Research Article

Optimal Insurance Indemnity for the Insured with Low Risk Tolerance

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This study focuses on the optimal insurance for the insured with low risk tolerance from the perspective of rank-dependent utility. Dissimilar to the widely employed assumption that the initial wealth of the insured can sufficiently cover any incurred premium and maximal possible loss, we assume that the policyholder may not possess sufficient initial wealth, thus exhibiting a low risk tolerance. Hence, we establish a corresponding insurance model in which the insured’s initial wealth is designed in such a manner that it only needs to cover the premium. Further, the model considers an introduced risk level that is less than the maximal possible loss. Owing to the lack of the condition of the sufficiency of initial wealth, the original optimization issue could not be converted into a canonical quantile-optimization issue, as usual, and the quantile formulation could not be obtained directly. To overcome this challenge, we apply the calculus of variations method employing a variable upper limit that is related to the maximal risk level. Further, the explicit optimal solution bearing Yaari’s dual criterion is given, demonstrating that the optimal insurance policy with sufficient initial wealth represents a special case. Finally, numerical examples are presented to compare the indemnity with different risk levels.

1. Introduction

Insurance constitutes a financial arrangement in which the insured and insurer share any incurred accidental loss. A purpose of insurance contracts is to maximize the benefits of the insured on the premise of satisfying the participation constraints of the insurer. Many studies have focused on determining the optimal insurance indemnity of the insured. For example, Ghossoub [1] re-examined the issue of insurance compensation demand in the wake of budget constraints, thus maximizing the subjective expected utility of the terminal wealth of the insured based on belief heterogeneity. Ghossoub [2] re-visited the optimal insurance design for maximizing the insured’s terminal wealth from the perspective of rank-dependent expected utility (RDEU), which imposes a “state-verification cost,” ensuring that the moral hazard caused by the potential misreporting of the insured’s losses is automatically excluded. By eliminating the nonatomic assumption and imposing the “incentive compatibility” constraint, Xu et al. [3] resolved the issue involving false reports of actual losses by the insured. The variational method is employed to characterize the optimal solution and obtain the optimal insurance contract. Zhang et al. [4] obtained the optimal indemnity design for health insurance via the Lagrange multiplier method and optimal control technology.

Most extant studies on insurance contracts assume that the insured are risk-averse, who aim to maximize the expected utility. The expected utility theory (EUT) describes the decision-making behaviors of rational people under risk conditions. However, people are not purely rational in reality, and their decisions are generally influenced by complex psychological mechanisms. Additionally, an information asymmetry—which accounts for adverse selections—exists in the insurance market [5, 6]; this inconsistent reality casts a dark cloud on the effectiveness of EUT in describing human decision-making in connection with risks [7, 8]. For example, EUT fails to explain the Allais, preference-reversal,
and probabilistic insurance paradoxes [9]. Furthermore, EUT is challenged by the purchases of lottery tickets and insurance policies by the same decision maker.

Since EUT cannot explain the foregoing and other related phenomena, different measures have been proposed to capture human decision-making behaviors, particularly with regard to risks. Yaari [10] investigated the consequences of the modified EUT, which requires the direct mixing of payments that is independent of the risk prospects, and derived a new risk-selection theory—namely, the dual theory of risk. Hong et al. [11] employed the expected-utility rank-dependent probabilistic risk-decision model to analyze the portfolio selection issue. Jeleva [12] employed the Choquet expected utility model to study the influence of background risks on people’s insurance purchase decisions. Quiggin [13] summarized some of the generalized applications of non-expected utility models in risk decision-making. Among these outstanding contributions, we highlight Quiggin’s [14] highly notable RDEU model, which comprises the following two components: the concave utility and inverse S-shaped probability distortion functions [15] (Footnote 4 of Bernard et al. [16] provides more details). RDEU weights the results based on their ranks and probabilities to ensure that the probability weighting correlates with the superiority principle, i.e., RDEU adds a probability weighting function to the classical expected utility model to capture the risk behaviors of decision-makers toward tail events. RDEU, which has been constantly explored, developed, and applied, derives its popularity from its flexibility, which can explain most observed phenomena, and this is dissimilar to the prediction of EUT. Abouda and Chateauneuf [17] characterized the symmetrical monotone risk aversion in the RDEU model. Ryan [18] discussed the relationship among five distinguishable risk-aversion notions for RDEU preferences. Hu and Mao [19] extended the characterization of left-mono-
tone risk aversion in the RDEU model to Ryan’s unbounded random variables. Polkovnichenko [20] demonstrated that RDEU could explain why most American families do not invest in stocks. He and Zhou [21] employed the Choquet integral of the utility function with respect to the distorted probability function; it represented a natural extension of RDEU from the definition of discrete random variables to integral continuous intervals.

Compared with EUT, RDEU can describe human behaviors more reasonably, while the inverse S-shaped distortion function renders the objective function nonconcave and nonconvex, making it challenging for classical optimization technology to solve the optimal solution. A “quantile formulation” has been introduced to resolve this challenge, and it changes the wealth decision variable into its quantile function, perfectly transforming the nonconvex and nonconcave objective challenges into one that is concave-optimization [22]. The quantile formulation has been employed in the insurance and portfolio selection literature by authors such as Carlier and Dana [23] and Jin and Zhou [24], who have solved the quantile-optimization issue under the monotonicity assumption. Xu [25] proposed a change-of-variable and relaxation method to solve the quantile-optimization issue without applying the variational method or making any monotonic assumptions. Ghossoub [2] employed a similar methodology to Xu’s [25] so as to convert the decision variable from a random variable to a quantile function.

Bernard et al. [16] might be the first to have studied the insurance contract from the perspective of the inverse S-shaped distortion function (the probability distortion function was previously assumed to be convex [3, 16]). Based on the quantile formulation, they presented a sufficient and necessary representation of the optimal contract, proved that generalized layered insurance is the optimal strategy, which avoids indemnities for large losses that exceed deductibles and small ones. Based on Bernard et al.’s work [16], Xu et al. [3] removed the nonatomic assumption and added a constraint in which the compensation and retention functions increased with increasing losses. Ghossoub [2] re-visited Bernard et al.’s and Xu et al.’s optimal insurance design. However, rather than applying “incentive compatibility” constraints, Ghossoub’s design required the insurance company to bear the state-verification costs of verifying the severity of losses, thereby automatically excluding the chances of a moral hazard that might arise when the insured reports exaggerated losses.

Bernard et al. [16], Xu et al. [3], and Ghossoub [2] commonly assume that the policyholder possesses sufficient initial wealth to cover the premium and maximal possible losses. However, since wealth is generally affected by age, family income, gender, and many other factors, the wealth levels of many of those insured cannot meet this assumption, and they exhibit inadequate risk tolerance. Quan et al. [26] demonstrated that people generally become less tolerant of risks and are more cautious as they age. Fang et al. [27] indicated that the accumulation of household wealth can significantly increase the risk tolerance of households and that this relationship diminishes with age. Ling et al. [28] believed that an individual’s risk behavior is affected by wealth. The research on social wealth distribution reveals that most people only possess a small share of the total social wealth. In reality, when major losses such as house fires, severe ailments, third-party liability accidents, and other disasters occur, the social wealth distribution indicates that the initial wealth of most policyholders would not be sufficient to alleviate their financial distress [29–31], which is about the reason they had purchased insurance policies. Considering that the actual risk tolerance of most policyholders constitutes a major factor in formulating insurance contracts [32] and that the optimal insurance contract under sufficient initial wealth is nearly impossible to apply to policyholders with insufficient initial wealth and low risk tolerance, we propose a new insurance model that can protect the insured with a low risk tolerance as the main objective of this paper.

Dissimilar to the case involving sufficient initial wealth, the quantile formulation of the case involving insufficient initial wealth could not be directly obtained because the optimization objective therein could not be converted into a calculus defined by [0, 1]. We denote \( I(x) \) as the generally defined indemnity function without the maximum risk level, \( v \), and \( T(x) \) as the indemnity function including \( v \). Therefore,
\(\bar{T}(x)\) is divided into two parts. First, when the loss is less than \(v\), \(\bar{T}(x)\) follows Xu et al.’s [3] indemnity design; second, when the loss exceeds \(v\), the compensation is a given linear function to protect the terminal wealth of the insured from exhibiting a negative value. Further, by applying the calculus of variation method with a variable upper limit, the explicit optimal solution with Yaari’s [10] dual criterion is obtained, demonstrating that Xu et al.’s [3] insurance policy represents a special case in which \(v\) tends to the maximal possible loss. Finally, a numerical example is presented to verify the result of this study, after which the indemnities under different \(v\) sets are compared.

The remainder of this paper is organized as follows: Section 2 introduces the optimal insurance model (under the RDEU framework) in which policyholders exhibited a low risk tolerance, Section 3 presents a general sufficient and necessary condition for obtaining the optimal solution, Section 4 presents in detail the optimal insurance policies of different cases, Section 5 provides numerical examples to compare our results with the known indemnity in the literature as well as indemnity plans with different premiums under the same initial wealth setting, and Section 6 presents the study conclusions.

### 2. Model Setting

Let \((\Omega, F, P)\) represent a probability space. An insured possesses an initial wealth \(\omega\) and faces a random loss, \(X\), that is supported in the interval \([0, M]\) (where \(M\) is a given positive scalar). When the insured selects an insurance contract with a premium, \(p_0\), such an insured will obtain an insurance indemnity, \(I(x)\), with \(I(0) = 0\), and satisfies the following expression:

\[
0 \leq I(x) - I(y) \leq x - y, \quad \text{for } 0 \leq x \leq y \leq M. \tag{1}
\]

To formulate insurance contracts for the insured with low risk tolerance, we introduce a maximal risk level, \(v\), which equals to \(\omega - p\), where \(p\) is the premium that the insured is willing to pay (it may be more expensive than \(p_0\)). The insurance indemnity plan, \(\bar{T}(\cdot)\), follows the expression below:

\[
\bar{T}(x) = \begin{cases} 
I(x), & 0 \leq x < v, \\
x - \omega + p, & v \leq x \leq M. 
\end{cases} \tag{2}
\]

The constraint indicates that \(\bar{T}(\cdot)\) satisfies the same condition as \(I(\cdot)\) if the loss, \(x\), is less than \(v\). Thus, the following expression is obtained:

\[
\% \omega - p - x + \bar{T}(x) = \omega - p - x + I(x) \geq \omega - p - x \geq \omega - p - v = 0. \tag{3}
\]

When \(x > v\), the terminal wealth will be equal to zero, indicating that the insurance contract covers for large risks and small losses and ensures that the insureds’ terminal wealth will be non-negative. Notably, the initial wealth is only required to satisfy the condition, \(\omega = p + v\), which can make up for the insufficient initial wealth of the insured since the condition \(\omega \geq p + M\) might be too challenging.

Such a contract will appeal to consumers with low risk tolerance.

Here, we assume that the RDEU framework dictates the insured’s risk preference, the insurer is risk-neutral, and the insurance premium is calculated by the expected value principle. Thus, the premium, \(p\), paid by the insured must not be less than \((1 + \theta)E(\bar{T}(X))\) to avoid the bankruptcy of the insurer in the long run (according to Raviv [33], \(\theta(0 \leq \theta \leq 1)\) is the insurer’s safety loading). The retention function, \(R(X) = X - \bar{T}(X)\), is the part of a loss that the insured bears, and it is specifically defined in Equation (7).

At this point, we propose the following assumptions to characterize and solve the optimal insurance issue.

**Assumption 1.** \(X\) is non-negative; the cumulative distribution function is \(F_X\), which is strictly increasing. Moreover, its quantile function, \(F_X^{-1}\), is absolutely continuous on \([0, 1]\).

**Assumption 2.** (Inverse S-shaped weighting). The probability weighting function \(T: [0, 1] \rightarrow [0, 1]\), is a continuous and strictly increasing mapping that is twice differentiable on \((0, 1)\). Moreover, a constant \(b \in (0, 1)\), exists such that \(T' = \text{strictly decreasing and increasing on} (0, b)\) and \((b, 1)\), respectively, and \(T''(0+) > 1\).

**Assumption 3.** The utility function, \(u: R^+ \rightarrow R^+\), is a strictly increasing function with a continuous derivative function, \(u'; u'\) is a decreasing function.

The first part of Assumption 1 is standard and common in the insurance literature (e.g., Raviv [33], Bernard et al. [16], and Xu et al. [3]). The second part is very mild and purely technical, and it is satisfied in the aforementioned papers. Assumption 1 ensures that \(F_X^{-1}(F_X(x)) = x, \forall x \in [0, M]\), which will be subsequently utilized frequently. Assumption 2 is satisfied for many proposed or applied weighting functions in the literature. Bernard et al. [16] and Xu et al. [3] included a slightly stronger condition, \(T''(1-) = \infty\), in the assumption of the inverse S-shape distortion function; however, the condition is not required in this study. Assumption 3 is the standard customary requirement for a utility function. Assumption 2 and 3 are the combination of the inverse S-shape distortion function and concave utility function in the RDEU framework.

Finally, an insurance contract to maximize the interests of the insured is designed. The measure of the preference of the final (random) wealth of the insured is formulated as \(V(\cdot)\), which can be expressed by the following optimization model:

\[
\max_{\bar{T}(\cdot)} V(\omega - p - X + \bar{T}(X)), \\
\text{s.t. } (1 + \theta)E(\bar{T}(X)) \leq p. \tag{4}
\]

The left-continuous generalized inverse of any \(F_X\) is defined as

\[
F_X^{-1}(z) = \inf\{s \in R: F_X(s) > z\}, \quad z \in (0, 1),
\]

where \(F_X^{-1}\) is the quantile function of \(X\).

Thus,
Moreover, the condition for obtaining an optimal function to Equation (8) is the upper limit of the expected value of the risk retained by the insured, where $\alpha(x) = F^{-1}(1 - \theta)(x)$ and consider the following auxiliary problem: max
\[
\int_0^1 u(\omega - p - \overline{R}(z))(1 - z)dz,
\]
\[\text{s.t.} \int_0^1 \overline{R}(z)dz \geq p_1.\] (8)

3. Characterization of the Solutions

Here, we apply the calculus of variations method with a variable upper limit to derive a necessary and sufficient condition for obtaining an optimal function to Equation (8).

Lemma 1. Based on Assumption 1, we derive
\[\overline{G} = \left\{ \overline{G} : \overline{G}(0) = 0, \quad 0 \leq \overline{G}(z) \leq \left(F^{-1}_X\right)'(z), \text{a.e.} \in [0,1] \right\},\] (9)

where $\alpha = \Pr(X < v)$.

\[\overline{G}_a = \left\{ \overline{G} : \overline{G} \text{ defined on } [0,a] | \overline{G} \text{ is abs. cont.}\overline{G}(0) = 0, \quad 0 \leq \overline{G}(z) \leq \left(F^{-1}_X\right)'(z), \text{a.e.} \in [0,a] \right\},\] (14)

\[\max_{\overline{G} \in \overline{G}_a} \int_0^a u(\omega - p - \overline{G}(z))T'(z)dz - \lambda \overline{P}_1.\] (15)

For convenience, we consider $\overline{R}(X)$ rather than $\overline{I}(X)$ in the subsequent study. We define
\[\overline{R} = \{\overline{R} : 0 \leq \overline{R}(x) - \overline{R}(y) \leq x - y, \quad \forall 0 \leq x \leq y \leq v & \overline{R}(x) = \omega - p, \quad \forall v < x \leq M\},\] (7)

and denote $\overline{G} = \{\overline{G} : \overline{G}(0) = 0, \overline{G}(z) \leq \left(F^{-1}_X\right)'(z), \text{a.e.} \in [0,1] \}$. Thus, the objective function of the insured transforms from the quantile function of $\overline{R}(X)$ into a functional of $\overline{G}$ hereafter. Moreover, $E(\overline{R}(X)) = \int_0^1 F^{-1}_R(x)dx = \int_0^1 \overline{G}(z)dz$ and inequality, $(1 + \theta)E(\overline{I}(X)) \leq p$, are equivalents of $E(\overline{I}(X)) = E(X - \overline{R}(X)) \leq p/(1 + \theta)$. Thus, we obtain $E(\overline{R}(X)) \leq E(X) - p/(1 + \theta)$ and denote $p_1 = E(X) - p/(1 + \theta)$, which is the upper limit of the expected value of the risk retained by the insured.

Therefore, the optimization problem (4) becomes
\[\max_{\overline{G} \in \overline{G}} \int_0^1 u(\omega - p - \overline{G}(z))T'(z)dz,\]
\[\text{s.t.} \int_0^1 \overline{G}(z)dz \geq p_1.\] (8)

Proof. Following Lemma 2.5 in Xu et al.’s [3] paper, we simply prove that condition (a), namely $\overline{R}(x) = \omega - p$ on $[v, M]$, is equivalent to (b), $\overline{G}(z) = \omega - p$ on $[a, 1]$.

If (a) holds, we can derive the following by combining it with the monotonicity of $\overline{R}(\cdot)$:
\[z = F_{\overline{R}(X)}(x) = \Pr(\overline{R}(X) \leq x),\]
\[= \begin{cases} 1, & x \geq \omega - p, \\
\Pr(\overline{R}(X) \leq x, X < v), & x < \omega - p.\end{cases}\] (10)

Thus, for any $z \in [\Pr(X < v), 1]$,
\[\overline{G}(z) = F^{-1}_{\overline{R}(X)}(z) = \inf \{ x \in R^* : F_{\overline{R}(X)}(x) \geq z \} = \omega - p,\] (11)

indicating that (b) holds.

Further, $a = \Pr(X < v) \leq F_X(x) \leq 1$ holds for any $x \in [v, M]$ if (b) holds. Therefore, following Lemma 2.4 of Xu et al.’s [3] paper, $\overline{R}(x) = F^{-1}_{\overline{R}(X)}(F_X(x)), \forall x \in [0, M]$, we derive
\[\overline{R}(x) = \overline{G}(F_X(x)) = \omega - p \quad \forall x \in [v, M].\] (12)

Thereafter, (8) becomes
\[\max_{\overline{G} \in \overline{G}} \int_0^a u(\omega - p - \overline{G}(z))T'(z)dz,\]
\[\text{s.t.} \int_0^a \overline{G}(z)dz \geq \overline{P}_1,\] (13)

where $\overline{P}_1 = E(X) - p/(1 + \theta) - (1 - a)(\omega - p)$.

To solve (13), we define
\[\overline{G}_a = \left\{ \overline{G} : \overline{G} \text{ is abs. cont.}\overline{G}(0) = 0, \quad 0 \leq \overline{G}(z) \leq \left(F^{-1}_X\right)'(z), \text{a.e.} \in [0,a] \right\},\] (14)

and consider the following auxiliary problem:
Lemma 2. A function, $\overline{G}_\lambda(\cdot)$, is an optimal solution to (15) if and only if $\overline{G}_\lambda \in \mathcal{G}$ and

$$
\overline{G}_\lambda(z) = \begin{cases} 
0, & \text{if } N_\lambda(z) < 0, \\
\left[0, \left(F_{X}^{-1}\right)'(z)\right], & \text{if } N_\lambda(z) = 0, \\
\left(F_{X}^{-1}\right)'(z), & \text{if } N_\lambda(z) > 0,
\end{cases}
$$

(16)

where $N_\lambda(z) = \int_{0}^{z} (\lambda - u'((\omega - p - \overline{G}_\lambda(t))T'(t))dT, z \in [0, a)$.

The proof of Lemma 2 can be completed by replacing the integral upper limit 1 in the proof of Theorem 3.1 in Xu et al.’s [3] report with $\alpha$.

Proposition 1. For any $\overline{G}_\lambda(\cdot) \in \mathcal{G}$, we denote the function $G_\lambda(\cdot)$ that is limited on $[0, a)$ by $\overline{G}_\lambda(\cdot)|_{[0,a)}$. If $\overline{G}_\lambda(\cdot)|_{[0,a)}$ satisfies (16),

$$
G_\lambda(\cdot)|_{[0,a)} = \begin{cases} 
G_\lambda(z)|_{[0,a)}, & z \in [0, a), \\
\omega - p, & z \in [a, 1],
\end{cases}
$$

(17)

will be an optimal solution to (8), where $\lambda^* \in \mathbb{R}^+$ is determined by binding the constraint, $\int_{0}^{a} G_\lambda(z)dz = \overline{p}_1$.

Proof. Lemma 2 revealed that $\overline{G}_\lambda(\cdot)|_{[0,a)}$ optimizes (15). Further, a standard duality argument deduces that $\overline{G}_\lambda(\cdot)|_{[0,a)}$ is an optimal solution to (14), i.e., we obtain

$$
\int_{0}^{a} G_\lambda(z)dz = \overline{p}_1
$$

and

$$
\int_{0}^{a} u(\omega - p - G_\lambda(z))T'(z)dz \geq \int_{0}^{a} u(\omega - p - \overline{G}_\lambda(z))T'(z)dz, \ \forall \overline{G} \in \mathcal{G}.
$$

(18)

Note that

$$
G_\lambda(z) = \begin{cases} 
0, & \text{if } \int_{z}^{\lambda - T'(t)} dt = \lambda(\alpha - z) - (T'(a) - T(z)) < 0, \\
\left[0, \left(F_{X}^{-1}\right)'(z)\right], & \text{if } \int_{z}^{\lambda - T'(t)} dt = \lambda(\alpha - z) - (T'(a) - T(z)) = 0, \\
\left(F_{X}^{-1}\right)'(z), & \text{if } \int_{z}^{\lambda - T'(t)} dt = \lambda(\alpha - z) - (T'(a) - T(z)) > 0.
\end{cases}
$$

(22)

To apply (22), we compare $\lambda$ and $T'(a) - T(z)/a - z$, and define

$$
g(z) = \frac{T'(a) - T(z)}{\alpha - z}, \ \ z \in [0, a).
$$

(23)

Lemma 3. Under Assumption 2, we derive the following:

(a) $g(z)$ is strictly decreasing on $[0, a)$ if $\alpha \leq b$.

(b) A unique $a \in (0, b)$ exists if $\alpha > b$, such that $g(\cdot)$ is strictly decreasing and increasing on $[0, a)$ and $[a, \alpha)$, respectively. Furthermore, $a$ can be uniquely determined by $T'(a) = g(a)$.

Proof. We derive

$$
\int_{0}^{1} G_{\lambda'}(z)dz = \int_{0}^{1} \overline{G}_{\lambda'}(z)dz + \int_{a}^{1} G_{\lambda'}(z)dz
$$

(19)

Moreover, for any $G \in \mathcal{G}$,

$$
\int_{0}^{1} u(\omega - p - \overline{G}(z))T'(z)dz = \int_{0}^{1} u(0)T'(z)dz
$$

(20)

We obtain

$$
\int_{0}^{1} u(\omega - p - \overline{G}(z))T'(z)dz
$$

$$
= \int_{0}^{1} u(\omega - p - \overline{G}(z))T'(z)dz + \int_{a}^{1} u(\omega - p - \overline{G}(z))T'(z)dz
$$

$$
\int_{0}^{1} u(\omega - p - \overline{G}(z))T'(z)dz + \int_{a}^{1} u(\omega - p - \overline{G}(z))T'(z)dz
$$

(21)

Thus, we prove the proposition.

In this section, we completely describe the optimal solution to the insurance issue. However, thus far, it is only an implicit description, and the result of this problem is not specific and clear. Thus, in the next section, we solve the explicit solution—that is, the optimal insurance policy—by employing Yaari’s dual criterion.

4. Optimal Insurance Policy

Based on Yaari’s dual criterion (Yaari, 1987), the utility function $u(x) \equiv x$ and condition (15) are simplified as follows:

(b) A unique $a \in (0, b)$ exists if $\alpha > b$, such that $g(\cdot)$ is strictly decreasing and increasing on $[0, a)$ and $[a, \alpha)$, respectively. Furthermore, $a$ can be uniquely determined by $T'(a) = g(a)$.
\[ g'(z) = \frac{(T(a) - T(z)) - T'(z)(a-z)}{(a-z)^2} = \frac{t(z)}{(a-z)^2}, \]  
(24)  
\[ t'(z) = -T''(z)(a-z). \]

If \( \alpha \leq b \), Assumption 2 indicates that
\[ t(z) = \frac{T(a) - T(z)}{\alpha - z} - T'(z)(a-z) < 0, \quad z \in (0, \alpha). \]  
(25)

Since \( T(\cdot) \) is strictly concave on \([0, b]\). Thus, \( g(z) \) is strictly decreasing on \([0, \alpha]\).

Remark 1. Here, Lemma 3 deduces to Lemma 4.1 of Xu et al. [3], where \( \alpha = 1 \); however, the condition, \( T'(1-) = +\infty \), is not required in this study.

When \( \alpha > b \), we set
\[ \tilde{\lambda} := \frac{g(a) - g(0)}{T(a)/\alpha}. \]  
(28)

If \( g(\alpha-) \geq g(0) \), i.e., \( T'(\alpha-) \geq T(a)/\alpha \), we further set \( c \in (\alpha, a] \) to be the unique scalar, such that \( g(c) = g(0) \) or \( T(c)/c = T(a)/\alpha \).

Next, we classify according to the sizes of \( \alpha \) and \( b \) and consider all the cases based on the value of \( \lambda \).

Case A. \( \alpha \leq b \& \lambda \leq g(\alpha-) \)

In this case,
\[ N_\lambda(z) = (a-z)(\lambda - g(z)) < 0, \quad \forall z \in [0, \alpha]. \]  
(29)

It follows from (21) that the optimal solution \( \overline{G}_\lambda(z) = 0 \) for all \( z \in [0, \alpha) \). Thus,
\[ \overline{R}_\lambda(x) = \overline{G}_\lambda(F_X(x)) = 0, \quad 0 \leq x < v. \]  
(30)

and
\[ \overline{T}_\lambda(x) = x - \overline{R}_\lambda(x) = x, \quad 0 \leq x < v. \]  
(31)

Case A II \( \alpha \leq b \& g(\alpha-) < \lambda < g(0) \)

Lemma 3 indicates that a unique \( x_0 \in (0, \alpha) \) exists, such that \( g(x_0) = \lambda \). Accordingly, we obtain
\[ N_\lambda(z) = \begin{cases} < 0, & 0 < z < x_0, \\ > 0, & x_0 < z < \alpha. \end{cases} \]  
(32)

Hence, (22) produces the following function:
\[ \overline{G}_\lambda(z) = \begin{cases} 0, & 0 \leq z < x_0, \\ F_X^{-1}(z) - F_X^{-1}(x_0), & x_0 \leq z \leq \alpha. \end{cases} \]  
(33)

The corresponding indemnity function is
\[ \overline{T}_\lambda(x) \equiv x - \overline{R}_\lambda(x) = \begin{cases} x, & 0 \leq x < F_X^{-1}(x_0), \\ F_X^{-1}(x_0), & F_X^{-1}(x_0) \leq x < v. \end{cases} \]  
(34)

Case A III \( \alpha \leq b \& \lambda > g(\alpha-) \)

From Lemma 3,
\[ N_\lambda(z) > 0, \quad \forall 0 < z < \alpha. \]  
(35)

Hence, (22) yields the following function:
\[ \overline{G}_\lambda(z) = F_X^{-1}(z), \quad \forall 0 \leq z < \alpha. \]  
(36)

The corresponding indemnity function is
\[ \overline{T}_\lambda(x) \equiv x - \overline{R}_\lambda(x) = 0, \quad 0 \leq x < v. \]  
(37)

Case B I \( \alpha > b \& \lambda \leq \tilde{\lambda} \)

In this case, \( N_\lambda(z) = (a-z)(\lambda - g(z)) < 0, \forall z \in [0, a) \cup t(\alpha, a] \), follows from (22) that. \( \overline{G}_\lambda(z) = 0 \).

Hence,
Thus,
\[ T_\lambda(x) = x - R_\lambda(x) = x, \quad 0 \leq x < v. \quad (39) \]

Case B. \( \alpha > b \& \lambda < g(0) \leq g(\alpha^\prime) \)
Lemma 3 reveals that unique \( x_0 \in (0, a) \) and \( y_0 \in (a, c) \) exist, such that \( g(x_0) = g(y_0) = \lambda \). Accordingly, we obtain
\[
N_\lambda(z) = \begin{cases} 
< 0, & 0 < z < x_0, \\
> 0, & x_0 < z < y_0, \\
< 0, & y_0 < z < a.
\end{cases} \quad (40)
\]

Hence, (22) yields the following function:
\[
\overline{G}_\lambda(z) = \begin{cases} 
0, & 0 \leq z < x_0, \\
F^{-1}_X(z) - F^{-1}_X(x_0), & x_0 \leq z < y_0, \\
F^{-1}_X(y_0) - F^{-1}_X(x_0), & y_0 \leq z \leq a.
\end{cases} \quad (41)
\]

The corresponding indemnity function is thus:
\[
\overline{T}_\lambda(x) \equiv x - R_\lambda(x) = \begin{cases} 
x, & 0 \leq F^{-1}_X(x_0), \\
F^{-1}_X(x_0) \leq x < F^{-1}_X(y_0), \\
x - F^{-1}_X(y_0) + F^{-1}_X(x_0), & F^{-1}_X(y_0) \leq x < v.
\end{cases} \quad (42)
\]

Case B. III \( \alpha > b \& \lambda < g(\alpha^\prime) < g(0) \)
According to Lemma 3, unique \( x_1 \in (0, a) \) and \( y_1 \in (a, a) \) exist, such that \( g(x_1) = g(y_1) = \lambda \). Accordingly, we obtain the following:
\[
N_\lambda(z) = \begin{cases} 
< 0, & 0 < z < x_1, \\
> 0, & x_1 < z < y_1, \\
< 0, & y_1 < z < a.
\end{cases} \quad (43)
\]

Hence, (22) produces the following function:
\[
\overline{G}_\lambda(z) = \begin{cases} 
0, & 0 \leq z < x_1, \\
F^{-1}_X(z) - F^{-1}_X(x_1), & x_1 \leq z < y_1, \\
F^{-1}_X(y_1) - F^{-1}_X(x_1), & y_1 \leq z \leq a.
\end{cases} \quad (44)
\]

The corresponding indemnity function is thus:
\[
\overline{T}_\lambda(x) \equiv x - R_\lambda(x) = \begin{cases} 
x, & 0 \leq F^{-1}_X(x_1), \\
F^{-1}_X(x_1) \leq x < F^{-1}_X(y_1), \\
x - F^{-1}_X(y_1) + F^{-1}_X(x_1), & F^{-1}_X(y_1) \leq x < v.
\end{cases} \quad (45)
\]

Case B. IV \( \alpha > b \& \lambda < g(\alpha^\prime) < g(0) \)
According to Lemma 3, a unique exists, such that \( g(x_2) = \lambda \). Accordingly, we obtain
\[
N_\lambda(z) = \begin{cases} 
< 0, & 0 < z < x_2, \\
> 0, & x_2 < z < a.
\end{cases} \quad (46)
\]

Hence, (22) produces the following function:
\[
\overline{G}_\lambda(z) = \begin{cases} 
0, & 0 \leq z < x_2, \\
F^{-1}_X(z) - F^{-1}_X(x_2), & x_2 \leq z \leq a.
\end{cases} \quad (47)
\]

The corresponding indemnity function is
The corresponding indemnity function is thus:
\[
I_0(x) \equiv x - R_0(x) = \begin{cases} 
0, & 0 \leq x < F_X^{-1}(x_0), \\
x - F_X^{-1}(x_0), & F_X^{-1}(x_0) \leq x < \nu.
\end{cases}
\] (51)

Case B. VI $\alpha > b \& \max\{g(\alpha)-, g(0)\} \leq \lambda \leq +\infty$

Based on Lemma 3,
\[
N_1(z) > 0, \quad \forall 0 < z < \alpha. \tag{52}
\]

Hence, (22) yields the following function:
\[
\bar{G}_1(z) = F_X^{-1}(z), \quad \forall 0 \leq z < \alpha. \tag{53}
\]

The corresponding indemnity function is thus:
\[
I_0(x) \equiv x - R_0(x) = 0, \quad 0 \leq x < \nu. \tag{54}
\]

The optimal solutions for all the cases based on the value of $\lambda$ are presented above. To state the main result in terms of the premium, $p$, and indemnity function, $I(\cdot)$, we further describe the relationship between them ($p$ and $I(\cdot)$).

We define
\[
p_l = (1 + \theta)(E[X] - (1 - \alpha)(\omega - p)) - \int_0^a F_X^{-1}(z)dz.
\] (55)

and
\[
p_u = (1 + \theta)(E[X] - (1 - \alpha)(\omega - p)). \tag{56}
\]

Therefore,
\[
p_u = (1 + \theta)(E[X] - (1 - \alpha)(\omega - p)) \geq (1 + \theta)(E[X] - \int_0^a \bar{G}(z)dz - (1 - \alpha)(\omega - p)) = (1 + \theta)E[I(X)] = p_l
\]
\[
\geq (1 + \theta)(E[X] - \int_0^a F_X^{-1}(z)dz - (1 - \alpha)(\omega - p)) = p_v.
\] (57)

This inequality can be interpreted as a scenario in which a risk-neutral insurer adopts the safety loading, $\theta$, and $p_l$ is the lowest price at which the insurer participates in the business and is willing to pay the indemnity $I(\cdot)$.

Any admissible prepaid $p$ lies within the interval $[p_l, p_v]$.

Comparing the foregoing cases with that of Xu et al. [3] reveals that the cases reported in this paper are more complicated, and this is because of the introduction of $\nu$ and the absence of condition $T' (1) = +\infty$. Thus, we can now state the main result with a given $\nu$.

**Theorem 1.** Under Yaari's criterion, $u(x) \equiv x$, and Assumptions 1 and 2, the optimal indemnity function $T^* (\cdot)$ to the original objective problem (4) is expressed as follows:

(a) $\alpha < b$

\[
\begin{cases} 
x, & x < \nu, \\
(x - \omega + p), & x \geq \nu.
\end{cases}
\] (58)

(ii) If $p_l \leq p < p_u$

\[
T^*(x) \equiv x - \bar{G}^*(F_X(x)) = \begin{cases} 
x, & 0 \leq x < F_X^{-1}(d), \\
F_X^{-1}(d), & F_X^{-1}(d) \leq x < F_X^{-1}(e), \\
x - F_X^{-1}(e) + F_X^{-1}(d), & F_X^{-1}(e) \leq x < \nu, \\
x - \omega + p, & \nu \leq x \leq M.
\end{cases}
\] (62)

(b) $\alpha > b$

We define
\[
\tilde{G}(z) = \begin{cases} 
F_X^{-1}(z), & 0 \leq z < c, \\
F_X^{-1}(c), & c \leq z \leq \alpha,
\end{cases}
\]

and $p_c = (1 + \theta)(E[X] - (1 - \alpha)(\omega - p) - K_c)$.

(B.1) $T(\alpha) \leq T^*(\alpha)-$

(i) If $p = p_w$

\[
T^*(x) = \begin{cases} 
x, & x < \nu, \\
(x - \omega + p), & x \geq \nu,
\end{cases}
\] (61)

(ii) If $p_l < p < p_u$
where \((d, e)\) is the unique pair satisfying 
\[0 \leq d < a < e \leq c, \quad g(d) = g(e), \quad \text{and} \quad E[T^*(X)] = p/1 + \theta.\]

(iii) If \(p_1 < p < p_o\),

\[T^*(x) \equiv x - \overline{G}^*(F_X(x)) = \begin{cases} 
0, & 0 \leq x < F^{-1}_X(d), \\
F^{-1}_X(d), & F^{-1}_X(d) \leq x < x - \omega + p, \\
x - \omega + p, & x = M, \\
x - \omega + p, & \forall x \leq M.
\end{cases} \]

(63)

where \(d\) is the unique scalar satisfying \(0 \leq d < a \leq \alpha\) and \(E[T^*(X)] = p/1 + \theta\).

If a unique pair \((d, e)\) that satisfies \(0 \leq d < a < e \leq \alpha\), where \(e = g^{-1}(g(d))\) exists, such that \(E[T^*(X)] = p/1 + \theta\), \(T^*(x)\) would be equal to (60). Otherwise, the optimal indemnity solution will be given, as follows:

\[T^*(x) \equiv x - \overline{G}^*(F_X(x)) = \begin{cases} 
x, & 0 \leq x < F^{-1}_X(d), \\
F^{-1}_X(d), & F^{-1}_X(d) \leq x < x - \omega + p, \\
x - \omega + p, & \forall x \leq M.
\end{cases} \]

(66)

where \(q\) is the unique scalar satisfying \(c \leq q \leq \alpha\) and \(E[T^*(X)] = p/1 + \theta\).

(B.2) \(T(a)/\alpha > T^*(a)\) (–)

(i) If \(p = p_o\),

\[T^*(x) = \begin{cases} 
x, & x < v, \\
(x - \omega + p), & x \geq v.
\end{cases} \]

(64)

(ii) If \(p \leq p < p_o\), let

\[T^*(x) = \begin{cases} 
x, & 0 \leq x < F^{-1}_X(d), \\
F^{-1}_X(d), & F^{-1}_X(d) \leq x < x - \omega + p, \\
x - \omega + p, & \forall x \leq M.
\end{cases} \]

(70)

Case B. \(1 \alpha > b\) and \(T(a)/\alpha \leq T^*(a)\) (–)

It is easy to observe that \(p_1 \leq p \leq p_o\).

(i) If \(p = p_o\), \(p_1 = 0\). Therefore, the optimal solution to (14) is trivially \(\overline{G}^*(z) = 0, \forall z \in [0, a]\), and

\[T^*(x) = \begin{cases} 
x, & x < v, \\
(x - \omega + p), & x \geq v.
\end{cases} \]

(71)

(ii) If \(p_1 \leq p < p_o\), \(0 < p_1 < \int_0^\alpha \overline{G}^*(z)dz\). In this case, a unique \(d \in (0, a)\) exists, such that \(\int_0^d \overline{G}^*(z)dz = p_1\), where \(\overline{G}^*\) is defined, as follows:

\[\overline{G}^*(z) = \begin{cases} 
0, & 0 \leq z < d, \\
F^{-1}_X(z) - F^{-1}_X(d), & \forall z \leq a.
\end{cases} \]

(68)

The existence derives from the fact that the function,

\[h(d) = \int_d^\alpha \left(F^{-1}_X(z) - F^{-1}_X(d)\right)dz, \quad 0 \leq d \leq a. \]

(69)

Satisfies \(h(0) = 0\) and \(h(\alpha) = \int_0^\alpha F^{-1}_X(z)dz\). Further, the uniqueness derives from \(\int_0^\alpha \overline{G}^*(z)dz = p_1\). By letting \(\lambda = g(d)\), it is easy to demonstrate that \(\overline{G}^*(z)\) satisfies (22) under \(\lambda\), corresponding to the aforementioned Case A.II. This result indicates that \(\overline{G}^*\) was optimal for (14) under \(p_1\). Thereafter, from Proposition 3.3, we obtain the optimal indemnity function of the original problem (4), as follows:

\[T^*(x) = \begin{cases} 
x, & 0 \leq x < F^{-1}_X(d), \\
F^{-1}_X(d), & F^{-1}_X(d) \leq x < x - \omega + p, \\
x - \omega + p, & \forall x \leq M.
\end{cases} \]

(70)
letting $\lambda = g(d)$, it is easy to prove that $\overline{G}^* (\cdot)$ satisfies (10) under $\lambda$, corresponding to the aforementioned Case B.II. This indicates that $\overline{G}^* (\cdot)$ is optimal for (14) under $\overline{p}_1$. Therefore, the optimal indemnity function is as follows:

$$ T(x) = \begin{cases} x, & 0 \leq x < v, \\ x - \omega + p, & x \geq v. \end{cases} \tag{75} $$

(iii) If $p_1 < p < p_o$, $K_e < \overline{p}_1 \leq \int_0^a \overline{G}^* (z)dz$. In this case, a unique $c \leq q \leq a$ exists, such that $\int_0^c \overline{G}^* (z)dz = \overline{p}_1$, where $\overline{G}^* (\cdot)$ is defined, as follows:

$$ \overline{G}^* (z) = \begin{cases} F_X^{-1} (z), & 0 \leq z < q, \\ F_X^{-1} (q), & q \leq z \leq a. \tag{76} \end{cases} $$

The existence derives from the fact that the function,

$$ l(x) = \int_0^x F_X^{-1} (z)dz + \int_x^a F_X^{-1} (x)dz, \quad c \leq x \leq a, $$

satisfies

$$ l(c) = \int_0^c F_X^{-1} (z)dz + \int_c^a F_X^{-1} (c)dz = K_e, \tag{77} $$

and

$$ l(a) = \int_0^a F_X^{-1} (z)dz. \tag{78} $$

The uniqueness derives from $\int_0^a \overline{G}^* (z)dz = \overline{p}_1$. By letting $\lambda = g(d)$, it is easy to demonstrate that $\overline{G}^* (\cdot)$ satisfies (22) under $\lambda$, corresponding to the aforementioned Case B.V. This indicates that $\overline{G}^* (\cdot)$ is optimal for (14) under $\overline{p}_1$. Therefore, the optimal indemnity function is given, as follows:

$$ T^* (x) = x - \overline{G}^* (F_X (x)) = \begin{cases} x, & 0 \leq x < F_X^{-1} (d), \\ F_X^{-1} (d), & F_X^{-1} (d) \leq x < F_X^{-1} (e), \\ x - F_X^{-1} (e) + F_X^{-1} (d), & F_X^{-1} (e) \leq x < v, \\ x - \omega + p, & v \leq x \leq M. \tag{79} \end{cases} $$

(i) If $p = p_o$, $\overline{p}_1 = 0$. Therefore, the optimal solution to (22) is trivially $\overline{G}^* (z) = 0 \forall z \in [0, a]$, and the existence derives from the definitions of $e$ and $\int_0^a \overline{G}^* (z)dz = \overline{p}_1$. By letting $\lambda = g(d)$, it is easy to determine that $\overline{G}^* (\cdot)$ satisfied Equation (22) under $\lambda$, corresponding to the aforementioned Case B.III. This indicates that $\overline{G}^* (\cdot)$ is optimal for Equation (14) under $\overline{p}_1$. Thus, the optimal indemnity function is
Employing Theorem 1, we assume that the optimal indemnity function is given thus:

\[
\mathcal{T}^* (x) = x - \mathcal{G}^* (F_X (x)) = \begin{cases} 
  x, & 0 \leq x < F_X^{-1} (d), \\
  F_X^{-1} (d), & F_X^{-1} (d) \leq x < F_X^{-1} (e), \\
  x - F_X^{-1} (e) + F_X^{-1} (d), & F_X^{-1} (e) \leq x < v, \\
  x - \omega + p, & v \leq x \leq M.
\end{cases}
\]  \quad (86)

(ii.2) If \( g (d) \geq \mathcal{T}' (\alpha -) \), it is easy to prove that \( \mathcal{G}^* (\cdot) \) deduces to

\[
\mathcal{G}^* (z) = \begin{cases} 
  0, & 0 \leq z < d, \\
  F_X^{-1} (z) - F_X^{-1} (d), & d \leq z < \alpha,
\end{cases}
\]  \quad (87)

satisfies (22) under \( \lambda \), corresponding to the aforementioned Case B.IV. This indicates that \( \mathcal{G}^* (\cdot) \) is optimal for (14) under \( \mathcal{P}_1 \). Therefore, the optimal indemnity function is given thus:

\[
\mathcal{T}^* (x) = x - \mathcal{G}^* (F_X (x)) = \begin{cases} 
  x, & 0 \leq x < F_X^{-1} (d), \\
  F_X^{-1} (d), & F_X^{-1} (d) \leq x < v, \\
  x - \omega + p, & v \leq x \leq M.
\end{cases}
\]  \quad (88)

Thus, the proof of Theorem 1 is completed.

Theorem 1 obtains the optimal insurance indemnity when the maximum risk level, \( v = \omega - p \), is given. Next, we analyze the optimal insurance indemnities for different insured with different initial wealth sizes, i.e., \( v \) is a variable. Namely, we consider the following optimization problem:

\[
\max_{\mathcal{T}^* (x)} \max_{v \in \mathcal{D}} \nu \cdot (X - \mathcal{T} (X)) \\
\text{s.t. } (1 + \theta) \mathcal{E} (\mathcal{T} (X)) \leq \mathcal{P},
\]  \quad (89)

Proof. Employing Theorem 1, we assume that the optimization problem (14) admits a solution, \( \mathcal{G}^* (z) \), i.e.,

\[
\mathcal{T}^* (x) = x - \mathcal{G}^* (F_X (x)) = \begin{cases} 
  x, & 0 \leq x < F_X^{-1} (d), \\
  F_X^{-1} (d), & F_X^{-1} (d) \leq x < F_X^{-1} (e), \\
  x - F_X^{-1} (e) + F_X^{-1} (d), & F_X^{-1} (e) \leq x < v, \\
  x - \omega + p, & v \leq x \leq M,
\end{cases}
\]  \quad (90)

where \( (d, e) \) is the unique pair satisfying

\[ 0 \leq d < a < e \leq c, \quad g (d) = g (e), \quad \text{and} \quad E [T^* (X)] = p/(1 + \theta). \]

(iii) If \( 0 < p < p_{c,1} \),

\[
\mathcal{T}^* (x) = x - \mathcal{G}^* (F_X (x)) = \begin{cases} 
  0, & 0 \leq x < F_X^{-1} (q), \\
  F_X^{-1} (q), & F_X^{-1} (q) \leq x \leq M,
\end{cases}
\]  \quad (91)

where \( q \) is the unique scalar satisfying \( c \leq q \leq 1 \) and \( E [T^* (X)] = p/(1 + \theta). \)

\[
\mathcal{V}^* (\alpha) = \max_{\mathcal{D}} \int_{0}^{\alpha} u (\omega - p - \mathcal{G}^* (z)) T^* (z) dz
\]  \quad (92)

\[
+ \int_{0}^{\alpha} u (\omega - p - \mathcal{G}^* (z)) T^* (z) dz.
\]  \quad (93)

Thus,

\[
\frac{\partial \mathcal{V}^*}{\partial \alpha} (\omega - p - \mathcal{G}^* (\alpha)) T^* (\alpha).
\]  \quad (94)

Note \( T^* (\alpha) > 0 \) and

\[
\mathcal{G}^* (\alpha) = F_X^{-1} (\alpha) = \inf \{ x \in [0, M] : \Pr (\mathcal{R} (X) \leq x) \geq \alpha \} \leq \omega - p.
\]  \quad (95)

We know that \( \partial \mathcal{V}^* / \partial \alpha \geq 0 \); thus, the objective function increases with respect to \( \alpha \). Therefore, \( \alpha^* = 1 \) maximizes the
value of the objective function, and the optimal maximal risk level, $v^* = F_X^{-1}(\alpha^*) = F_X^{-1}(1) = M$. Under the conditions of $\alpha^* = 1$, $v^* = M$, and $T^0(1-) > 1$, Corollary 1 can be immediately obtained from Theorem 4.2.

Remark 2. Corollary 1 demonstrates that the optimal solution of $v$, as a decision variable, can be obtained in the special case of $v = M$. In fact, Corollary 1 is simply Theorem 1 of Xu et al. [3]. Comparing Theorem 1 in this study with Theorem 4.2 of Xu et al. [3] reveals that the latter requires the insured to have sufficient initial wealth to ensure that such an insured will not go bankrupt after a loss. Thus, the result of this study can be adapted to protect policyholders with insufficient initial wealth from bankruptcy.

5. Numerical Example

Numerical examples are presented in this section to illustrate our results. First, we adopt the same numerical setting as that of Xu et al. [3] to verify that the optimal insurance indemnity of this study is the same as that of Xu et al. [3]. The initial wealth $\omega$ is set to 15, and premium $p$ takes 5 in both models. The inequality $\omega - p - M \geq 0$ demonstrates that the initial wealth in this case is sufficient. The result is illustrated in Figure 1.

Figure 1 shows that the two indemnity function lines coincide when the initial wealth is sufficient and the same.

For comparison, we next employ the same numerical setting as that of Xu et al. [3] and Bernard et al. [16] except for the initial wealth $\omega$, and newly introduced maximum risk level, $v$. Given that $p = 3.5\omega = 5$, and $M = 10$, the optimal indemnity plan cannot be applied since the inequality (9) $(\omega - p - M \geq 0)$ in Xu et al.’s [3] study and Bernard et al.’s [16] assumption that the initial wealth is sufficient to cover the insurance premium and maximum losses are not satisfied. According to Theorem 1 in this paper, the indemnity plan containing the introduced $v = 1.5$ (Figure 2), as well as the optimal indemnities of Xu et al. [3] and Bernard et al. [16] with initial wealth, $\omega = 15$, can be obtained employing the same premium, $p = 3.5$.

Figure 2 depicts the optimal compensation function line in our study when the risk level $v = 1.5$, and shows its comparison with the compensation function line of Bernard et al. [16] and Xu et al. [3] under the same premium $p = 3.5$.

When the loss is less than $v$, the indemnity function line of this study is higher than that of Xu et al.’s [3], and this higher compensation for a small loss is appropriate for a policyholder with low risk tolerance. Although Bernard et al.’s [16] study proposed a full compensation for a small loss at first, it included a deductible before the maximum level, $v$, and this cannot satisfy the low risk tolerance of the insured.

When the loss exceeds $v$, the compensation function line becomes slightly lower than that of Xu et al. [3], with a difference of 0.0788, which could still guarantee that the policyholder will not go bankrupt after a great loss. Notably, the initial wealth in this study ($\omega = 5$) is far lesser than theirs ($\omega = 15$). Additionally, for different insured with different $\omega$, the corresponding $v$ might also differ. Thus, $v$ varies, and

![Figure 1](image1.png)

**Figure 1**: Optimal indemnities in our study and that of Xu et al. [3] with the same numerical setting.

![Figure 2](image2.png)

**Figure 2**: Optimal indemnities in our study and those of Xu et al. [3] and Bernard et al. [16].

![Figure 3](image3.png)

**Figure 3**: Optimal indemnity functions with different values of premium $p$. 
policyholders can select appropriate insurance policies based on their risk tolerance. When $v$ equals to $M$, the compensation function line in this study correlates with that of Xu et al. [3].

Figure 3 shows that the indemnity functions with different premiums, $p = 3, 3.5, and 4$, are compared for a fixed initial wealth ($\omega = 5$). Intuitively, the higher the value of premium $p$, the more the compensation for the loss. It can be further clarified that the insured with lower risk tolerance is more sensitive to small losses; thus, such an insured would be more eager for higher compensation for small losses and must pay higher premiums accordingly.

6. Conclusion

In this paper, we introduce an optimal insurance model in which a maximal risk level $v$ is designed to fit the actual financial situation of the insured with insufficient initial wealth. The non-requirement of sufficient initial wealth from the insured would complicate the application of the canonical quantile-optimization method to solve the original optimization problem. We resolve this challenge by applying the calculus of variations method with a variable upper limit, which is related to $v$. The optimal solutions under Yaari’s dual criterion are given with very detailed cases. In a special case in which the risk tolerance of the policyholder is sufficient, i.e., $v = M$, the optimal insurance policy is similar to that of Xu et al. [3]. For the general case in which the utility function is strictly concave, several results are obtained and would be included in future discussions.

Data Availability

There is a numerical example in this article, the parameter data used to support the findings of this study are included within the article and reference [3].

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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