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Retraction

Retracted: Classical and Bayesian Inference of Marshall-Olkin Extended Gompertz Makeham Model with Modeling of Physics Data

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] R. A. H. Mohamed, A. A. Al-Babtain, I. Elbatal, E. M. Almetwally, and H. M. Almongy, "Classical and Bayesian Inference of Marshall-Olkin Extended Gompertz Makeham Model with Modeling of Physics Data," *Journal of Mathematics*, vol. 2022, Article ID 2528583, 14 pages, 2022.

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Research Article

Classical and Bayesian Inference of Marshall-Olkin Extended Gompertz Makeham Model with Modeling of Physics Data

Rania A. H. Mohamed,[1] Abdulhakim A. Al-Babtain ,[2] I. Elbatal ,[3] Ehab M. Almetwally ,[4,5] and Hisham M. Almongy [6]

[1]*Department of Statistics Mathematics and Insurance, Faculty of Commerce, Port Said University, Port Fuad 42526, Egypt*
[2]*Department of Statistics and Operations Research, King Saud University, Riyadh 11362, Saudi Arabia*
[3]*Department of Mathematics and Statistics-College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11362, Saudi Arabia*
[4]*Department of Statistics, Faculty of Business Administration, Delta University of Science and Technology, Gamasa 11152, Egypt*
[5]*The Scientific Association for Studies and Applied Research, Al Manzalah, Egypt*
[6]*Department of Applied Statistics and Insurance, Faculty of Commerce, Mansoura University, Mansoura 35516, Egypt*

Correspondence should be addressed to Ehab M. Almetwally; ehabxp_2009@hotmail.com

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The purpose of this study is to present the Marshall- Olkin extended Gompertz Makeham (MOEGM) lifetime distribution, which has four parameters. As a result, we will describe some of the structural elements that are introduced for this model. The maximum likelihood approach is used to estimate the model parameters, and it is well known that likelihood estimators for unknown parameters are not always available. As a result, we examine the prior distributions, which allow for prior dependence among the components of the parameter vector, as well as the Bayesian estimators derived with respect to the squared error loss function. A Monte Carlo simulation research is carried out to examine the performance of the likelihood estimators and the Bayesian technique. Finally, we demonstrate the significance of the new model. And to conclude, we illustrate the importance of the new model by exploring some of the empirical applications of physics to show it's flexibility and potentiality of a new model.

1. Introduction

Gompertz distribution has been obtained by Gompertz [1]. It is critical in the analysis of survival periods in several areas, including marketing, gerontology, biology, and computer science. It was used to characterize human mortality, develop growth models, and create actuarial tables. The Gompertz distribution's hazard rate function (hrf) is an increasing function used by actuaries and demographers to characterize the distribution of adult life lengths. Makeham [2] looked at the Gompertz distribution's fit to actuarial data and found that by modifying it, he could enhance the fit. This change is now known as the Gompertz- Makeham (GM) distribution. The Gompertz - Makeham (GM) distribution studied by Bailey et al. [3]. The GM distribution has been frequently utilised in actuarial tables and growth models to describe human mortality.

Missov and Lenart [4] discovered closed-form solutions to the life-expectancy integral in homogeneous and gamma-heterogeneous populations, as well as in the presence or absence of the Makeham factor. Chukwu and Ogunde [5] introduced Kumaraswamy Gompertz-Makeham, a five-parameter generalized version of the GM with decreasing, rising, and bathtub-shaped failure rate functions. For the GM model, Wrycza [6] developed a straightforward formulation of life table entropy.

The cumulative distribution function (c.d.f.) of the Gompertz- Makeham (GM) distribution is given by

$$G_{\mathrm{GM}}(x,\theta,\alpha,\lambda) = 1 - e^{-\theta x - (\alpha/\lambda)\left(e^{\lambda x}-1\right)}, \quad x>0, \tag{1}$$

where $\lambda > 0$ is a scale parameter, $\theta > 0$ and $\alpha > 0$ are shape parameters.The corresponding probability density function (p.d.f) and hrf are given by

$$g_{GM}(x,\theta,\alpha,\lambda) = \left(\alpha e^{\lambda x} + \theta\right) e^{-\theta x - (\alpha/\lambda)\left(e^{\lambda x} - 1\right)},$$

$$h_{GM}(x,\theta,\alpha,\lambda) = \frac{g_{GM}(x,\theta,\alpha,\lambda)}{\overline{G}_{GM}(x,\theta,\alpha,\lambda)} = \alpha e^{\lambda x} + \theta, \quad (2)$$

respectively.

There has lately been a resurgence of interest in developing innovative generators for univariate continuous distributions by introducing one or more additional shape factors into the baseline model. This parameter induction has been demonstrated to be useful in analyzing tail characteristics and increasing the goodness-of-fit of the recommended generator family. These asymmetric distributions were formed by adding new parameters to a baseline c.d.f., resulting in a new family of more analytically flexible asymmetric distributions. In the statistical literature, several classes have been proposed for constructing new distributions by adding one or more parameters. The beta-G by Eugene et al. [7], Kumaraswamy-G by Cordeiro and de Castro [8], new extended cosine-G distributions by Muhammad et al. [9], new truncated muth generated family by Almarashi et al. [10], odd Perks-G class by Elbatal et al. [11], and the Zografos-Balakrishnan-G family by Nadarajah et al. [12] are just a few examples of well-known generators.

Marshall and Olkin [13] suggested a general approach for adding a new positive shape parameter to a baseline distribution, resulting in the Marshall-Olkin family of distributions (abbreviated as "MO" for short). The baseline distribution is included in this family as a fundamental instance, and some distributions have more flexibility for representing diverse types of data. The proportional odds family with tilt parameter are other names for the MO family of distributions (Marshall and Olkin [14]). The Marshall and Olkin family's c.d.f. is defined as:

$$F(x,\gamma) = \frac{G(x)}{1 - \overline{\gamma}\overline{G}(x)}, \quad x > 0,\ \gamma > 0. \quad (3)$$

The survival function $\overline{F}(x,\gamma)$ is given by

$$\overline{F}(x,\gamma) = \frac{\gamma \overline{G}(x)}{1 - \overline{\gamma}\overline{G}(x)}, \quad (4)$$

where $\overline{\gamma} = (1-\gamma)$, for $\gamma = 1$, we get the baseline distribution, i.e., $\overline{F}(x) = \overline{G}(x)$, where the shape parameter γ is called tilt parameter, since the hazard rate function $h(x;\gamma)$ of the transformed distribution is shifted below when $\gamma \geq 1$ or shifted below when $0 < \gamma \leq 1$ from the baseline hazard rate function $h_G(x)$. In fact, $h(x;\gamma) \leq h_G(x)$ when $\gamma \geq 1$ and $h(x;\gamma) \geq h_G(x)$ when $0 < \gamma \leq 1$. The corresponding p.d.f becomes

$$f(x,\gamma) = \frac{\gamma g(x)}{\left[1 - \overline{\gamma}\overline{G}(x)\right]^2}, \quad (5)$$

the hrf is given by

$$h(x,\gamma) = \frac{f(x,\gamma)}{\overline{F}(x,\gamma)} = \frac{\gamma g(x)}{\overline{G}(x)\left[1 - \overline{\gamma}\overline{G}(x)\right]}. \quad (6)$$

In recent years, several authors have used this method to extend well-known distributions. A few examples include Ghitany et al. [15] presented censored scheme of MO extended Weibull distribution, Jayakumar and Mathew [16] introduced on a generalization to MO with application of Burr type XII distribution, Pérez-Casany and Casellas [17] presented MO extended Zipf Distribution, Krishna et al. [18] proposed the MO Fréchet distribution, Gui [19] introduced the MO power log - normal distribution and and its applications to survival data, Idika et al. [20] introduced the MO generalized Erlange - truncated exponential distribution, MirMostafaee et al. [21] represented the MO extended generalized Rayleigh distribution, among others. The aim of this paper is to propose a new class of lifetime distributions called "The MO extended Gompertz-Makeham" distribution, as referred to as (MOEGM).

In this paper, the Marshall- Olkin extended Gompertz Makeham (MOEGM) lifetime distribution has been presented, which has four parameters. As a result, we will describe some of the structural properties that are introduced for this model. The maximum likelihood approach is used to estimate the model parameters, and it is well known that likelihood estimators for unknown parameters are not always available. As a result, we examine the prior distributions, which allow for prior dependence among the components of the parameter vector, as well as the Bayesian estimators derived with respect to the squared error loss function. A Monte Carlo simulation and two real data sets are carried out to examine the performance of the model and likelihood estimators and the Bayesian technique.

The rest of the paper is organized as follows: The Marshall-Olkin extended Gompertz Makeham distribution and its technique are defined in Section 2. Section 3 introduces and investigates numerous structural characteristics properties of the MOEGM distribution. Section 4 shows the likelihood estimates for the unknown parameters. Section 5 shows the Bayesian estimates of the unknown parameters. Simulation results are carried out in Section 6. Section 7 depicts two real-world data applications. Finally, we demonstrate the significance of this study's closing remarks.

2. The MOEGM Model

In this section, we introduce the four parameter Marshall-Olkin extended Gompertz-Makeham (MOEGM) distribution. Using equations (1), (3) and (4) shown in the previous section, the c.d.f. and survival function can be written as follows,

$$F(x;\theta,\alpha,\lambda,\gamma)=\frac{G(x)}{1-\overline{\gamma}\overline{G}(x)}=\frac{1-e^{-\theta x-(\alpha/\lambda)\left(e^{\lambda x}-1\right)}}{1-\overline{\gamma}e^{-\theta x-(\alpha/\lambda)\left(e^{\lambda x}-1\right)}},$$
$$\overline{F}(x,\theta,\alpha,\lambda,\gamma)=\frac{\overline{G}(x)}{1-\overline{\gamma}\overline{G}(x)}=\frac{e^{-\theta x-(\alpha/\lambda)\left(e^{\lambda x}-1\right)}}{1-\overline{\gamma}e^{-\theta x-(\alpha/\lambda)\left(e^{\lambda x}-1\right)}}, \tag{7}$$

respectevely. The corresponding p.d.f given by

$$f(x;\theta,\alpha,\lambda,\gamma)=\frac{\gamma\left(\alpha e^{\lambda x}+\theta\right)e^{-\theta x-(\alpha/\lambda)\left(e^{\lambda x}-1\right)}}{\left[1-\overline{\gamma}e^{-\theta x-(\alpha/\lambda)\left(e^{\lambda x}-1\right)}\right]^{2}}. \tag{8}$$

Henceforth, Let $X\sim \mathrm{MOEGM}(\varphi)$, having p.d.f. (8) where $\varphi=(\theta,\alpha,\lambda,\gamma)$. Figure 1 display some plots of the p.d.f. of MOEGM model for some different parameter values.

The failure (hazard) rate function in event time analysis quantifies the current likelihood of failure for the population that has not yet failed. The hrf is essential when dealing with lifetime data in reliability analysis, survival analysis, and demography, as well as when building and creating models. The hrf for the Marshall-Olkin extended Gompertz-Makeham distribution is as follows in Figure 2. Figure 2 display some plots of the hrf of MOEGM model for some different parameter values.

$$h(x,\varphi)=\frac{f(x;\varphi)}{\overline{F}(x;\varphi)}=\frac{\gamma\left(\alpha e^{\lambda x}+\theta\right)}{1-\overline{\gamma}e^{-\theta x-(\alpha/\lambda)\left(e^{\lambda x}-1\right)}}. \tag{9}$$

2.1. Expansion of p.d.f. In this subsection, we present the expansion of the *MOEGM* density function in terms of an infinite linear combination of Gompertz-Makeham distribution. using the power series expansion

$$(1-z)^{-n}=\sum_{i=0}^{\infty}\frac{\Gamma(n+i)}{\Gamma(n)i!}z^{i},\quad n>0,\ |z|<1. \tag{10}$$

We get

$$\left[1-\overline{\gamma}e^{-\theta x-(\alpha/\lambda)\left(e^{\lambda x}-1\right)}\right]^{-2}=\sum_{i=0}^{\infty}(i+1)\overline{\gamma}^{i}e^{-i\left[\theta x+(\alpha/\lambda)\left(e^{\lambda x}-1\right)\right]}, \tag{11}$$

substituting equation (11) into equation (8), we get

$$f(x;\varphi)=\gamma\sum_{i=0}^{\infty}(i+1)\overline{\gamma}^{i}\left(\alpha e^{\lambda x}+\theta\right)e^{-(i+1)\left[\theta x+(\alpha/\lambda)\left(e^{\lambda x}-1\right)\right]}. \tag{12}$$

Using the series expansion of $e^{-(i+1)e^{\lambda x}}$ as follows

$$e^{-(i+1)(\alpha/\lambda)e^{\lambda x}}=\sum_{j=0}^{\infty}\frac{(-1)^{j}(i+1)^{j}(\alpha/\lambda)^{j}}{j!}e^{\lambda jx}, \tag{13}$$

thus after some algebra (12) can be written as

$$f(x;\varphi)=\sum_{k=0}^{\infty}\delta_{k}\left(\alpha(j+1)^{k}+\theta j^{k}\right)x^{k}e^{-(i+1)\theta x}, \tag{14}$$

where

$$\delta_{k}=\sum_{i,j=0}^{\infty}\frac{\gamma(i+1)^{j+1}(-1)^{j+k}\overline{\gamma}^{i}e^{(i+1)(\alpha/\lambda)}(\alpha/\lambda)^{j}\lambda^{k}}{j!k!}. \tag{15}$$

3. Statistical Features

3.1. Quantile Function. For a random variable X has c.d.f. of Marshall- Olkin power generalized Weibull distribution, the quantile function $Q(p)$ is given by the relation

$$\theta x_{p}+\frac{\alpha}{\lambda}\left(e^{\lambda x_{p}}-1\right)+\log\left[\frac{1-p}{1-p\overline{\gamma}}\right]=0,\quad p\in(0,1). \tag{16}$$

By equation (16), in addition to using the qf to obtain the Bowley's skewness and the Moors' kurtosis, is highly useful for generating MOEGM random variate and can be simply applied. Bowley's skewness is based on quartiles, as described by Kenney and Keeping [22], it's given by

$$B_{S}=\frac{Q(3/4)-2Q(2/4)+Q(1/4)}{Q(3/4)-Q(1/4)}, \tag{17}$$

and the Moor's kurtosis, see Moors [23], is given by

$$M_{k}=\frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}, \tag{18}$$

where $Q(\cdot)$ is the quantile function given by equation (16).

3.2. Moments. The r_{th} moment of the MOEGM distribution is discussed in this subsection. In any statistical analysis, especially in applications, moments are crucial and important. It can be used to investigate a distribution's most essential properties and qualities (e.g., tendency, dispersion, skewness and kurtosis).

3.3. Theorem Quantile Function. If X has $X\sim \mathrm{MOEGM}(\varphi)$, where $\varphi=(\theta,\alpha,\lambda,\gamma)$ then the r_{th} moment of X is given by

$$\mu_{r}'(x)=\sum_{k=0}^{\infty}\delta_{k}\left(\alpha(j+1)^{k}+\theta j^{k}\right)\frac{\Gamma(r+k+1)}{[(i+1)\theta]^{r+k+1}}. \tag{19}$$

Proof. Let X be a random variable with the distribution MOEGM. The well-known formula can be used to calculate the r_{th} ordinary moment.

$$\begin{aligned}\mu_{r}'(x)&=\int_{0}^{\infty}x^{r}f(x,\varphi)\mathrm{d}x\\&=\sum_{k=0}^{\infty}\delta_{k}\left(\alpha(j+1)^{k}+\theta j^{k}\right)\int_{0}^{\infty}x^{r+k}e^{-(i+1)\theta x}\mathrm{d}x,\end{aligned} \tag{20}$$

setting $y=(i+1)\theta x$, after some algebra, the r_{th} ordinary moment can be written as

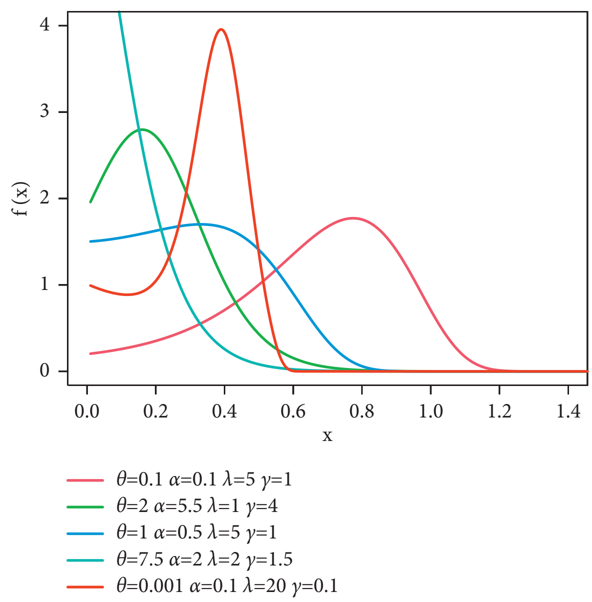


Figure 1: The p.d.f. plot for the MOEGM model.

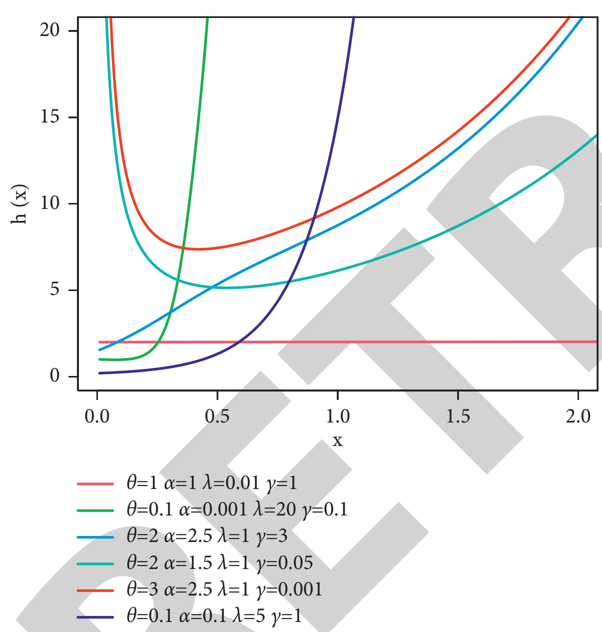


Figure 2: The hrf plot for the MOEGM model.

$$\mu_r'(x) = \sum_{k=0}^{\infty} \delta_k\left(\alpha(j+1)^k + \theta j^k\right) \frac{\Gamma(r+k+1)}{[(i+1)\theta]^{r+k+1}}, \tag{21}$$

where $\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} \mathrm{d}x$ denotes the gamma function. □

3.4. Moment Generating Function. Moment generating functions are helpful for a variety of reasons, one of which being their usage in sums of random variables analysis. When compared to working directly with the probability function or c.d.f. of a random variable, it provides the foundation for an alternative approach to analytic solutions.

Theorem 1. *If X has the MOEGM$(\theta, \alpha, \lambda, \gamma)$, then the the moment generating function (mgf) of X is given as follows*

$$M_X(t) = \sum_{k=0}^{\infty} \delta_k\left(\alpha(j+1)^k + \theta j^k\right) \frac{\Gamma(k+1)}{[\theta(i+1) - t]^{k+1}}. \tag{22}$$

Proof. We begin with the well-known simplification of the moment generating function, which is as follows:

$$\begin{aligned} M_X(t) &= E\left(e^{tX}\right) \\ &= \int_0^{\infty} e^{tx} f(x)\mathrm{d}x \\ &= \sum_{k=0}^{\infty} \delta_k\left(\alpha(j+1)^k + \theta j^k\right) \int_0^{\infty} x^k e^{-[(i+1)\theta - t]x}\mathrm{d}x \\ &= \sum_{k=0}^{\infty} \delta_k\left(\alpha(j+1)^k + \theta j^k\right) \frac{\Gamma(k+1)}{[\theta(i+1) - t]^{k+1}}, \end{aligned} \tag{23}$$

which completes the proof. □

3.5. Conditional Moments. The s_{th} lower incomplete moment of MOEGM distribution is

$$\begin{aligned} \eta_s(t) &= \int_0^t x^s f(x)\mathrm{d}x \\ &= \sum_{k=0}^{\infty} \delta_k\left(\alpha(j+1)^k + \theta j^k\right) \int_0^t x^{s+k} e^{-(i+1)\theta x}\mathrm{d}x \\ &= \sum_{k=0}^{\infty} \delta_k\left(\alpha(j+1)^k + \theta j^k\right) \frac{\gamma(s+k+1, \theta(i+1)t)}{[\theta(i+1)]^{s+k+1}}, \end{aligned} \tag{24}$$

where $\gamma(s,t) = \int_0^t x^{s-1} e^{-x}\mathrm{d}x$ is the lower incomplete gamma function. The first incomplete moment of X, denoted by, $\eta_1(t)$, is computed using equation (24) by setting $s = 1$ as

$$\eta_1(t) = \sum_{k=0}^{\infty} \delta_k\left(\alpha(j+1)^k + \theta_j^k\right) \frac{\gamma(k+2, \theta(i+1)t)}{[\theta(i+1)]^{k+2}}. \tag{25}$$

Similarly, the s_{th} upper incomplete moment of MOEGM distribution is

$$\begin{aligned} \xi_s(t) &= \int_t^{\infty} x^s f(x)\mathrm{d}x \\ &= \sum_{k=0}^{\infty} \delta_k\left(\alpha(j+1)^k + \theta j^k\right) \int_t^{\infty} x^{s+k} e^{-(i+1)\theta x}\mathrm{d}x \\ &= \sum_{k=0}^{\infty} \delta_k\left(\alpha(j+1)^k + \theta j^k\right) \frac{\Gamma(s+k+1, \theta(i+1)t)}{[\theta(i+1)]^{s+k+1}}, \end{aligned} \tag{26}$$

where $\Gamma(s,t) = \int_t^{\infty} x^{s-1} e^{-x}\mathrm{d}x$ is the upper incomplete gamma function.

The mean residual lifespan (MRL) has a diverse set of uses and applications see Lai and Xie [24]. The expected extended life length for a unit alive at age t is represented by the MRL (or life expectancy at age t). The MRL is given by

$$\mu(t) = E(X|X > t) = \frac{\xi_1(t)}{\overline{F}(t)} - t, \tag{27}$$

where $\xi_1(t)$ is the first incomplete moment of X and by setting $s = 1$ in equation (26), we get

$$\mu(t) = \frac{1}{\overline{F}(t)} \sum_{k=0}^{\infty} \delta_k\left(\alpha(j+1)^k + \theta j^k\right) \frac{\Gamma(k+2, \theta(i+1)t)}{[\theta(i+1)]^{k+2}} - t. \tag{28}$$

In addition, the mean inactivity time (MIT) shows the amount of time that has passed after an item has failed, assuming that the failure happened in $(0; t)$. For $t > 0$, the MIT of X is defined by

$$\begin{aligned}\tau(t) = E(X|X < t) &= t - \frac{\eta_1(t)}{F(t)} \\ &= t - \frac{1}{F(t)} \sum_{k=0}^{\infty} \delta_k\left(\alpha(j+1)^k + \theta j^k\right) \frac{\gamma(k+2, \theta(i+1)t)}{[\theta(i+1)]^{k+2}}.\end{aligned} \tag{29}$$

4. Estimation and Inference

Only full samples are used to calculate the maximum likelihood estimates (MLEs) of the parameters of the MOEGM distribution in this section. Let $X_1, \ldots, X_n$ be a random sample of size n from MOEGM(φ) where $\varphi = (\theta, \alpha, \lambda, \gamma)$. Let $\varphi = (\theta, \alpha, \lambda, \gamma)^T$ be the parameter vector. The log-likelihood function for the vector of parameters $\varphi = (\theta, \alpha, \lambda, \gamma)$ can be written as

$$\begin{aligned}\log L(\varphi) = n \log(\gamma) + \sum_{i=1}^{n} \log\left(\alpha e^{\lambda x_i} + \theta\right) - \theta \sum_{i=1}^{n} x_i - \frac{\alpha}{\lambda} \sum_{i=1}^{n} \log\left(e^{\lambda x_i} - 1\right) \\ - 2 \sum_{i=1}^{n} \log\left[1 - \overline{\gamma} e^{-\theta x_i - (\alpha/\lambda)\left(e^{\lambda x_i} - 1\right)}\right].\end{aligned} \tag{30}$$

The following is the associated score function:

$$U_n(\varphi) = \left[\frac{\partial L(\varphi)}{\partial \theta}, \frac{\partial L(\varphi)}{\partial \alpha}, \frac{\partial L(\varphi)}{\partial \lambda}, \frac{\partial L(\varphi)}{\partial \gamma}\right]^T. \tag{31}$$

Either directly or by solving the nonlinear likelihood equations derived by differentiating equation (30), the log-likelihood can be maximized. The score vector's components of likelihood are as follows:

$$\begin{aligned}
\frac{\partial \log L(\varphi)}{\partial \theta} &= \sum_{i=1}^{n} \frac{1}{\alpha e^{\lambda x_i} + \theta} - \sum_{i=1}^{n} x_i - 2 \sum_{i=1}^{n} \frac{\overline{\gamma} x_i e^{-\theta x_i - (\alpha/\lambda)\left(e^{\lambda x_i} - 1\right)}}{1 - \overline{\gamma} e^{-\theta x_i - (\alpha/\lambda)\left(e^{\lambda x_i} - 1\right)}}, \\
\frac{\partial \log L(\varphi)}{\partial \alpha} &= \sum_{i=1}^{n} \frac{e^{\lambda x_i}}{\alpha e^{\lambda x_i} + \theta} - \frac{1}{\lambda} \sum_{i=1}^{n} \log\left(e^{\lambda x_i} - 1\right) - \frac{2}{\lambda} \sum_{i=1}^{n} \frac{\overline{\gamma} e^{-\theta x_i - (\alpha/\lambda)\left(e^{\lambda x_i} - 1\right)}\left(e^{\lambda x_i} - 1\right)}{1 - \overline{\gamma} e^{-\theta x_i - (\alpha/\lambda)\left(e^{\lambda x_i} - 1\right)}}, \\
\frac{\partial \log L(\varphi)}{\partial \lambda} &= \sum_{i=1}^{n} \frac{\alpha x_i e^{\lambda x_i}}{\alpha e^{\lambda x_i} + \theta} - \alpha \sum_{i=1}^{n} \left(\frac{x_i e^{\lambda x_i}}{\lambda} - \frac{e^{\lambda x_i} - 1}{\lambda^2}\right) \\
&\quad + \frac{2\alpha}{\lambda^2} \sum_{i=1}^{n} \frac{\overline{\gamma} e^{-\theta x_i - (\alpha/\lambda)\left(e^{\lambda x_i} - 1\right)}\left(\lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1\right)}{1 - \overline{\gamma} e^{-\theta x_i - (\alpha/\lambda)\left(e^{\lambda x_i} - 1\right)}}, \\
\frac{\partial \log L(\varphi)}{\partial \gamma} &= \frac{n}{\gamma} - 2 \sum_{i=1}^{n} \frac{e^{-\theta x_i - (\alpha/\lambda)\left(e^{\lambda x_i} - 1\right)}}{1 - \overline{\gamma} e^{-\theta x_i - (\alpha/\lambda)\left(e^{\lambda x_i} - 1\right)}}.
\end{aligned} \tag{32}$$

The maximum likelihood estimation (MLE) of φ, say $\widehat{\varphi}$, is obtained by solving the nonlinear system $U_n(\varphi) = 0$.

5. Bayesian Estimation

The Bayesian technique deals with the parameters because random and parameter uncertainties are represented by a previous joint distribution that was formed before the failure data was collected. The flexibility of the Bayesian technique to incorporate past knowledge into research makes it particularly useful in the study of reliability, as one of the major challenges with reliability analysis is a lack of data. Prior gamma distributions are used in the θ, λ, α and γ parameters of the MOEGM distribution, where θ, λ, α, and γ are non-negative values. As separate joint prior density functions, the θ, λ, α, and γ parameters as follows:

$$\prod(\theta, \lambda, \alpha, \gamma) \propto \theta^{q_1 - 1} \lambda^{q_2 - 1} \alpha^{q_3 - 1} \gamma^{q_4 - 1} e^{-\left(w_1 \theta + w_2 \lambda + w_3 \alpha + w_4 \gamma\right)}. \tag{33}$$

The likelihood function of the MOEGM distribution and joint prior density (30) are used to produce the joint posterior density function of θ, λ, α, and γ.

$$\pi(\theta,\lambda,\alpha,\gamma|\underline{x}) = \frac{L(\underline{x}|\theta,\lambda,\alpha,\gamma)\prod(\theta,\lambda,\alpha,\gamma)}{\int_\theta\int_\lambda\int_\alpha\int_\gamma L(\underline{x}|\theta,\lambda,\alpha,\gamma)\prod(\theta,\lambda,\alpha,\gamma)\mathrm{d}\theta\mathrm{d}\lambda\mathrm{d}\alpha\mathrm{d}\gamma},$$
$$\propto L(\underline{x}|\theta,\lambda,\alpha,\gamma)\prod(\theta,\lambda,\alpha,\gamma), \propto \theta^{q_1-1}\lambda^{q_2-1}\alpha^{q_3-1}\gamma^{n+q_4-1}e^{-\theta\left(w_1+\sum_{i=1}^n x_i\right)-\alpha\left[w_3+\left(\sum_{i=1}^n\left(e^{\lambda x_i}-1\right)/\lambda\right)\right]}\times \quad (34)$$
$$\prod_{i=1}^n \frac{e^{-w_4\gamma}\left(\alpha e^{-\lambda(w_2-x_i)}+\theta e^{-w_2\lambda}\right)}{\left[1-\overline{\gamma}e^{-\theta x_i-\alpha/\lambda\left(e^{\lambda x_i}-1\right)}\right]^2}.$$

The majority of Bayesian inference algorithms are based on symmetric loss functions. A prominent symmetric loss function is the squared-error loss function (SELF). The Bayesian estimators of θ, λ, α, and γ, say $(\widetilde{\theta}_B, \widetilde{\lambda}_B, \widetilde{\alpha}_B, \widetilde{\gamma}_B)$ based on SELF.

$$\begin{aligned}\widetilde{\theta} &= E(\theta\,|\,\alpha,\lambda,\gamma,\underline{x}),\\ \widetilde{\lambda} &= E(\lambda\,|\,\alpha,\theta,\gamma,\underline{x}),\\ \widetilde{\alpha} &= E(\alpha\,|\,\theta,\lambda,\gamma,\underline{x}),\\ \widetilde{\gamma} &= E(\gamma\,|\,\theta,\lambda,\alpha,\underline{x}).\end{aligned} \quad (35)$$

It should be noted that the integrals supplied by equation (35) cannot be deduced clearly. As a result, we use Markov-Chain-Monte-Carlo (MCMC) to approximate the value of expectations in equation (35).

An observation was made that the integrals are given by equation (35) are not possible to derive explicitly. As a result, we employ the MCMC technique to approximate the value of integrals in equation (35). Many of studies used MCMC technique such as Al-Babtain et al. [25], Tolba et al. [26, 27], and Bantan et al. [28].

In Gibbs samplers, more general Metropolis algorithms are important subclasses of MCMC algorithms. Two of the most prevalent MCMC methodologies are the Metropolis-Hastings (MH) and Gibbs sampling methods. The MH technique, like acceptance-rejection sampling, assumes that each algorithm iteration can yield a candidate value from a proposal distribution. We apply the MH in the Gibbs sampling phases to get random samples of conditional posterior densities from the MOEGM distribution:

$$\begin{aligned}\pi(\theta|\lambda,\alpha,\gamma,\underline{x}) &\propto \theta^{n+q_1-1}e^{-\theta\left(w_1+\sum_{i=1}^n x_i\right)}\prod_{i=1}^n\frac{1}{\left[1-\overline{\gamma}e^{-\theta x_i-(\alpha/\lambda)\left(e^{\lambda x_i}-1\right)}\right]^2},\\ \pi(\lambda|\theta,\alpha,\gamma,\underline{x}) &\propto \lambda^{q_2-1}e^{-\left(\alpha\sum_{i=1}^n\left(e^{\lambda x_i}-1\right)/\lambda\right)}\prod_{i=1}^n\frac{e^{-\lambda(w_2-x_i)}+\theta e^{-w_2\lambda}}{\left[1-\overline{\gamma}e^{-\theta x_i-(\alpha/\lambda)\left(e^{\lambda x_i}-1\right)}\right]^2},\\ \pi(\alpha|\theta,\lambda,\gamma,\underline{x}) &\propto \alpha^{q_3-1}e^{-\alpha w_3}\prod_{i=1}^n\frac{\alpha}{\left[1-\overline{\gamma}e^{-\theta x_i-(\alpha/\lambda)\left(e^{\lambda x_i}-1\right)}\right]^2},\\ \pi(\gamma|\theta,\lambda,\alpha,\underline{x}) &\propto \gamma^{n+q_4-1}\prod_{i=1}^n\frac{e^{-w_4\gamma}}{\left[1-\overline{\gamma}e^{-\theta x_i-(\alpha/\lambda)\left(e^{\lambda x_i}-1\right)}\right]^2}.\end{aligned} \quad (36)$$

6. Simulation

The Monte-Carlo simulation approach is used in this section to compare the likelihood estimation method with the Bayesian estimation method. The *R* language is used to estimate MOEGM distribution parameters using MLE and a Bayesian estimation approach based on MCMC under SELF. Monte-Carlo experiments are carried out using 10000 randomly generated MOEGM distribution samples, where x represents the MOEGM lifetime for various parameter actual values and sample sizes n: $(30, 70, 150, \text{and } 200)$. The best estimator approaches could be described as minimizing estimator bias $(A1)$ and mean squared error $(A2)$. The MOEGM distribution's true parameters have been determined.

Tables 1–3 describe the simulation results of the approaches presented in this paper for point estimate. In order to do the essential comparison between various point estimating methods, we examine the A1 and A2 values. As a result, the following conclusions were drawn:

(1) For parameters of the MOEGM distribution, the $A1$ and $A2$ decrease as sample size n grows.

(2) The best estimating method is Bayesian estimation.

(3) The $A1$ and $A2$ for all parameters diminish as γ increases.

(4) The $A1$ and $A2$ for all parameters increase as θ increases.

7. Applications of Physics

In this section, two real-world data applications are used to demonstrate the significance of the MOEGM distribution. We employ the Akaike information criterion measures (AICM), Bayesian information criterion measures (BICM), Consistent Akaike information Criterion (AICCM), Kolmogorov-Smirnov statistics (KSS), and the PVKSS test to compare the models. Smaller values of these statistical metrics equate to a better fit to the data set. The maximum likelihood approach is used to estimate the parameters of each distribution, while the Bayesian estimation method is used to estimate the parameters of the *MOEGM* distribution.

7.1. First Real Data of Flood Peaks. In this subsection, the first application of real data set is employed to illustrate the importance of the MOEGM distribution. This data set represents 72 excrescences of flood peaks for the years 1958–1984 (rounded to one decimal place) of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The first data set is: "1.7, 2.5, 27.4, 1.0, 27.1, 2.2, 22.9, 1.7, 0.1, 1.1, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 21.5, 27.6, 36.4, 2.7, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 64.0, 1.5, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5 and 7.0." The fit of the proposed model are compared with the transmuted Gompertz-Makeham (TGM) (Abd El-Bar [29]), beta generalized Gompertz (BGG) (Benkhelifa [30]), kumaraswamy gompertz makeham (KGM) (Chukwu and Ogunde [5]), Gompertz Lomax (GL) (Oguntunde et al. [31]), exponentiated generalized Weibull-Gompertz (EGWG) (El-Bassiouny et al. [32]), generalized Gompertz (GG) (El-Gohary et al. [33]) and Gompertz models.

Table 4 presents the MLEs with standard error (SE) of the model parameters for the first data set. The values of AICM, BICM, AICCM, HQICM, KSS and the PVKS are presented for the MOEGM model and the other models.

From Table 4, we conclude that the MOEGM model gives the best fit, where the values of AICM, BICM, AICCM, HQICM, and KSS are smaller and the PVKS is higher for the MOEGM model when compared with those values of the other models. Figures 3(a), 3(b) illustrate the p.d.f., empirical c.d.f.s and probability plots, respectively, of the comparative models to show the over fitting of the MOEGM distribution. Figures 3(a)–3(c) illustrate estimated p.d.f. with histogram, estimated c.d.f. with empirical c.d.f., and Q-Q plot of the MOEGM distribution, respectively. Figures 4 and 5 clarify probability plots of the comparative models to show the over fitting of the MOEGM distribution.

Based on the results in Table 4 and Figures 3 and 4, we conclude that the MOEGM distribution is a better fit than comparative models for this data set.

Table 5 discussed MLE and Bayesian estimation methods comparing by SE, we note that the Bayesian estimation has smaller SE than MLE. The trace plots and the convergence plots of parameters by MCMC results of the MOEGM distribution are obtained in right and left Figure 6. The posterior density of MCMC findings for each parameter is shown in the center of Figure 6, which indicates a symmetric normal distribution comparable to the proposed distribution.

7.2. Second Real Data of Stochastic Processes. In this subsection, we discuss data set of stochastic processes which was first introduced by Aarset [34] and represents the lifetimes of 50 devices (in weeks). This data set, also reported in Benkhelifa [30] BGG distribution, and Abd El-Bar [29] to discuss TGM distribution, is: "0.1, 0.2, 1, 1, 1, 67, 67, 67, 72, 75, 79, 1, 1, 2, 3, 6, 7, 11, 60, 63, 63, 12, 18, 18, 85, 85, 85, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 67, 82, 82, 83, 84, 84, 84, 85, 85, 86, 86." The TGM distribution is better than Gompertz, shifted Gompertz, transmuted Lindley, Gompertz Makeham, transmuted Burr type III, transmuted Gompertz, transmuted exponentiated exponential, and transmuted generalized linear exponential distributions, for more details see Abd El-Bar [29]. The BGG distribution is better than Gompertz, beta generalized exponential, generalized exponential, beta Gompertz, exponential, beta exponential, and GG, for more details see Benkhelifa [30].

Table 6 presents the MLEs of the model parameters for the stochastic processes data set. The values of AICM, BICM, AICCM, HQICM, KSS and the PVKS are presented for the MOEGM model and the TGM, and the BGG distribution.

From Table 6, we conclude that the MOEGM model gives the best fit, where the values AICM, BICM, AICCM, HQICM, and KSS are smaller for MOEFM distribution than TGM and BGG distribution, and the PVKS is higher for the MOEGM model than TGM and BGG distribution. Figures 7(a), 7(b) illustrate the p.d.f.s, empirical c.d.f.s and probability plots, respectively, of the comparative models to show the over fitting of the MOEGM distribution.

Table 1: $A1$ and $A2$ of MOEGM parameters by MLE, and Bayesian when. $\alpha = 0.5$, $\lambda = 0.5$.

			$\gamma = 0.75$				$\gamma = 3$			
$\alpha = 0.5$, $\lambda = 0.5$			MLE		Bayesian		MLE		Bayesian	
θ	n		$A1$	$A2$	$A1$	$A2$	$A1$	$A2$	$A1$	$A2$
0.75	30	θ	0.0663	0.2090	−0.0105	0.0807	0.1024	0.1833	−0.0064	0.0839
		α	0.2976	2.6962	−0.0118	0.0642	−0.0389	0.3818	0.0092	0.0688
		λ	0.5169	2.1535	−0.0483	0.0773	0.4304	1.0915	−0.0201	0.0679
		γ	0.6221	5.1127	−0.0147	0.0593	0.2325	3.3815	−0.0333	0.1281
	70	θ	0.0361	0.1570	−0.0009	0.0252	0.1016	0.1289	0.0094	0.0244
		α	0.0892	1.0049	−0.0178	0.0238	−0.0932	0.1873	0.0020	0.0174
		λ	0.3333	1.1596	−0.0219	0.0284	0.3409	0.4543	−0.0251	0.0271
		γ	0.2307	1.4412	−0.0108	0.0224	−0.0001	1.3741	−0.0164	0.0293
	150	θ	0.0413	0.1089	0.0009	0.0204	0.0643	0.0681	−0.0001	0.0187
		α	−0.0081	0.3685	−0.0046	0.0162	−0.0659	0.1047	0.0050	0.0119
		λ	0.2005	0.5740	−0.0189	0.0251	0.2062	0.2413	−0.0174	0.0172
		γ	0.0851	0.5711	−0.0052	0.0172	0.0176	0.6182	0.0087	0.0219
	200	θ	0.0605	0.0764	0.0019	0.0056	0.0504	0.0349	−0.0053	0.0051
		α	−0.0431	0.0853	−0.0059	0.0048	−0.0346	0.0603	0.0026	0.0045
		λ	0.1002	0.2251	−0.0083	0.0062	0.1023	0.1092	−0.0004	0.0045
		γ	0.0373	0.1438	−0.0064	0.0049	0.0306	0.4561	−0.0045	0.0064
3	30	θ	−0.6898	1.7822	−0.0026	0.1054	−0.4916	1.0311	−0.0101	0.1357
		α	0.9339	10.4192	−0.0348	0.0685	0.5875	5.2078	−0.0263	0.0934
		λ	1.5259	10.5863	−0.0581	0.1057	0.9608	5.9980	−0.0252	0.0891
		γ	0.3771	2.7854	0.0451	0.0909	0.8456	9.9552	−0.0161	0.1125
	70	θ	−0.6936	1.3025	−0.0004	0.0283	−0.4240	0.7049	−0.0079	0.0323
		α	0.5995	5.5597	−0.0336	0.0319	0.7208	4.3036	−0.0227	0.0276
		λ	0.8973	4.5855	−0.0218	0.0362	0.4344	3.7110	−0.0188	0.0314
		γ	0.0863	0.9616	−0.0041	0.0221	0.7759	9.5984	0.0030	0.0353
	150	θ	−0.4045	0.7562	−0.0024	0.0246	−0.2864	0.4528	−0.0088	0.0243
		α	0.5395	3.1483	−0.0043	0.0252	0.5922	2.7309	−0.0113	0.0204
		λ	0.3132	2.7684	−0.0180	0.0268	0.1767	2.6308	−0.0174	0.0270
		γ	0.1129	0.6149	0.0007	0.0130	0.5947	5.4817	0.0037	0.0255
	200	θ	−0.3160	0.4131	0.0007	0.0060	−0.1378	0.2110	0.0019	0.0070
		α	0.2804	0.8734	−0.0102	0.0067	0.2864	0.9394	−0.0035	0.0055
		λ	0.1636	1.3895	−0.0075	0.0081	0.0428	1.4592	−0.0057	0.0072
		γ	0.0105	0.1054	0.0013	0.0043	0.2357	2.0228	−0.0033	0.0065

Table 2: $A1$, and $A2$ of MOEGM parameters by MLE, and Bayesian when.$\alpha = 2$, $\lambda = 0.75$.

			$\gamma = 0.75$				$\gamma = 3$			
$\alpha = 2$, $\theta = 0.75$			MLE		Bayesian		MLE		Bayesian	
λ	n		$A1$	$A2$	$A1$	$A2$	$A1$	$A2$	$A1$	$A2$
0.5	30	θ	0.2234	1.3192	−0.0571	0.1101	0.1057	0.5171	−0.0292	0.0957
		α	0.2155	8.8678	0.0025	0.1119	−0.2657	2.6640	0.0056	0.1101
		λ	0.7288	6.6163	−0.0177	0.0883	0.6797	2.8948	−0.0474	0.0687
		γ	0.5717	5.5573	0.0139	0.0535	0.6411	9.8986	−0.0285	0.1085
	70	θ	0.0565	0.9048	−0.0155	0.0298	0.0954	0.3111	−0.0119	0.0304
		α	0.0168	3.6365	−0.0194	0.0301	−0.1759	1.2053	−0.0051	0.0262
		λ	0.2718	2.7239	−0.0203	0.0316	0.3094	1.1250	−0.0149	0.0300
		γ	0.2262	1.9272	0.0014	0.0197	0.3733	7.0686	−0.0003	0.0335
	150	θ	0.0826	0.6743	−0.0187	0.0269	0.0326	0.1921	−0.0013	0.0216
		α	0.1547	3.0759	−0.0044	0.0227	−0.2055	0.4865	−0.0038	0.0205
		λ	0.0107	1.8665	−0.0170	0.0246	0.2044	0.4764	−0.0012	0.0200
		γ	0.2081	1.3476	−0.00001	0.0155	−0.0034	2.7353	0.0001	0.0258
	200	θ	0.0121	0.3445	0.00004	0.0064	0.0015	0.0763	−0.0033	0.0063
		α	−0.0127	0.9029	−0.0012	0.0058	−0.1218	0.2391	0.0017	0.0059
		λ	0.0517	0.6778	−0.0063	0.0066	0.1161	0.1799	−0.0056	0.0053
		γ	0.0401	0.3068	0.0013	0.0043	−0.0460	1.2654	−0.0002	0.0059

Table 2: Continued.

			$\gamma = 0.75$				$\gamma = 3$			
$\alpha = 2$, $\theta = 0.75$			MLE		Bayesian		MLE		Bayesian	
λ	n		$A1$	$A2$	$A1$	$A2$	$A1$	$A2$	$A1$	$A2$
3	30	θ	0.5940	2.3168	−0.0579	0.1390	0.1581	1.38285	−0.0576	0.11203
		α	0.0865	5.9463	−0.0084	0.0992	−0.2814	2.56117	0.0197	0.10938
		λ	0.9393	9.0583	−0.0141	0.1159	1.1403	5.1957	−0.0045	0.1068
		γ	0.5021	3.3756	0.0013	0.0544	0.2552	9.34188	−0.0265	0.12716
	70	θ	0.4157	1.2562	−0.0165	0.0327	0.1110	0.45985	−0.0051	0.03112
		α	0.0020	3.1157	−0.0081	0.0297	−0.1964	1.6083	−0.0062	0.0274
		λ	0.4155	3.8968	−0.0107	0.0295	0.6467	2.39636	−0.0168	0.02602
		γ	0.2856	1.5833	−0.0040	0.0207	0.1558	4.69425	−0.0031	0.03366
	150	θ	0.2229	0.6404	−0.0268	0.0249	0.1088	0.1939	−0.0139	0.0266
		α	−0.1318	1.2423	0.0052	0.0212	−0.1068	0.99542	0.0015	0.02033
		λ	0.3486	2.0025	−0.0055	0.0249	0.3275	1.06376	−0.0004	0.02312
		γ	0.1034	0.6022	0.0004	0.0142	0.1821	2.7484	−0.0116	0.0256
	200	θ	0.0952	0.2498	−0.0003	0.0058	0.0622	0.06397	−0.0057	0.00671
		α	−0.1570	0.5557	−0.0010	0.0061	−0.0534	0.58078	−0.0028	0.00549
		λ	0.2533	0.7987	0.0012	0.0061	0.1733	0.5270	−0.0043	0.0058
		γ	0.0112	0.1641	0.0020	0.0042	0.1292	1.52845	−0.0012	0.00647

Table 3: $A1$ and $A2$ of MOEGM parameters by MLE, and Bayesian when.$\alpha = 1.5$, $\lambda = 2$.

			$\gamma = 0.75$				$\gamma = 3$			
$\theta = 1.5$, $\lambda = 2$			MLE		Bayesian		MLE		Bayesian	
α	n		$A1$	$A2$	$A1$	$A2$	$A1$	$A2$	$A1$	$A2$
2	30	θ	0.3758	2.0520	−0.0504	0.1184	0.1751	1.3836	−0.0540	0.1333
		α	0.1679	7.5009	−0.0406	0.1219	−0.0944	3.4698	−0.0152	0.1037
		λ	1.2045	10.8415	−0.0046	0.1205	1.0588	6.0416	0.0114	0.1023
		γ	0.4812	3.8034	0.0184	0.0523	0.8747	9.6510	−0.0330	0.1342
	70	θ	0.1450	1.1987	−0.0010	0.0338	0.2552	0.7051	−0.0007	0.0292
		α	−0.1459	2.9704	−0.0025	0.0295	−0.1261	1.7495	−0.0012	0.0275
		λ	0.7292	5.2783	−0.0206	0.0307	0.5441	2.7962	−0.0113	0.0273
		γ	0.0962	0.7837	0.0040	0.0190	0.5367	7.7801	−0.0064	0.0333
	150	θ	0.1380	0.7847	0.0020	0.0247	0.1120	0.2308	−0.0063	0.0263
		α	−0.0373	1.3426	−0.0046	0.0252	−0.1260	0.9165	−0.0007	0.0208
		λ	0.2790	2.4414	−0.0041	0.0288	0.3392	1.2381	−0.0091	0.0231
		γ	0.0879	0.4284	0.0054	0.0142	0.2194	3.2863	−0.0066	0.0243
	200	θ	0.0384	0.3915	−0.0017	0.0065	0.0734	0.1321	−0.0037	0.0061
		α	−0.1123	0.4354	0.0009	0.0069	−0.0612	0.5020	0.0042	0.0064
		λ	0.2016	0.9034	−0.0012	0.0064	0.1647	0.5829	−0.0015	0.0059
		γ	0.0069	0.1181	−0.0038	0.0046	0.1241	1.5795	0.0004	0.0066
4	30	θ	0.2236	3.2384	−0.0636	0.1240	0.2958	2.5922	−0.0100	0.1290
		α	−0.1682	7.2254	−0.0192	0.1180	−0.6578	7.8654	−0.0266	0.1177
		λ	1.1611	9.1960	−0.0561	0.1317	1.5862	9.9355	0.0020	0.1226
		γ	0.2173	1.4663	0.0239	0.0529	0.9513	9.4847	−0.0503	0.1214
	70	θ	0.2432	2.8862	−0.0067	0.0291	0.2506	1.9412	−0.0095	0.0343
		α	0.0337	5.1864	−0.0036	0.0326	−0.5213	4.9119	−0.0199	0.0336
		λ	0.3411	6.1987	−0.0065	0.0278	0.8382	5.8843	−0.0292	0.0346
		γ	0.1539	0.7071	0.0074	0.0197	0.4678	10.9169	−0.0150	0.0357
	150	θ	0.0727	1.3884	−0.0028	0.0246	0.0395	0.9230	−0.0098	0.0262
		α	−0.1128	1.7539	−0.0143	0.0257	−0.3259	2.2341	0.0030	0.0261
		λ	0.2263	2.2326	−0.0131	0.0249	0.4201	2.2449	−0.0039	0.0222
		γ	0.0540	0.2351	0.0067	0.0117	0.0980	3.8122	−0.0031	0.0239
	200	θ	0.0722	0.9579	0.0036	0.0074	0.0215	0.6763	0.0000	0.0060
		α	−0.0290	1.1019	−0.0035	0.0062	−0.2542	1.3980	−0.0041	0.0062
		λ	0.0586	1.4141	0.0012	0.0065	0.2818	1.1992	−0.0040	0.0073
		γ	0.0331	0.1423	0.0021	0.0043	0.0397	2.1774	0.0021	0.0069

Table 4: MLEs of the models parameters with SE and different measures for fitting: flood peaks data.

		MLEs	SE	KSS	PVKS	CVMS	ADS	AICM	AICCM	BICM	HQICM
MOEGM	θ	0.0141	0.0569	0.0985	0.4873	0.1014	0.5756	506.9077	507.5047	516.0144	510.5331
	α	0.0185	0.0379								
	λ	0.0319	0.0382								
	γ	0.2966	0.3969								
GL	θ	1.7323	3.7299	0.0999	0.4691	0.1193	0.6709	507.6678	508.2648	516.7744	511.2931
	α	0.4989	3.6786								
	λ	1.2268	5.6233								
	γ	0.1877	0.6431								
Gompertz	λ	0.0013	0.0081	0.1463	0.0918	0.1120	0.6459	508.2565	508.429	518.8094	512.8534
	α	0.0852	0.0148								
EGWG	θ	1.6924	19.7580	0.1026	0.4351	0.1027	0.6039	509.3512	510.2603	520.7345	513.8829
	λ	0.4178	4.5617								
	α	0.1118	4.4805								
	γ	0.5656	3.0847								
	β	0.3331	0.4674								
BGG	λ	0.0123	0.0191	0.0998	0.4799	0.4754	0.5876	508.5600	509.4691	519.9433	513.0917
	α	0.0331	0.0851								
	γ	0.5632	1.6298								
	θ	1.4283	4.6792								
	β	1.6742	3.1784								
KGM	λ	0.0229	0.0442	0.1034	0.4251	0.1102	0.5928	508.7930	509.7020	520.1763	513.3247
	α	0.0160	0.0508								
	γ	0.0399	0.0788								
	a	0.7910	0.1410								
	b	1.1351	0.9661								
TGM	λ	0.0036	0.0596	0.1290	0.1821	0.1112	0.6376	508.4915	509.0885	517.5982	512.1169
	α	0.0655	0.0499								
	θ	0.0053	0.0060								
	γ	0.2544	0.4056								

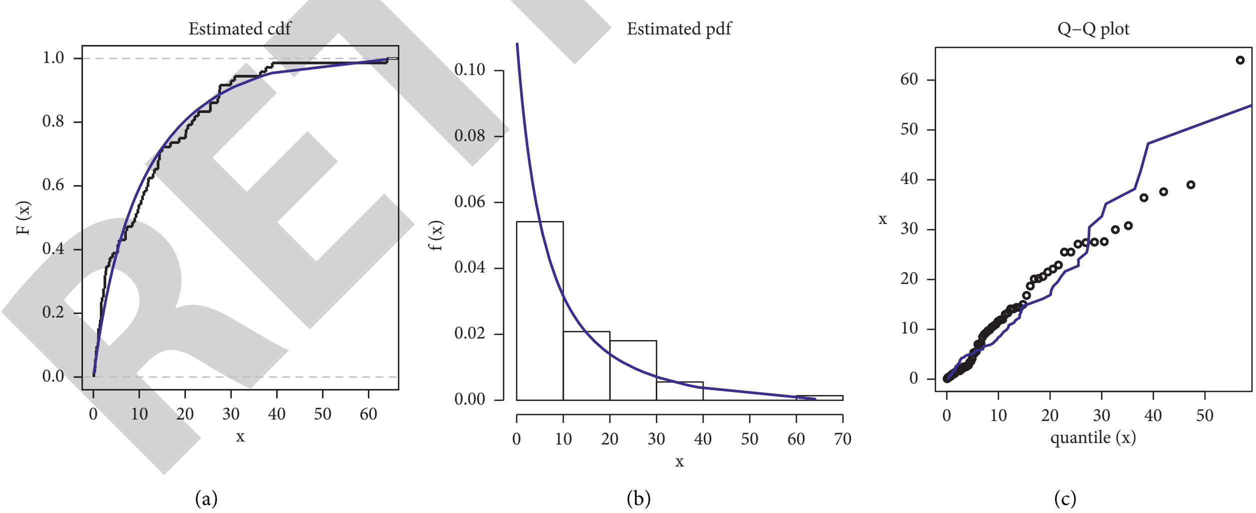


Figure 3: (a) Histogram and estimated p.d.f., (b) Estimated c.d.f. and the empirical c.d.f., (c) generate qunatile and data: flood peaks data.

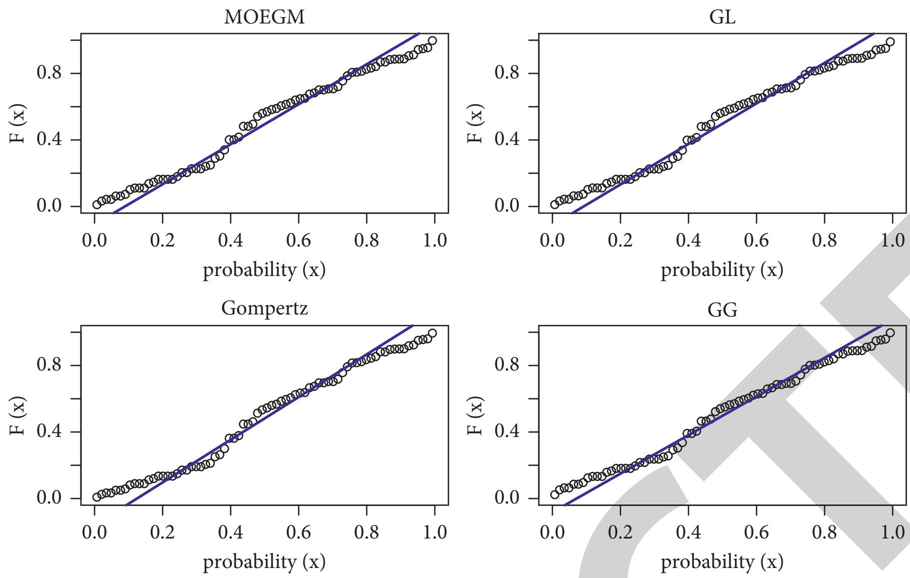


Figure 4: PP plot for different models.

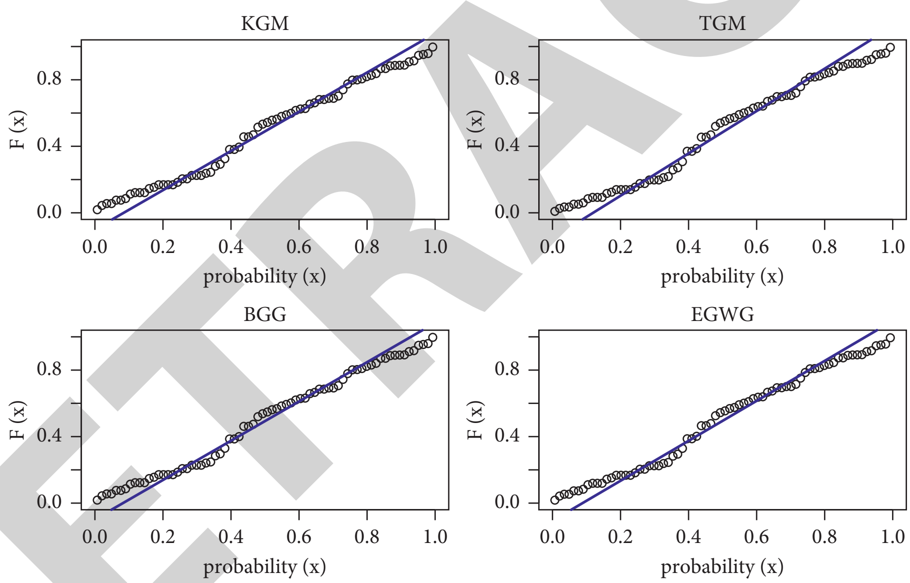


Figure 5: PP plot for different models.

Table 5: MLE and Bayesian estimation methods comparing by SE: flood peaks data.

	MLE		Bayesian	
	Estimates	SE	Estimates	SE
θ	0.0141	0.0569	0.014041	0.003221
α	0.0185	0.0379	0.018479	0.001432
λ	0.0319	0.0382	0.031859	0.001461
γ	0.2966	0.3969	0.294978	0.148231

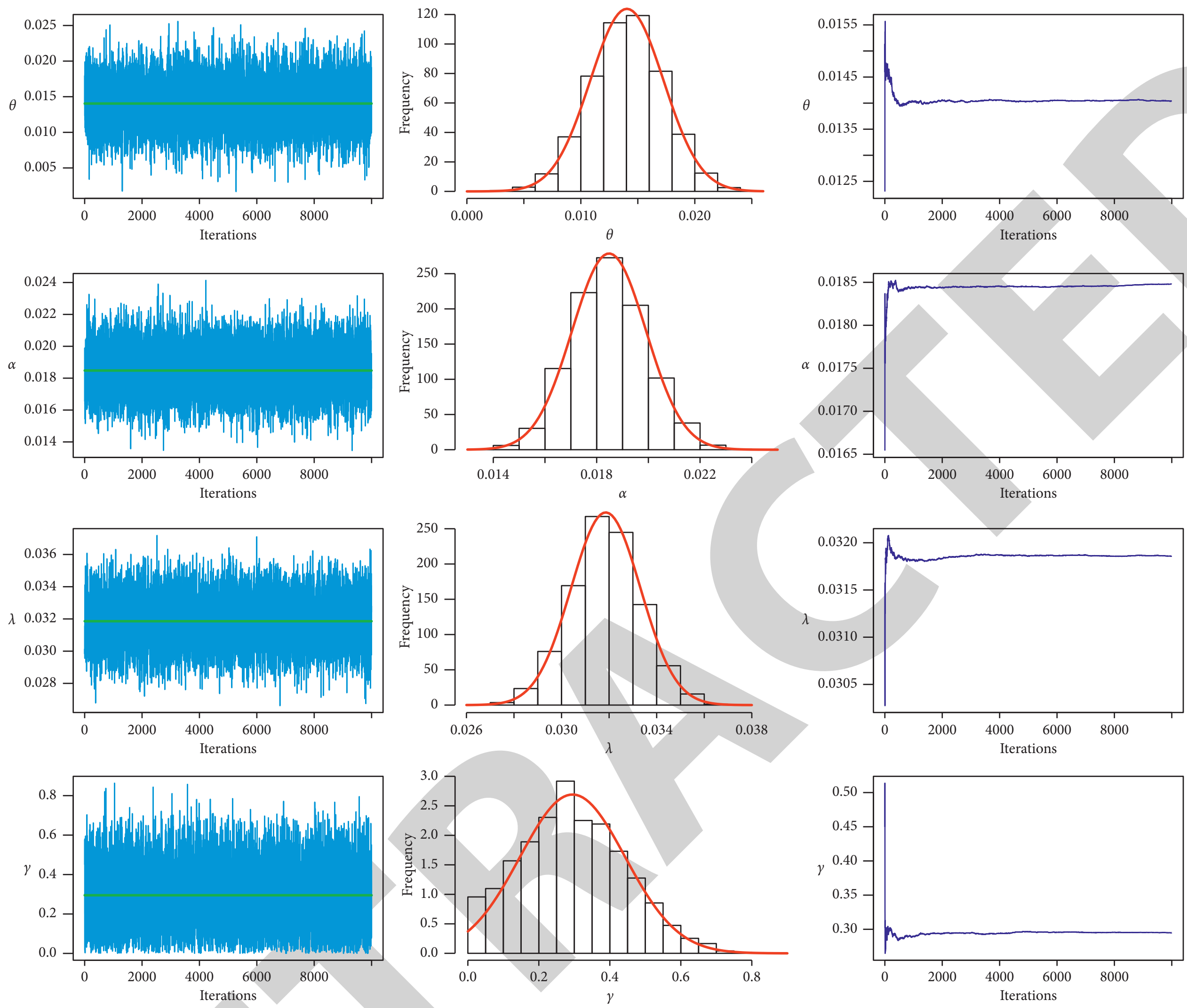
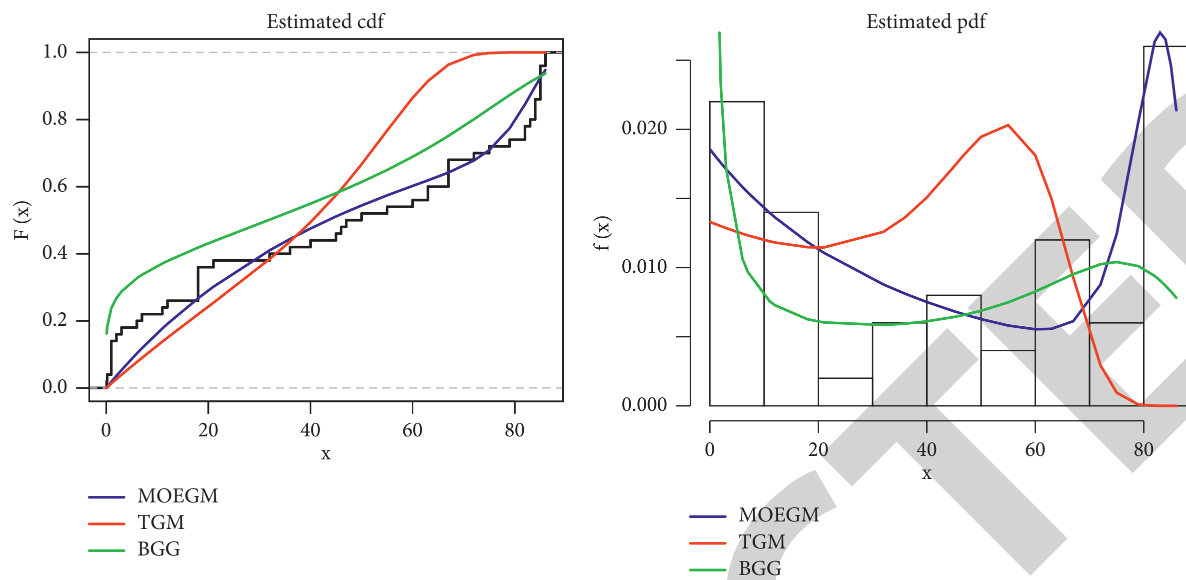


Figure 6: The trace plots, posterior density and the convergence for parameters θ, α, λ and γ: flood peaks data.

Table 6: Comparison between MOEGM, BGG and TGM distributions:stochastic processes data.

	θ	α	λ	γ	b	KSS
MOEGM	0.0096	$1.593E-10$	0.2511	0.5168		0.1265
TGM	0.0998	0.0003	0.0115	0.1280		0.1517
BGG	0.0638	0.0029	1.3644	0.1198	0.1776	0.1266
	CVMS	AICM	AICCM	BICM	HQICM	PVKS
MOEGM	0.1214	440.6824	441.5713	448.3305	443.5948	0.4003
TGM	0.2751	454.0590	454.9480	461.7070	456.9710	0.2002
BGG	0.157543	461.0024	462.366	470.5625	464.6429	0.3871



(a) (b)

FIGURE 7: (a) Histogram and estimated p.d.f., (b) Estimated c.d.f. and the empirical c.d.f.: stochastic processes data.

TABLE 7: MLE and Bayesian estimation methods comparing by SE: stochastic processes data.

	MLE		Bayesisna	
	Estimates	SE	Estimates	SE
θ	0.0096	0.3547	0.4195	0.2138
α	$1.593E-10$	$8.680E-07$	$3.34E-09$	$4.13E-09$
λ	0.2511	0.0945	0.2514	0.0338
γ	0.5168	0.0568	0.5015	0.0191

Table 7 discussed MLE and Bayesian estimation methods comparing by SE, we note Bayesian estimation has smaller SE than MLE.

8. Conclusion

Based on Marshall and Olkin approach, a new four-parameter extended Gompertz Makeham distribution was developed, the Marshall-Olkin extended Gompertz Makeham distribution. It includes special models, the Marshall-Olkin extended Makeham, Marshall-Olkin Gompertz Makeham, Gompertz Makeham, and Makeham distributions. Depending on the shape parameters, the MOEGM density function can take on a variety of shapes. Furthermore, depending on the design parameters, its hazard rate function might take on various shapes. We have included some statistical features. The method of likelihood and Bayesian estimation methods are used to estimate the unknown parameters of the proposed distribution. An MCMC technique is used to give a comparison for the estimated parameters. These comparisons were made using bias and MSE as criteria. The MSE and Bias of the Bayesian-based SELF are superior to both MLE in our simulation case. Real data sets were observed and it was noted that the MOEGM distribution resulted in the best fit. To summaries, the MOEGM distribution may provide a relatively flexible mechanism for fitting a wide range of positive real-world data sets. The novel distribution may be a feasible alternative to existing models now available in the literature for modeling actual data in domains like as engineering, survival analysis, hydrology, economics, and others.

Data Availability

The data used to corroborate the study's conclusions is supplied in the paper.

Conflicts of Interest

The authors declare no conflicts of interest.

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