_{\infty} \leq b^{u}\left(\frac{\left(G_{\varsigma_0}^{u} H_{\varsigma_0}^{u}\right)^{k}}{1-G_{\varsigma_0}^{u} H_{\varsigma_0}^{u}}\right)\beta. \tag{70}$$

Proof. Consider

$$\begin{aligned}\left\|P^{k}-P^{\infty}\right\|_{\infty} &= \left\|P^{k}-P^{k+1}+P^{k+1}-P^{k+2}+P^{k+2}+\cdots\right\|_{\infty} \\ &\leq \left\|P^{k}-P^{k+1}\right\|_{\infty}+\left\|P^{k+1}-P^{k+2}\right\|_{\infty}+\cdots,\end{aligned} \tag{71}$$

Using (69), the following inequality holds

$$\left\|P^{k}-P^{\infty}\right\|_{\infty} \leq b^{u}\sum_{m=k}^{\infty}\left(G_{\varsigma_0}^{u} H_{\varsigma_0}^{u}\right)^{m}\beta. \tag{72}$$

Sum of infinite geometric series implies

$$\left\|P^{\infty}-P^{k}\right\|_{\infty} \leq b^{u}\left(\frac{\left(G_{\varsigma_0}^{u} H_{\varsigma_0}^{u}\right)^{k}}{1-G_{\varsigma_0}^{u} H_{\varsigma_0}^{u}}\right)\beta. \tag{73}$$

This complete the proof. □

Theorem 6. *Let k be subdivision depth and let $\nabla^{k}=\|P^{k+1}-P^{k}\|$ be the error bound between uniform Doo-Sabin subdivision surface P^{∞} and its k-th level control polygon P^{k}. For arbitrary $\epsilon>0$, if*

$$k \geq \log_{\left(G_{\varsigma_0}^{u} H_{\varsigma_0}^{u}\right)^{-1}}\left(\frac{b^{u}\beta}{\epsilon\left(1-G_{\varsigma_0}^{u} H_{\varsigma_0}^{u}\right)}\right). \tag{74}$$

then $\|P^{\infty}-P^{k}\| \leq \epsilon$.

Proof. See in [13]. □

4. Numerical Experiments

Some numerical tests for calculating the Doo-Sabin subdivision depths (number of iterations) for distinct valences n are presented. Before proceeding to the main task, the associated constants $G_{\varsigma_0}^{u} H_{\varsigma_0}^{u}$, $\varsigma_0 \geq 1$ are estimated using Corollary 2, which are shown in Table 1 for $n=3,4,5,6,7$.

4.1. Experiment 1. Theorem 3 and Theorem 5 were used to estimate error bounds and subdivision depth in this experimental test for valence number $n=3$. From (1) and (2)

$$\begin{aligned}P_{0}^{k+1} &= \frac{2}{3}P_{0}^{k}+\frac{1}{6}P_{1}^{k}+\frac{1}{6}P_{2}^{k}, \\ P_{1}^{k+1} &= \frac{1}{6}P_{0}^{k}+\frac{2}{3}P_{1}^{k}+\frac{1}{6}P_{2}^{k}, \\ P_{2}^{k+1} &= \frac{1}{6}P_{0}^{k}+\frac{1}{6}P_{1}^{k}+\frac{2}{3}P_{2}^{k}.\end{aligned} \tag{75}$$

From (41)

$$\begin{aligned}g_{0}^{u} &= \frac{1}{2}, \\ g_{1}^{u} &= 0, \\ h_{0}^{u} &= -\frac{1}{2}, \\ h_{1}^{u} &= -\frac{1}{2}.\end{aligned} \tag{76}$$

From (10) and (11)

$$G_{\varsigma_0}^{u} = \max_{i\in\Sigma(\varsigma_0,3)}\left\{\sum_{m=0}^{\lfloor i/2^{\varsigma_0}\rfloor}\left|C_{m,i}^{\varsigma_0}\right|\right\}, \tag{77}$$

$$H_{\varsigma_0}^{u} = \max_{j\in\Sigma(\varsigma_0,3)}\left\{\sum_{l=0}^{\lfloor j/2^{\varsigma_0}\rfloor}\left|C_{l,j}^{\varsigma_0}\right|\right\}. \tag{78}$$

(i) First convolution $(\varsigma_0=1)$

Here $\Omega(1,3)=5$ and $\Sigma(1,3)=\{4,5\}$, $G_{1}^{u}H_{1}^{u}$ is given by

$$\begin{aligned}G_{1}^{u}H_{1}^{u} &= \max_{i,j\in\Sigma\{1,3\}}\left\{\sum_{p=0}^{[j/2]}\sum_{q=0}^{[i/2]}\left|C_{q,i}^{[1,g^{u}]}\right|\left|C_{p,j}^{[1,h^{u}]}\right|\right\} \\ &= \max_{i,j\in\{4,5\}}\left\{\sum_{p=0}^{[j/2]}\sum_{q=0}^{[i/2]}\left|C_{q,i}^{[1,g^{u}]}\right|\left|C_{p,j}^{[1,h^{u}]}\right|\right\} \\ &= \max\left\{\sum_{p=0}^{[4/2]}\sum_{q=0}^{[4/2]}\left|C_{q,4}^{[1,g^{u}]}\right|\left|C_{p,4}^{[1,h^{u}]}\right|, \sum_{p=0}^{[5/2]}\sum_{q=0}^{[5/2]}\left|C_{q,5}^{[1,g^{u}]}\right|\left|C_{p,5}^{[1,h^{u}]}\right|\right\} \\ &= \max\left\{\sum_{p=0}^{2}\sum_{q=0}^{2}\left|g_{4-2q}^{u}\right|\left|h_{4-2p}^{u}\right|, \sum_{p=0}^{2}\sum_{q=0}^{2}\left|g_{5-2q}^{u}\right|\left|h_{5-2p}^{u}\right|\right\} \\ &= \max\{0.2500,0\},\end{aligned} \tag{79}$$

$$G_{1}^{u}H_{1}^{u}=0.2500. \tag{80}$$

Table 1: Associated constants of Doo-Sabin subdivision scheme for distinct valence numbers.

$n/G^u_{\varsigma_0}H^u_{\varsigma_0}$	$G^u_1H^u_1$	$G^u_2H^u_2$	$G^u_3H^u_3$	$G^u_4H^u_4$	$G^u_5H^u_5$
3	0.25000	0.06250	0.01563	0.00391	0.0009765
4	0.28125	0.04394	0.00632	0.00087	0.0000889
5	0.15547	0.01817	0.00152	0.00013	0.0000110
6	0.16667	0.02177	0.00295	0.00039	0.0000237
7	0.23106	0.04966	0.01001	0.00204	0.0001244

Table 2: Error bound of the uniform Doo-Sabin subdivision surfaces. Here, k presents the subdivision level, and n presents the valence number.

n/k	1	2	3	4	5	6	7
$\varsigma_0 = 1$							
3	0.0166	0.0042	0.0010	0.0003	0.00007	0.00002	0.000004
4	0.0343	0.0096	0.0027	0.0008	0.0002	0.00006	0.000017
10	0.3811	0.2094	0.1151	0.0632	0.0347	0.01909	0.0105
$\varsigma_0 = 2$							
3	0.0033	0.0002	0.00001	8.14×10^{-7}	5.09×10^{-8}	3.18×10^{-9}	1.99×10^{-10}
4	0.0040	0.0002	0.000008	3.41×10^{-7}	1.49×10^{-8}	6.59×10^{-10}	2.89×10^{-10}
10	0.1011	0.0247	0.00603	0.0015	0.00036	0.000088	0.000021
$\varsigma_0 = 3$							
3	0.0008	0.00002	1.94×10^{-7}	3.03×10^{-9}	4.74×10^{-11}	7.40×10^{-11}	1.16×10^{-14}
4	0.0005	0.000003	2.22×10^{-8}	1.41×10^{-10}	8.88×10^{-13}	5.61×10^{-15}	3.50×10^{-17}
10	0.0345	0.0034	0.00003	0.000003	0.00036	3.27×10^{-7}	3.23×10^{-8}
$\varsigma_0 = 4$							
3	0.0002	7.67×10^{-7}	3.03×10^{-9}	1.17×10^{-11}	4.58×10^{-14}	1.79×10^{-16}	7.01×10^{-19}
4	0.00008	6.63×10^{-8}	5.77×10^{-11}	5.02×10^{-14}	4.37×10^{-17}	3.80×10^{-20}	3.30×10^{-23}
10	0.0129	0.00052	0.000002	8.28×10^{-7}	3.30×10^{-8}	1.32×10^{-9}	5.27×10^{-11}
$\varsigma_0 = 5$							
3	0.00005	4.77×10^{-8}	4.66×10^{-11}	4.55×10^{-14}	4.44×10^{-17}	4.34×10^{-17}	4.23×10^{-23}
4	0.000008	6.91×10^{-10}	6.15×10^{-14}	5.47×10^{-18}	4.86×10^{-22}	4.32×10^{-26}	3.84×10^{-30}
10	0.0047	0.00007	0.000001	1.49×10^{-8}	2.19×10^{-10}	3.23×10^{-12}	4.75×10^{-14}

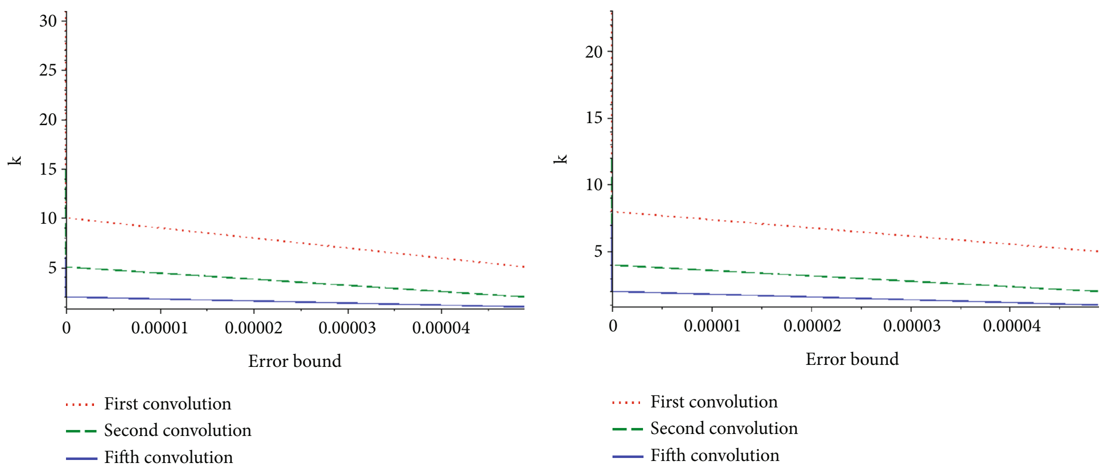
(ii) Second convolution ($\varsigma_0 = 2$)

Now for $\Omega(2,3) = 15$ and $\Sigma(2,3) = \{12,13,14,15\}$, $G^u_2H^u_2$ is given by

$$
\begin{aligned}
G^u_2H^u_2 &= \max_{i,j\in\{12,13,14,15\}}\left\{\sum_{p=0}^{[j/2^2]}\sum_{q=0}^{[i/2^2]}\left|C^{[2,g^u]}_{q,i}\right|\left|C^{[2,h^u]}_{p,j}\right|\right\}\\
&= \max\left\{\sum_{p=0}^{[12/4]}\sum_{q=0}^{[12/4]}\left|C^{[2,g^u]}_{q,12}\right|\left|C^{[2,h^u]}_{p,12}\right|, \sum_{p=0}^{[13/4]}\sum_{p=0}^{[13/4]}\left|C^{[2,g^u]}_{p,13}\right|\left|C^{[2,h^u]}_{p,13}\right|,\right.\\
&\quad\left.\sum_{p=0}^{[14/4]}\sum_{q=0}^{[14/4]}\left|C^{[2,g^u]}_{q,20}\right|\left|C^{[2,h^u]}_{p,14}\right|, \sum_{p=0}^{[15/4]}\sum_{q=0}^{[15/4]}\left|C^{[2,gu]}_{q,15}\right|\left|C^{[2,h^u]}_{p,15}\right|\right\}\\
&= \max\left\{\sum_{p=0}^{3}\sum_{q=0}^{3}\left|\sum_{r=2q}^{[12/2]}C^{[1,g^u]}_{q,r}C^{[1,g^u]}_{r,12}\right|\left|\sum_{s=2p}^{[12/2]}C^{[1,h^u]}_{p,s}C^{[1,h^u]}_{s,12}\right|,\right.\\
&\quad\sum_{p=0}^{3}\sum_{q=0}^{3}\left|\sum_{r=2q}^{[13/2]}C^{[1,g^u]}_{q,r}C^{[1,g^u]}_{r,13}\right|\left|\sum_{s=2p}^{[13/2]}C^{[1,h^u]}_{p,s}C^{[1,h^u]}_{s,13}\right|,\\
&\quad\sum_{p=0}^{3}\sum_{q=0}^{3}\left|\sum_{r=2q}^{[14/2]}C^{[1,g^u]}_{q,r}C^{[1,g^u]}_{r,14}\right|\left|\sum_{s=2p}^{[14/2]}C^{[1,h^u]}_{p,s}C^{[1,h^u]}_{s,14}\right|,\\
&\quad\left.\sum_{p=0}^{3}\sum_{q=0}^{3}\left|\sum_{r=2q}^{[15/2]}C^{[1,g^u]}_{q,r}C^{[1,g^u]}_{r,15}\right|\left|\sum_{s=2p}^{[15/2]}C^{[1,h^u]}_{p,s}C^{[1,h^u]}_{s,15}\right|\right\}
\end{aligned}
$$

Table 3: Subdivision depth of the uniform Doo-Sabin subdivision surface at face valence $n = 3$.

$G^u_{\varsigma_0} H^u_{\varsigma_0} / \epsilon$	0.0000488	4.77×10^{-8}	4.66×10^{-11}	4.55×10^{-14}	4.44×10^{-17}	4.34×10^{-20}	4.12×10^{-22}
$G^u_1 H^u_1$	5	10	15	20	25	30	31
$G^u_2 H^u_2$	2	5	7	10	13	15	15
$G^u_3 H^u_3$	2	3	5	7	8	10	10
$G^u_4 H^u_4$	1	3	4	5	6	7	8
$G^u_5 H^u_5$	1	2	3	4	5	6	7



(a) Graph for valence $n = 3$

(b) Graph for valence $n = 6$

Figure 3: Presentation of subdivision depth by taking first, second, and fifth level of convolutions.

$$= \max \left\{ \sum_{p=0}^{3}\sum_{q=0}^{3} \left| \sum_{r=2q}^{6} g^u_{r-2q} g^u_{12-2r} \right| \left| \sum_{s=2p}^{6} h^u_{s-2p} h^u_{12-2s} \right|, \right.$$
$$\sum_{p=0}^{3}\sum_{q=0}^{3} \left| \sum_{r=2q}^{6} g^u_{r-2q} g^u_{13-2r} \right| \left| \sum_{s=2p}^{6} h^u_{s-2p} h^u_{13-2s} \right|,$$
$$\sum_{p=0}^{3}\sum_{q=0}^{3} \left| \sum_{r=2q}^{7} g^u_{r-2q} g^u_{14-2r} \right| \left| \sum_{s=2p}^{7} h^u_{s-2p} h^u_{14-2s} \right|, \tag{81}$$
$$\left. \sum_{p=0}^{3}\sum_{q=0}^{3} \left| \sum_{r=2q}^{7} g^u_{r-2q} g^u_{15-2r} \right| \left| \sum_{s=2p}^{7} h^u_{s-2p} h^u_{15-2s} \right| \right\}$$
$$= \max \{0.062500, 0.062500, 0.062500, 0.062500\},$$

$$G^u_2 H^u_2 = 0.062500. \tag{82}$$

Similarly, compute $G^u_{\varsigma_0} H^u_{\varsigma_0}$ values for $\varsigma_0 \geq 3$. Table 2 displays the values computed up to $\varsigma_0 = 5$ for convenience.

After the calculation of the convolution constant of the uniform Doo-Sabin Subdivision scheme, it can be observed that the values of $G^u_{\varsigma_0} H^u_{\varsigma_0}$ decrease with the increase of the level of convolution ς_0 which is the main approach of the proposed work.

Calculate the errors of the proposed results using Theorem 4 for different subdivision levels k and distinct face valence numbers $n = 3, 4, 10$, as shown in Table 2. Estimate the subdivision depth of uniform Doo-Sabin subdivision at valence $n = 3$ using Theorem 5 and Table 3.

Figure 3(a) depicts a graphical representation of the subdivision depth for valance $n = 3$ at various convolution levels. From the figure, it is clear that the subdivision depth (the number of iterations) decreases by increasing the order of convolution ς_0.

4.2. Experiment 2. Consider the following uniform Doo-Sabin surface scheme for face valance number $n = 6$ given in (1) and (2).

$$
\begin{aligned}
P^{k+1}_0 &= \frac{11}{24}P^k_0 + \frac{1}{6}P^k_1 + \frac{1}{12}P^k_2 + \frac{1}{24}P^k_3 + \frac{1}{12}P^k_4 + \frac{1}{6}P^k_5,\\
P^{k+1}_1 &= \frac{1}{6}P^k_0 + \frac{1}{24}P^k_1 + \frac{1}{6}P^k_2 + \frac{1}{12}P^k_3 + \frac{1}{24}P^k_4 + \frac{1}{12}P^k_5,\\
P^{k+1}_2 &= \frac{1}{12}P^k_0 + \frac{1}{6}P^k_1 + \frac{11}{24}P^k_2 + \frac{1}{6}P^k_3 + \frac{1}{12}P^k_4 + \frac{1}{24}P^k_5,\\
P^{k+1}_3 &= \frac{1}{24}P^k_0 + \frac{1}{12}P^k_1 + \frac{1}{6}P^k_2 + \frac{11}{24}P^k_3 + \frac{1}{6}P^k_4 + \frac{1}{12}P^k_5,\\
P^{k+1}_4 &= \frac{1}{12}P^k_0 + \frac{1}{24}P^k_1 + \frac{1}{12}P^k_2 + \frac{1}{6}P^k_3 + \frac{11}{24}P^k_4 + \frac{1}{6}P^k_5,\\
P^{k+1}_5 &= \frac{1}{6}P^k_0 + \frac{1}{12}P^k_1 + \frac{1}{24}P^k_2 + \frac{1}{12}P^k_3 + \frac{1}{6}P^k_4 + \frac{11}{24}P^k_5.
\end{aligned}
\tag{83}
$$

Table 4: Subdivision depth of uniform Doo-Sabin subdivision surface at valence $n = 6$.

$G^u_{\varsigma_0} H^u_{\varsigma_0} / \epsilon$	0.0000488	4.77×10^{-8}	4.66×10^{-11}	4.55×10^{-14}	4.44×10^{-17}	4.34×10^{-20}	4.12×10^{-22}
$G^u_1 H^u_1$	5	8	12	16	20	24	26
$G^u_2 H^u_2$	2	4	6	8	9	11	12
$G^u_3 H^u_3$	2	3	4	5	6	7	9
$G^u_4 H^u_4$	1	2	3	4	5	7	8
$G^u_5 H^u_5$	1	2	3	4	5	6	7

To find the error bounds and subdivision depth of above scheme, first consider

$$\begin{aligned} g^u_0 &= \frac{7}{24}, \\ g^u_1 &= 0, \\ g^u_2 &= \frac{-1}{12}, \\ g^u_3 &= \frac{-1}{8}, \\ g^u_4 &= \frac{-1}{12}, \\ g^u_5 &= 0, \\ h^u_0 &= \frac{-5}{24}, \\ h^u_1 &= \frac{-7}{24}, \\ h^u_2 &= \frac{-1}{3}, \\ h^u_3 &= \frac{-7}{24}, \\ h^u_4 &= \frac{5}{24}, \\ h^u_5 &= \frac{5}{8}. \end{aligned} \tag{84}$$

From (10) and (11), we have

$$G^u_{\varsigma_0} = \max_{i \in \Sigma(\varsigma_0, 6)} \left\{ \sum_{m=0}^{\lfloor i/2^{\varsigma_0} \rfloor} \left| C^{\varsigma_0}_{m,i} \right| \right\}, \tag{85}$$

$$H^u_{\varsigma_0} = \max_{j \in \Sigma(\varsigma_0, 6)} \left\{ \sum_{l=0}^{\lfloor j/2^{\varsigma_0} \rfloor} \left| C^{\varsigma_0}_{l,j} \right| \right\}. \tag{86}$$

Now, we apply Theorem 4, the error bounds $G^u_{\varsigma_0} H^u_{\varsigma_0}$, for $\varsigma_0 \geq 1$, are shown in Table 2 at distinct valence numbers. Further, the subdivision depth of the subdivision scheme for $n = 6$ is shown in Table 4. The graphical representation of the subdivision depth for distinct valences n at distinct convolution level is shown in Figure 3(b). From the figure, it is clear that the subdivision depth (number of iterations) decreases by increasing the order of convolution ς_0.

Data Availability

The data used to support the findings of the study is available within this paper.

Conflicts of Interest

"The authors declare that there are no conflicts of interest regarding the publication of this paper."

Authors' Contributions

Conceptualization and methodology were done by Faheem Khan. Formal analysis was done by Bushra Shakoor. Writing original draft and review were done by Ghulam Mustafa. Editing and programming were done by Sidra Razaq.

References

[1] Y. Liu, H. Shou, and K. Ji, "Review of subdivision schemes and their applications," *Recent Patents on Engineering.*, vol. 16, no. 4, pp. 50–62, 2022.

[2] R. Hameed, G. Mustafa, and R. Majeed, "Variations in 2D and 3D models by a new family of subdivision schemes and algorithms for its analysis," *Mathematical Problems in Engineering*, vol. 2022, Article ID 3661307, 18 pages, 2022.

[3] U. Labsik and G. Greiner, "Interpolatory $\sqrt{3}$ subdivision," *Eurographics*, vol. 19, no. 3, 2000.

[4] D. Doo and M. Sabin, "Behaviour of recursive division surfaces near extraordinary points," *Computer Aided Design*, vol. 10, no. 6, pp. 356–360, 1978.

[5] E. Catmull and J. Clark, "Recursively generated B-spline surfaces on arbitrary topological meshes," *Computer Aided Design*, vol. 10, no. 6, pp. 350–355, 1978.

[6] C. Loop, *Smooth Subdivision Surfaces Based on Triangles [Master's Thesis]*, Department of Mathematics, University of Utah, 1987.

[7] L. Velho and D. Zorin, "Quasi 4-8 subdivision," *Computer Aided Geometric Design*, vol. 18, no. 4, pp. 345–357, 2001.

[8] J. Stam and C. Loop, "Quad/triangle subdivision," *Computer Graphics Forum*, vol. 22, no. 1, pp. 79–85, 2003.

[9] J. Peters and L. Shiue, "Combining 4- and 3-direction subdivision," *ACM Transactions on Graphics*, vol. 23, no. 4, pp. 980–1003, 2004.

[10] N. Dyn and D. Levin, "A butterfly subdivision scheme for surface interpolation with tension control," *ACM Transactions on Graphics*, vol. 9, no. 2, pp. 160–169, 1990.

[11] D. Zorin, P. Schroder, and W. Sweldens, "Interpolating subdivision for meshes with arbitrary topology, SIGGRAPH'96," in

Proceedings of the 23rd Annual Conference on Computer Graphics and Interactive Techniques, ACM, pp. 189–192, New York, NY, USA, 1996.

[12] M. Moncayo and S. Amat, "Error bounds for a class of subdivision schemes based on the two-scale refinement equation," *Journal of Computational and Applied Mathematics*, vol. 236, no. 2, pp. 265–278, 2011.

[13] A. Shahzad, K. Faheem, A. Ghaffar, G. Mustafa, K.-S. Nisar, and D. Baleanu, "A novel numerical algorithm to estimate the subdivision depth of binary subdivision schemes," *Symmetry*, vol. 12, no. 1, p. 66, 2020.

[14] C. Fuhua and J.-H. Yong, "Subdivision depth computation for Catmull-Clark subdivision surfaces," *Computer-Aided Design and Applications*, vol. 3, no. 1-4, p. 485, 2006.

[15] G. Mustafa, F. Chen, and J. Deng, "Estimating error bounds for binary subdivision curves/surfaces," *Journal of Computational and Applied Mathematics*, vol. 193, no. 2, pp. 596–613, 2006.