

Research Article

Characterizations of the Weak Bivariate Failure Rate Order and Bivariate IFR Aging Class

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In this paper, two characterizations of the weak bivariate failure rate order over the bivariate Laplace transform order of two-dimensional residual lifetimes are given. The results are applied to characterize the weak bivariate failure rate ordering of random pairs by the weak bivariate mean residual lifetime ordering of the minima of pairs with exponentially distributed random pairs with unspecified mean. Moreover, a well-known bivariate aging term, namely, the bivariate increasing failure rate, is characterized by the weaker bivariate decreasing mean residual lifetime property of a random pair of minima.

1. Introduction and Preliminaries

Consider the random pair \( X^T = (X_1, X_2) \) as lifetimes of two devices with joint survival function (s.f.) \( F \). Let us assume that the first device is at age \( t_1 \) and the second device is at age \( t_2 \). The first device and the second device are assumed to be working at the times \( t_1 \) and \( t_2 \), respectively. The residual lifetime random pair is defined as

\[
X^T(t) = (X_1(t), X_2(t)) = (X_1 - t_1, X_2 - t_2 | X_1 > t_1, X_2 > t_2),
\]

in which \( t = (t_1, t_2) \) with \( t_i < X_i \), so that \( u_{X_i} = \sup \{ x \geq 0 \} \) \( F_{X_i}(x) < 1 \) is the upper bound of the support of \( X_i, i = 1, 2 \), and moreover

\[
X_i(t) = (X_i - t_i | X_1 > t_1, X_2 > t_2), \quad \forall (t_1, t_2): F(t_1, t_2) > 0.
\]

The couple \( t^T = (t_1, t_2) \) is called the pair of current ages. The pair \( X(t) \) is called the bivariate residual lifetime associated with \( X = (X_1, X_2) \). The residual lifetime random pair is adopted to measure the time-to-failure of the devices or the system composed of these devices in light of the ages of the devices. The random variables \( X_i(t), i = 1, 2 \), are the marginal random variables of \( X(t) \). It can be seen that \( X(t) \) has joint s.f.

\[
F_t(x) = \frac{F(x_1 + t_1, x_2 + t_2)}{F(t_1, t_2)},
\]

where \( x = (x_1, x_2) \). If \( F \) is assumed to be linked with an absolutely continuous joint cumulative distribution function (c.d.f.) with the corresponding joint probability density function (p.d.f.) \( f \), then the joint p.d.f. of \( X(t) \) is obtained by

\[
f_t(x) = \frac{f(x_1 + t_1, x_2 + t_2)}{F(t_1, t_2)}.
\]

If we assume that \( X_i \) has marginal s.f. \( F_{X_i} \), then the p.d.f. of \( X_i \) (when it exists) is denoted by \( f_{X_i}, i = 1, 2 \). The conditional s.f. of \( X_i(t) \) and \( X_2(t) \) is obtained, respectively, as

\[
F_{X_i(t)}(x_i) = \frac{F(x_i + t_1, t_2)}{F(t_1, t_2)},
\]

and the corresponding conditional p.d.f.s are, therefore, revealed as

\[
f_{X_i(t)}(x_i) = \frac{f(x_i + t_1, t_2)}{F(t_1, t_2)}.
\]
In the case when \( X_1 \) and \( X_2 \) are independent, the random variable \( X_i(t) \) defined in (2) is equal in distribution with \( (X_i - t|X_i > t), \quad i = 1, 2 \), which is the commonly used univariate residual lifetime which is known in the literature.

The failure gradient of \( X \) is given by

\[
f_X(t)(x_i) = \frac{-\partial f_X(t)}{c_X(t)} \frac{x_i + t_1, t_2}{c_X(t_1, t_2)}.
\]

In the following, two well-known bivariate stochastic orders are defined, which are frequently used in the sequel.

**Definition 1.** Let \( X \) and \( Y \) be two non-negative random pairs s.f.s \( F \) and \( G \) and failure gradients \( h_X \) and \( h_Y \), respectively. Then, it is said that \( X \) is smaller than \( Y \) in weak bivariate failure rate order (denoted as \( X \leq \text{wfr} Y \)) whenever

\[
h_{X_i}(t) \geq h_{Y_i}(t), \quad i = 1, 2, \text{ for all } t \in \mathbb{R}_+^2, \text{ or equivalently if } \frac{G(t)}{F(t)} \text{ is non-decreasing in } t = (t_1, t_2) \in \{ t \in \mathbb{R}_+^2; \ G(t) > 0 \}.
\]

For further properties of bivariate/multivariate failure rate functions and also some of their applications in different contexts, we refer the readers to Basu [1]; Johnson and Kotz [2]; Marshal [3]; Navarro and Ruiz [4]; Khaleedi and Kochar [5]; Misra and Naqvi [6]; Badia et al. [7]; Badia and Lee [8]; Nair and Vinesh [9]; and Gupta and Kirmani [10].

**Definition 2.** Let \( X \) and \( Y \) be two non-negative random pairs with s.f.s \( F \) and \( G \) and m.r.l. functions \( m_X \) and \( m_Y \), respectively. We then say that \( X \) is smaller than \( Y \) in weak bivariate mean residual lifetime order (denoted as \( X \leq \text{wmr} Y \)) if \( m_{X_i}(t) \leq m_{Y_i}(t), \quad i = 1, 2, \text{ for all } t \in \mathbb{R}_+^2, \text{ or equivalently if } \frac{\int_{t_1}^{\infty} G(x_1, x_2) dx_1}{\int_{t_1}^{\infty} F(x_1, x_2) dx_1} \text{ is non-decreasing in } t_1 \geq 0, \)

\[
\int_{t_1}^{\infty} G(x_1, x_2) dx_1
\]

(12)

Note that \( X \leq \text{wmr} Y \) implies that \( X \leq \text{wmr} Y \) (cf. Hu et al. [11]).

For any non-negative random variable \( X \) with c.d.f. \( F_X \), the Laplace–Stieltjes transform is defined by

\[
L_X(s) := \int_0^{\infty} e^{-st} dF_X(x), \quad s > 0.
\]

Evidently, \( L_X(s) \) is non-increasing in \( s \). Let \( F_X \equiv 1 - F_X \) denote the s.f. of \( X \) which measures the probabilities of selective right tails of the distribution of \( X \). The Laplace transform of \( F_X \) is given by

\[
L^*_X(s) := \int_0^{\infty} e^{-st} dF_X(x), \quad s > 0.
\]

It is worth mentioning that the Laplace transforms involved in (13) and (14) always exist (since \( X \) is non-negative).

The Laplace transform has been used to be regarded as actuarial amounts, such as indemnities associated with risks, incomes associated with financial transactions, or life premiums in life insurance (see Belzunce et al. [12]; Denuit [13]; and Belzunce et al. [14]). It can be easily verified that if \( F_X \) is absolutely continuous, then

\[
L^*_X(s) = \frac{1}{s} \left( 1 - L_X(s) \right), \quad s > 0.
\]

In view of (13) and also in the spirit of (14), one can rewrite \( L_X(s) = E[e^{-st}] \) and \( L^*_X(s) = E[\min(X, E)] \) where \( E \) stands for a random variable with exponential distribution with mean \( 1/s \).

Given two random variables \( X \) and \( Y \) with Laplace transforms \( L_X \) and \( L_Y \), respectively, it is said that \( X \) is smaller than \( Y \) in Laplace transform order (denoted by \( X \leq \text{lt} Y \)) whenever \( L_X(s) \geq L_Y(s) \), for all \( s \geq 0 \). Equivalently, from (15),

\[
X \leq \text{lt} Y \iff L^*_X(s) \leq L^*_Y(s), \quad \text{for all } s > 0.
\]

Let us assume that \( X_i = (X - t|X > t), \quad t < u_X, \) and \( Y_i = (Y - t|Y > t), \quad t < u_Y, \) denote the residual lifetime of \( X \) and \( Y \), respectively, where \( u_X \) and \( u_Y \) denote the upper bound of supports of \( X \) and \( Y \). Belzunce et al. [12] applied the Laplace transform to compare residual lifetimes in place of the original random variables. We say that \( X \) with p.d.f. \( f_X \) and s.f. \( F_X \) is smaller than \( Y \) with p.d.f. \( f_Y \) and s.f. \( F_Y \) in the failure rate order (denoted as \( X \leq \text{fr} Y \) if \( h_X(t) \geq h_Y(t), \) for all \( t < \min(u_X, u_Y), \text{ in which } h_X(t) = (f_X(t)/F_X(t)), \quad t < u_X, \) and \( h_Y(t) = (f_Y(t)/F_Y(t)), \quad t < u_Y, \) are the failure rate functions of \( X \) and \( Y \), respectively. Belzunce et al. [15] showed that
\[ X \leq_{fr} Y \iff X_i \leq_{fr} Y_i, \quad \text{for all } t < \min\{u_X, u_Y\}. \]  

It is said that \( X \) with m.r.l. function \( m_X \) given by \( m_X(t) = E(X) \) is smaller than \( Y \) with m.r.l.f. \( m_Y \) given by \( m_Y(t) = E(Y) \) (denoted as \( X \leq_{mrl} Y \)) whenever \( m_X(t) \leq m_Y(t) \), for all \( t < \min\{u_X, u_Y\} \). Belzunce et al. [15] also proved that
\[ X \leq_{fr} Y \iff 1 - e^{-sX} \leq_{mrl} 1 - e^{-sY}, \quad \text{for all } s > 0. \]

For the non-negative random pair \( X = (X_1, X_2) \) with joint p.d.f. \( f(x_1, x_2) \), the bivariate Laplace–Stieltjes transform is given by
\[
L_X(s) = \int_0^\infty \int_0^\infty e^{-s_1X_1 - s_2X_2} f(x_1, x_2) dx_1 dx_2, \quad s = (s_1, s_2) \in \mathbb{R}_+^2.
\]

Suppose that \( X^T = (X_1, X_2) \) and \( Y^T = (Y_1, Y_2) \) have respective Laplace transforms \( L_X \) and \( L_Y \). Then, it is said that \( X \) is smaller than \( Y \) in the bivariate Laplace transform (denoted by \( X \leq Y \)) whenever \( L_X(s) \geq L_Y(s) \), for all \( s = (s_1, s_2) \in \mathbb{R}_+^2 \). From Theorem 7.D.1. of Shaked and Shanthikumar [16], it is deduced that \( X \leq Y \) implies \( X_i \leq_{fr} Y_i, \quad i = 1, 2 \), but the reversed implication is not true in general.

The aim of this paper is to compare the characterizations from (17) and (18) given characterizations to bivariate distributions, enumerating also the dependence structure of random pairs as a new aspect in ordering bivariate distributions. By developing the theory for the bivariate lifetime distribution, it is found that the dependence structure of the random pairs considered has no influence on the characterization of the weak bivariate failure rate order and also on a characterization of a bivariate increasing failure rate property, as will be shown.

The paper is organized as follows. Section 2 presents the main results of the paper, which include a characterization of the weak bivariate failure rate order using the univariate Laplace transform order applied to conditional bivariate residual lifetimes, and then a development of this characterization to a stronger case. In Section 3, in the spirit of the previous characterizations, we present further characterizations of the weak bivariate failure rate order in terms of the weak bivariate mean residual life order and also a characterization result for the increasing bivariate failure rate class using the decreasing bivariate mean residual life. In Section 4, we conclude the paper with a detailed summary of the paper and also a future perspective of the developments of the results on the multivariate cases.

### 2. Characterizations of Weak Bivariate Failure Rate Order

In this section, the bivariate Laplace transform is applied to residual lifetime pair \( X(t) \) to compare the residual lifetimes of \( X \) and \( Y \). Denote by \( L_{X_i|X_1 = x_1}(s_2) \), \( s_2 > 0 \), the Laplace transform of the conditional random variable \( X_i | X_1 = x_1 \) and denote by \( L_{X_i|X_1}(s_2) \) the Laplace transform of \( X_2 \) given \( X_1 \) which is randomly drawn. By some routine calculation, one obtains
\[
L_X(s) = E\left[ e^{-s_1X_1} E\left[ e^{-s_2X_2} | X_1 \right] \right]
= E\left[ e^{-s_1X_1} L_{X_2|X_1}(s_2) \right]
= 1 - s_1 L_{X_1}(s_1) - s_2 L_{X_1}(s_2)
+ s_1 s_2 L_{X_1}(s_1, s_2), \quad s_2 > 0, \quad i = 1, 2,
\]

where
\[
L_X(s) = \int_0^\infty \int_0^\infty e^{-s_1x_1 - s_2x_2} F(x_1, x_2) dx_1 dx_2.
\]

To derive the Laplace transform of bivariate residual lifetime \( X(t) = (X_1(t), X_2(t)) \) first, let us get the expression of the Laplace transform of \( X_i(t) \), \( i = 1, 2 \). The formulas given in (20) and (21) can be fulfilled by the identities (5) to get
\[
L_{X_i}(s_1)(s_2) = \int_0^\infty e^{-s_1t_1} F(t_1, t_2) dt_1 \int_0^\infty e^{-s_2t_2} F(t_1, t_2) dt_2.
\]

Now, one can develop that
\[
L_{X_i}(s_1) = 1 - s_1 \int_0^\infty e^{-s_1t_1} F(t_1, t_2) dt_1
- s_2 \int_0^\infty e^{-s_1t_1} F(t_1, t_2) dt_2
+ s_1 s_2 \int_0^\infty \int_0^\infty e^{-s_1t_1 - s_2t_2} F(t_1, t_2) dt_1 dt_2.
\]

The following result presents equivalent conditions for the Laplace transform ordering of \( X_i(t) \) and \( Y_i(t) \), \( i = 1, 2 \).

**Proposition 1.** Let \( X^T = (X_1, X_2) \) and \( Y^T = (Y_1, Y_2) \) be two non-negative random pairs having joint s.f.s \( F \) and \( G \) with bivariate residual lifetimes \( X^T(t) = (X_1(t), X_2(t)) \) and \( Y^T(t) = (Y_1(t), Y_2(t)) \), respectively. Then,

(i) \( X_i(t) \leq_{fr} Y_i(t), \) for all \( t \in \mathbb{R}_+^2 \) if, and only if, \( \left( \int_0^\infty e^{-s_1t_1} F(t_1, t_2) dt_1 \right)^{s_1} \left( \int_0^\infty e^{-s_2t_2} F(t_1, t_2) dt_2 \right)^{s_2} \) is non-decreasing in \( t_1 \geq 0 \) for every \( t_2 \geq 0 \).

(ii) \( X_i(t) \leq_{fr} Y_i(t) \), for all \( t \in \mathbb{R}_+^2 \) if, and only if, \( \left( \int_0^\infty e^{-s_1t_1} G(t_1, t_2) dt_1 \right)^{s_1} \left( \int_0^\infty e^{-s_2t_2} G(t_1, t_2) dt_2 \right)^{s_2} \) is non-decreasing in \( t_2 \geq 0 \) for every \( t_1 \geq 0 \).

**Proof.** We prove only assertion (i). Assertion (ii) can be proved by an analogous method. It is easily verified that for all \( t_i \geq 0, \quad i = 1, 2 \) and \( s_i \geq 0, \)
\[
\frac{\partial}{\partial t_1} \int_{t_1}^{\infty} e^{-s_1x_1}G(x_1,t_2)dx_1 \text{sign} = \int_{t_1}^{\infty} e^{-s_1x_1}F(t_1,t_2)dx_1 = e^{-s_1t_1}F(t_1,t_2) \int_{t_1}^{\infty} e^{-s_1x_1}G(x_1,t_2)dx_1
\]
\[
- e^{-s_1t_1}G(t_1,t_2) \int_{t_1}^{\infty} e^{-s_1x_1} \frac{\partial}{\partial t_1} \int_{t_1}^{\infty} e^{-s_1x_1}F(x_1,t_2)dx_1, \]
(24)
in which \text{sign} stands for equality in sign. Thus, it follows that for all \(t_i \geq 0, \quad i = 1, 2,\)
\[
\frac{\partial}{\partial t_1} \int_{t_1}^{\infty} e^{-s_1x_1}G(x_1,t_2)dx_1 \geq 0, \quad \forall s_1 \geq 0, \quad \forall t_i \geq 0.
\]
(25)
if and only if (see (22))
\[
L^*_X(t_1)(s_1) = \frac{\int_{t_1}^{\infty} e^{-s_1x_1}F(x_1,t_2)dx_1}{e^{-s_1t_1}F(t_1,t_2)} \leq \frac{\int_{t_1}^{\infty} e^{-s_1x_1}G(x_1,t_2)dx_1}{e^{-s_1t_1}G(t_1,t_2)} = L^*_Y(t_1)(s_1), \quad \forall s_1 \geq 0.
\]
(26)

This is also equivalent to \(X_1(t) \leq Y_1(t).\)

Suppose that \(T_s = (T_s^x, T_s^y)\) with \(s = (s_1, s_2)\) where \(T_s^x\) denotes an exponentially distributed random variable with mean \(1/s_1, \quad i = 1, 2.\) Let us assume that \(X\) and \(T_s^x\) are independent and, further, \(T_s^x\) and \(T_s^y\) are also independent. Denote by \(\min\) the minimum of \(a\) and \(b.\) The random pair \(X \land T_s^x = (X_1 \land T_s^x, X_2 \land T_s^x)\) has joint s.f.
\[
F_s^x(x) = P(X \land T_s^x > x)
= P(X > x, T_s^x > x)
= e^{-s_1x_1-s_2x_2}F(x), \quad x = (x_1, x_2) \in \mathbb{R}^2.
\]
(27)

Likewise, provided that \(Y\) and \(T_s^y\) are also independent, \(Y \land T_s^y = (Y_1 \land T_s^x, Y_2 \land T_s^x)\) has joint s.f. \(G_s(x) = e^{-s_1x_1-s_2x_2}G(x).\) From definition of bivariate m.r.l. and by Proposition 1,
\[
X_1(t) \leq Y_1(t) \Leftrightarrow m_{1,X,T_s^x}(t_1,t_2) \leq m_{1,Y,T_s^y}(t_1,t_2),
X_2(t) \leq Y_2(t) \Leftrightarrow m_{2,X,T_s^x}(t_1,t_2) \leq m_{2,Y,T_s^y}(t_1,t_2).
\]
(28)

Therefore, applying the wrn order,
\[
X_i(t) \leq_w Y_i(t), \quad i = 1, 2 \Leftrightarrow X \land T_{s_i} \leq_w \text{wmt} Y \land T_{s_i}, \quad \text{for all } s_i \in \mathbb{R}_+^2.
\]
(29)

In the next result, the wrn order is characterized by the Laplace transform order of residual lifetimes of marginal distributions of \(X(t)\) and \(Y(t).\)

**Theorem 1.** Let \(X^* = (X_1, X_2)\) and \(Y^* = (Y_1, Y_2)\) be two non-negative random pairs having joint s.f.s \(F^*\) and \(G^*\), respectively. Then,
\[
X \leq_w Y \quad \text{if, and only if,} \quad X_i(t) \leq_w Y_i(t), \quad i = 1, 2, \text{ for all } t \in \mathbb{R}_+^2.
\]
(30)

**Proof.** We first prove that \(X_i(t) \leq_w Y_i(t), \quad i = 1, 2, \text{ for all } t \in \mathbb{R}_+^2\) implies that \(X \leq_w Y.\) It is known that, for every \(i = 1, 2,\)
\[
\begin{align*}
L_{X_i}(s) &= \frac{1}{s_1}(1 - L_{X_i}(s)), \\
L_{Y_i}(s) &= \frac{1}{s_1}(1 - L_{Y_i}(s)).
\end{align*}
\]
(31)

Using the identities given in (31), the identities given in (22), and by applying (6), one can get
\[
\begin{align*}
\int_{t_1}^{\infty} e^{-s_1x_1}G(x_1,t_2)dx_1 &= \frac{\int_{t_1}^{\infty} e^{-s_1x_1}F(x_1,t_2)dx_1}{e^{-s_1t_1}F(t_1,t_2)} \\
&\leq \frac{\int_{t_1}^{\infty} e^{-s_1x_1}G(x_1,t_2)dx_1}{e^{-s_1t_1}G(t_1,t_2)} = L_{Y_1}(s_1), \quad \forall s_1 \geq 0.
\end{align*}
\]
(26)
and similarly,
\[
\begin{align*}
\int_{t_1}^{\infty} e^{-s_1x_1}F(x_1,t_2)dx_1 &= \frac{\int_{t_1}^{\infty} e^{-s_1x_1}F(x_1,t_2)dx_1}{e^{-s_1t_1}F(t_1,t_2)} \\
&\leq \frac{\int_{t_1}^{\infty} e^{-s_1x_1}F(x_1,t_2)dx_1}{e^{-s_1t_1}F(t_1,t_2)} = L_{X_1}(s_1), \quad \forall s_1 \geq 0.
\end{align*}
\]
(27)

From Proposition 1, it is deduced that \(X_1(t) \leq_w Y_1(t), \text{ for all } t \in \mathbb{R}_+^2\), holds if, and only if, (32) is non-decreasing in \(t_1\) for every \(t_2 \geq 0\) and for all \(s_1 \geq 0,\) and in particular, when \(s_1 \rightarrow + \infty, X_1(t) \leq_w Y_1(t), \text{ for all } t \in \mathbb{R}_+^2\), concludes that (32) is non-decreasing in \(t_1\) for every \(t_2 \geq 0.\) In parallel, \(X_2(t) \leq_w Y_2(t), \text{ for all } t \in \mathbb{R}_+^2\), holds if, and only if, (33) is non-decreasing in \(t_1\) for every \(t_2 \geq 0\) and for all \(s_2 \geq 0,\) and in the special case when \(s_2 \rightarrow + \infty, X_2(t) \leq_w Y_2(t), \text{ for all } t \in \mathbb{R}_+^2\), implies that (33) is non-decreasing in \(t_2\) for every \(t_1 \geq 0\). Notice that for all \(s_1 \geq 0\) and, also, for all \(t \in \mathbb{R}_+^2,\)
\[ e^{-r_1(x_1-t_1)}I[x_1 > t_1] \left( -\frac{\partial}{\partial x_1} F(x_1, t_2) \right) \leq -\frac{\partial}{\partial x_1} F(x_1, t_2), \]  

in which \(- (\partial/\partial x_1) F(x_1, t_2)\) is an integrable function for all \(t \in \mathbb{R}_+^2\), since

\[
\int_0^{\infty} \left( -\frac{\partial}{\partial x_1} F(x_1, t_2) \right) dx_1 = F(x_1, t_2) \leq 1. \tag{35}
\]

By Lebesgue’s dominated convergence theorem, we have

\[
\lim_{s_i \to \infty} \int_{t_i}^{\infty} f(x_1, t_2) e^{-s_i(x_i-t_i)} dx_1 = \lim_{s_i \to \infty} \int_0^{\infty} f(x_1, t_2) e^{-s_i(x_i-t_i)} I[x_1 > t_1] dx_1, \tag{36}
\]

where \(f(x_1, t_2) = -(\partial/\partial x_1) F(x_1, t_2)\). In a similar manner,

\[
\lim_{s_i \to \infty} \int_{t_i}^{\infty} g(x_1, t_2) e^{-s_i(x_i-t_i)} dx_1 = 0, \tag{37}
\]

where \(g(x_1, t_2) = -(\partial/\partial x_1) G(x_1, t_2)\). Thus,

\[
X_1(t) \leq t_1 Y_1(t), \quad \forall t \in \mathbb{R}_+^2 \text{ yields } G(t_1, t_2) = \frac{G(t_1, t_2)}{F(t_1, t_2)} \text{ is non-decreasing in } t_1, \quad \forall t_2 \geq 0. \tag{38}
\]

Further,

\[
X_2(t) \leq t_2 Y_2(t), \quad \forall t \in \mathbb{R}_+^2 \text{ yields } G(t_1, t_2) = \frac{G(t_1, t_2)}{F(t_1, t_2)} \text{ is non-decreasing in } t_2, \quad \forall t_1 \geq 0. \tag{39}
\]

Making use of (38) together with (39) concludes that \(X \leq \text{wfr} Y\). To prove the reversed implication, we assume that \(X \leq \text{wfr} Y\). Select an arbitrary \(t_1 \geq 0\) and fix it and also \(s_1 \geq 0\) as an arbitrary value. Define the functions \(J\) and \(H\) as

\[
(t_1, x_1) \mapsto J(t_1, x_1) = e^{-s_1 x_1} I[x_1 > t_1], \quad t_1, x_1 \geq 0,
\]

\[
(x_1, x_2) \mapsto H(x_1, x_2) = J(t_1, x_1), \quad i = 1, 2, \quad x_i \geq 0, \tag{40}
\]

respectively, so that \(H(x_1, 1) = F(x_1, t_2)\) and \(H(x_1, 2) = G(x_1, t_2)\). From assumption, \((G(x_1, t_2)/F(x_1, t_2))\) is non-decreasing in \(x_1 \geq 0\). Hence, \(H\) is TP2 in \((x_1, i) \in \mathbb{R}_+ \times [1, 2]\). By general composition theorem of Karlin [17],

\[
\int_{t_i}^{\infty} f(t_1, x_1) H(t_1, x_1) dx_1 = \text{ is also TP2 in } (t_1, x_1) \in \mathbb{R}_+^2 \times \mathbb{R}_+. \tag{41}
\]

Since the conclusion is not affected by the choice of \(t_2 \geq 0\) and also \(s_1 \geq 0\), one concludes that \(\int_{t_1}^{\infty} e^{-s_1 x_1} G(x_1, t_2) dx_1 / \int_{t_1}^{\infty} e^{-s_2 x_1} F(x_1, t_2) dx_1\) is non-decreasing in \(t_1 \geq 0\), for all \(s_1 \geq 0\) and for every \(t_2 \geq 0\). By Proposition 1 (i), it follows that \(X_1(t) \leq Y_1(t)\), for all \(t \in \mathbb{R}_+^2\). By a similar discussion, it is realized that \(X \leq \text{wfr} Y\) also implies that \(X_2(t) \leq Y_2(t)\). The proof is completed.

The following example illustrates an application of Theorem 1.

Example 1. Let \(X = (X_1, X_2)\) follow the following s.f. associated with a mixture distribution:

\[
F(t) = pe^{-\lambda_1(t_1+t_2)} + qe^{-\lambda_1(t_1+t_2)}, \quad t = (t_1, t_2) \in \mathbb{R}_+^2, \tag{41}
\]

and, furthermore, let \(Y = (Y_1, Y_2)\) follow a mixture distribution having s.f.

\[
G(t) = pe^{-\lambda_2(t_1+t_2)} + qe^{-\lambda_2(t_1+t_2)}, \quad t = (t_1, t_2) \in \mathbb{R}_+^2, \tag{42}
\]

where \(q = 1 - p\) and \(p\) is an arbitrary value in \([0, 1]\) and also \(\lambda_i, \lambda_j > 0\) for every \(i, j = 1, 2\). By appealing to routine calculations, the bivariate failure rates of \(X\) and \(Y\) are specified, respectively, by

\[
h_{i, X}(t) = \frac{p\lambda_1 + q\lambda_1 e^{(\lambda_i - \lambda_1)(t_1+t_2)}}{p + q e^{(\lambda_i - \lambda_1)(t_1+t_2)}}, \tag{43}
\]

\[
h_{i, Y}(t) = \frac{p\lambda_2 + q\lambda_2 e^{(\lambda_j - \lambda_2)(t_1+t_2)}}{p + q e^{(\lambda_j - \lambda_2)(t_1+t_2)}}, \tag{44}
\]

for every \(i = 1, 2\). We also can get

\[
L_{X,i}(s_i) = \frac{(p/s_i + \lambda_i) + (q/s_i + \lambda'_i)e^{(\lambda_i - \lambda_1)(t_1+t_2)}}{p + q e^{(\lambda_i - \lambda_1)(t_1+t_2)}}, \tag{45}
\]

\[
L_{Y,i}(s_i) = \frac{(p/s_i + \lambda_2) + (q/s_i + \lambda'_i)e^{(\lambda_j - \lambda_2)(t_1+t_2)}}{p + q e^{(\lambda_j - \lambda_2)(t_1+t_2)}}, \tag{46}
\]

for every \(i = 1, 2\). We can observe that if \(\lambda_1 \geq \lambda_2\) and \(\lambda_1 \geq \lambda_1\) so that \((\lambda_1 - \lambda') (\lambda_2 - \lambda') \geq 0\), then one realizes that \(h_{i, X}(t) \geq h_{i, Y}(t)\) for all \(t \in \mathbb{R}_+^2\) and for every \(i = 1, 2\), i.e., \(X \leq \text{wfr} Y\), and also it is seen that \(L_{X,i}(s_i) \geq L_{Y,i}(s_i)\) for every \(i = 1, 2\) and for all \(t \in \mathbb{R}_+^2\) and for all \(s_i > 0\), \(i = 1, 2\), which means that \(X_1(t) \leq Y_1(t), \quad i = 1, 2, \quad \text{for all } t \in \mathbb{R}_+^2\). Thus, the result of Theorem 1 is confirmed by making ordering conditions on parameters of a parametric distribution.

By Theorem 7.D.5 in Shaked and Shanthikumar [16], \(X(t) \leq \text{BL} Y(t)\) implies \(X_1(t) \leq Y_1(t), \quad i = 1, 2\). From Theorem 1, we can, therefore, deduce that if \(X(t) \leq \text{BL} Y(t)\), for all \(t \in \mathbb{R}_+^2\), then \(X \leq \text{wfr} Y\). One may question whether the converse implication holds true. We show that the reversed implication is also satisfied.

Proposition 2. \(X \leq \text{wfr} Y\) implies \(X(t) \leq \text{BL} Y(t)\), for all \(t \in \mathbb{R}_+^2\).

**Proof.** By definition, the proof is obtained if we show that \(L_{X,i}(s) \geq L_{Y,i}(s)\), for all \(s \in \mathbb{R}_+^2\) and for all \(t \in \mathbb{R}_+^2\). From assumption, \(X \leq \text{wfr} Y\) holds. By Theorem 1, it follows that \(X_2(t) \leq Y_2(t)\), and thus \(1 - s_2 L_{X,i}(s_2) \geq 1 - s_2 L_{Y,i}(s_2)\), for all \(t \in \mathbb{R}_+^2\) and for every \(s_2 \geq 0\). We prove using this ordering condition that
\[ s_2 L_{X(t)}^*(s) - L_{X_1(t)}^*(s_1) \geq s_2 L_{Y(t)}^*(s) - L_{Y_1(t)}^*(s_1), \quad \forall t \in \mathbb{R}_+^2, \forall s_2 \geq 0, i = 1, 2. \]  

By applying (21), the expected result will be secured. We can get
\[
L_{X(t)}^*(s) = \int_{t_1}^{\infty} \int_{t_2}^{\infty} e^{-s_1x_1-s_2x_2} T(x_1, x_2) dx_1 dx_2
= e^{s_1t_1} \int_{t_2}^{\infty} e^{-s_1x_1} \left( \int_{t_1}^{\infty} e^{-s_2x_2} T(x_1, x_2) dx_2 \right) dx_1
= e^{s_1t_1} \int_{t_2}^{\infty} e^{-s_1x_1} \frac{\mathcal{F}(x_1, t_2)}{\mathcal{F}(t_1, t_2)} dx_1,
\]
where \( t^* = (x_1, t_2) \) and \( L_{X_1(t)}^*(s_1) \) is the Laplace transform of \( \mathcal{F}_{X_1(t)}(s) \) which can be obtained by substituting (5) when applied on \( X_1(t^*) = (X_2 - t_2)x_1 > x_1, X_2 > t_2 \) into (22).

Now, we develop that
\[
D_X(t, s) = L_{X(t)}^*(s_1) - s_2 L_{X(t)}^*(s) = \int_{t_1}^{\infty} \int_{t_2}^{\infty} e^{-s_1x_1-s_2x_2} T(x_1, x_2) dx_1 dx_2
= e^{s_1t_1} \int_{t_2}^{\infty} e^{-s_1x_1} \left( \int_{t_1}^{\infty} e^{-s_2x_2} T(x_1, x_2) dx_2 \right) dx_1
= e^{s_1t_1} \int_{t_2}^{\infty} e^{-s_1x_1} \mathcal{F}(x_1, t_2) \mathcal{F}(t_1, t_2) dx_1.
\]

\[
D_Y(t, s) - D_X(t, s) = \int_{t_1}^{\infty} e^{-s_1(x_1-t_1)} \left( \frac{\mathcal{F}(x_1, t_2)}{\mathcal{G}(t_1, t_2)} \right. \left. \overline{\mathcal{F}(x_1, t_2)} \right) dx_1
= \int_{t_1}^{\infty} e^{-s_1(x_1-t_1)} \left( \mathcal{F}(x_1, t_2) - \mathcal{G}(t_1, t_2) \right) dx_1,
\]
where \( \forall t \in \mathbb{R}_+^2, \forall s_2 \geq 0. \)

Thus, Theorem 1 together with Proposition 2 provide the following result as a characterization property of the wrf order by bivariate Laplace transform ordering of residual lives. The proof is explicit, and thus we omit it.

**Theorem 2.** Let \( X^T = (X_1, X_2) \) and \( Y^T = (Y_1, Y_2) \) be two non-negative random pairs having joint s.f.s \( \mathcal{F} \) and \( \mathcal{G} \), respectively. Then,
\[
X \leq_{wfr} Y \text{ if, and only if, } X(t) \leq_{BL} Y(t),
\]
where \( i = 1, 2 \), for all \( t \in \mathbb{R}_+^2 \).

3. **Characterizations of Bivariate Increasing Failure Rate Aging Class**

One of the attractive applications of stochastic orderings in the univariate, bivariate, and multivariate settings is the characterization of an aging class of lifetime distributions. This advantage arises from the stochastic comparison of the remaining lifetimes of units after a succession of ages.

In this section, we provide further characterizations of the wrf order between lifetime pairs of \( X \) and \( Y \) by means of the wrml order of typical transformations of \( X \) and \( Y \). Descriptions of some bivariate aging notions in terms of the Laplace transform ordering of residual lives of \( X \) and a characterization result for a bivariate IFR aging property using a weaker bivariate DMRL aging behavior are given.

**Theorem 3.** Let \( X \) and \( Y \) be two lifetime random pairs with s.f.s \( \mathcal{F} \) and \( \mathcal{G} \), respectively. Suppose that \( T \) is a non-negative random pair which is independent of both \( X \) and \( Y \). Then,
\[
X \leq_{wfr} Y \iff X \wedge T \leq_{wrmr} Y \wedge T.
\]

**Proof.** Denote by \( H(x) \) the joint s.f of \( T \). Suppose that \( X \wedge T \) and \( Y \wedge T \) have respective joint s.f.s \( \mathcal{F}' \) and \( \mathcal{G}' \) which are given, from assumption, as
\[
\mathcal{F}'(x) = H(x) \mathcal{F}(x), \quad \mathcal{G}'(x) = H(x) \mathcal{G}(x),
\]
where \( x = (x_1, x_2) \in \mathbb{R}^2 \). Trivially, \( X \leq_{\text{wfr}} Y \) is equivalent to \( X \perp T \leq_{\text{wfr}} Y \perp T \). Thus, \( X \perp T \leq_{\text{wfr}} Y \perp T \). Conversely, since for all non-negative random variables \( T \), \( X \leq_{\text{wfr}} Y \) if \( X \perp T \leq_{\text{wfr}} Y \perp T \). We have \( X \perp T \leq_{\text{wfr}} Y \perp T \). Conversely, since for all non-negative random variables \( T \), \( X \leq_{\text{wfr}} Y \) if \( X \perp T \leq_{\text{wfr}} Y \perp T \).

Proof. (55) and (56) given below being satisfied for all \( s, t \in \mathbb{R}^2 \).

\[
m_{1,X \perp T}(t) = \int_{t_1}^{\infty} e^{-x_1 t_1-x_2 t_2} F(t_1, t_2) \, dt_1 \leq \int_{t_1}^{\infty} e^{-x_1 t_1-x_2 t_2} \bar{F}(t_1, t_2) \, dt_1 \tag{55}
\]

and

\[
m_{2,X \perp T}(t) = \int_{t_2}^{\infty} e^{-x_1 t_1-x_2 t_2} F(t_1, t_2) \, dt_2 \leq \int_{t_2}^{\infty} e^{-x_1 t_1-x_2 t_2} \bar{F}(t_1, t_2) \, dt_2 \tag{56}
\]

It is evident that (55) is equivalent to \( L_{X \times Y}^*(s_1) \leq L_{Y \times X}^*(s_1) \) and also (56) is equivalent to \( L_{X \times Y}^*(s_2) \leq L_{Y \times X}^*(s_2) \) both of which hold for all \( t, s \in \mathbb{R}^2 \). Therefore, in the spirit of Theorem 1, it follows that \( X \leq_{\text{wfr}} Y \). This completes the proof.

The result of Theorem 1 can be used to characterize the wfr order between \( X \) and \( Y \) using the wfr order when applying on \( g_s(X) = (1 - e^{-x_1 s_1}, 1 - e^{-x_2 s_2}) \) and \( g_s(Y) = (1 - e^{-x_1 s_1}, 1 - e^{-x_2 s_2}) \) as two \( s \)-dependent random pairs on \( [0, 1] \).

Theorem 4. Let \( X \) and \( Y \) be two lifetime random pairs with s.f.s \( F \) and \( G \), respectively. Then,

\[
X \leq_{\text{wfr}} Y \iff g_s(X) \leq_{\text{wfr}} g_s(Y), \quad \text{for all } s \in \mathbb{R}^2. \tag{57}
\]

Proof. We first prove that \( X \leq_{\text{wfr}} Y \) implies \( g_s(X) \leq_{\text{wfr}} g_s(Y) \) for every non-negative \( s \). If \( X \leq_{\text{wfr}} Y \) and if we denote \( g_s(a) = 1 - e^{-x_1 s_1}, a > 0 \), and \( g_s(b) = 1 - e^{-x_2 s_2}, b > 0 \), then since \( g_s(a) \) and \( g_s(b) \) are increasing in \( a \) and \( b \), respectively, by Theorem 6.D.4. of Shaked and Shanthikumar [26], \( g_s(X) \leq_{\text{wfr}} g_s(Y) \). To prove the converse part, note that \( g_s(X) \) and \( g_s(Y) \) have respective s.s.f.s.

\[
F_{g_s(X)}(u) = \bar{F}\left(\frac{1}{s_1} \ln(1 - u_1), \frac{1}{s_1} \ln(1 - u_2)\right),
\]

\[
\bar{G}_{g_s(Y)}(u) = \bar{G}\left(\frac{1}{s_2} \ln(1 - u_1), \frac{1}{s_2} \ln(1 - u_2)\right), \tag{58}
\]

in which \( u = (u_1, u_2) \) and \( u_i \in [0, 1], \quad i = 1, 2 \). Therefore,

\[
m_{g_s(X)}(u) = \int_{u_1}^{\infty} \bar{F}(-\frac{1}{s_1} \ln(1 - u_1), -\frac{1}{s_1} \ln(1 - u_2)) \, du_1.
\]

and moreover,

\[
m_{g_s(Y)}(u) = \int_{u_2}^{\infty} \bar{F}(-\frac{1}{s_2} \ln(1 - u_1), -\frac{1}{s_2} \ln(1 - u_2)) \, du_2.
\]

By taking \( u_i = 1 - e^{-x_1 t_i} \) where \( t_i, i = 1, 2 \), (59) implies that for all \( s, t \in \mathbb{R}^2 \),

\[
L_{X \times Y}^*(s_1) = \frac{1}{s_1} m_{1,g_s(X)}(1 - e^{-x_1 s_1}, 1 - e^{-x_2 s_2}) \leq L_{Y \times X}^*(s_1) = \frac{1}{s_1} m_{1,g_s(Y)}(1 - e^{-x_1 s_1}, 1 - e^{-x_2 s_2}), \tag{60}
\]

and, in parallel, (60) concludes for all \( s, t \) that

\[
L_{X \times Y}^*(s_2) = \frac{1}{s_2} m_{2,g_s(X)}(1 - e^{-x_1 s_1}, 1 - e^{-x_2 s_2}) \leq L_{Y \times X}^*(s_2) = \frac{1}{s_2} m_{2,g_s(Y)}(1 - e^{-x_1 s_1}, 1 - e^{-x_2 s_2}). \tag{61}
\]
Hence, applying Theorem 1, the result follows.

The following classes of bivariate aging notions are quite well known in the literature (see Bassan et al. [18] and Lai and Xie [19]).

Definition 3. Suppose that $X^T = (X_1, X_2)$ is a random pair of lifetimes which has joint s.f. $F$, joint failure rate function $h \equiv (h_{1,X}, h_{2,X})$, and joint m.r.l. function $m \equiv (m_{1,X}, m_{2,X})$. Then,

(i) $X$ is said to have bivariate increasing failure rate (BIFR) aging property whenever $(F(t_1 + a_1, t_2 + a_2)/F(t_1, t_2))$ is non-increasing in $t_i \geq 0 \ (i = 1, 2)$ for every $a = (a_1, a_2) \in \mathbb{R}_+^2$ or equivalently if $h_{1,X}(t_1, t_2) \leq h_{1,X}(t_1 + a_1, t_2 + a_2)$, for all $t_i \geq 0$ and $a_i \geq 0, i = 1, 2$.

(ii) $X$ is said to have bivariate decreasing mean residual lifetime (BDMRL) aging property provided that $m_{1,X}(t_1, t_2) \geq m_{1,X}(t_1 + a_1, t_2 + a_2)$, for all $t_i \geq 0$ and $a_i \geq 0, i = 1, 2$.

Rajesh et al. [20] introduced and studied some notions of aging based on the Laplace transform of bivariate residual life. It is well known that the BIFR property implies the BDMRL property, but the converse is not true in general. In the next result, we characterize the BIFR class of joint lifetime distributions by means of the BDMRL property.

Theorem 5. Suppose that $X$ is a non-negative random pair. Then,

$X$ is BIFR $\iff X \wedge E_s$ is BDMRL for every $s \in \mathbb{R}_+^2$.  \hspace{1cm} (63)

Proof. It can be readily seen that $X$ is BIFR if, and only if, $X(a) \leq \text{wfr} X$ for every $a \in \mathbb{R}_+^2$. This together with Theorem 3 concludes that $X$ is BIFR if, and only if, $(X(a) \wedge E_s) \leq \text{wmr} (X \wedge E_s)$ for all $a, s \in \mathbb{R}_+^2$. It is not hard to prove that $(X \wedge E_s)(a) \wedge E_s$ and $(X \wedge E_s)(a)$ are identical in distribution.  \hspace{1cm} (64)

Therefore, $X$ is BIFR if, and only if, $(X \wedge E_s)(a) \leq \text{wmr} (X \wedge E_s)$ for all $a, s \in \mathbb{R}_+^2$, which, equivalently, holds whenever $X \wedge E_s$ is BDMRL for every $s \in \mathbb{R}_+^2$. \hfill \Box

4. Conclusions

Two objectives were pursued with this work. It is well known in the literature that the bivariate Laplace transform order and the weak bivariate mean residual life order are both weaker than the bivariate weak failure rate order in the sense that the latter implies the former ones. The first objective of this work was to characterize the bivariate weak failure rate order of two pairs of random lifetimes using the bivariate Laplace transform order applied to the residual lifetimes. The result provides further characterizations of the bivariate weak failure rate order between two random pairs of lifetimes by the weak bivariate mean residual lifetime order applied to certain classes of lifetime transformations. One of these is the minimum between lifetimes and an independent random lifetime, and another is the class of distribution functions of the exponential distribution with an unspecified mean. The second goal of the paper was to characterize a known aging property, namely, the bivariate increasing failure rate, using a weaker aging property, namely, the bivariate decreasing mean residual life property, of a specified transformation of the underlying random pair of lifetimes. This class is the minimum of the underlying random lifetimes with a random pair of independent exponential variables, each having an arbitrary mean.

In the future, the author would like to see whether the given characterizations extend to higher (more than two) dimensions. Consider two random vectors $X^T = (X_1, \ldots, X_p)$ and $Y^T = (Y_1, \ldots, Y_p)$ of lifetimes with partially differentiable s.f.s $F$ and $G$, respectively. According to Hu et al. [11], it is said that $X$ is smaller than $Y$ in the weak multivariate failure rate order (denoted by $X \leq \text{wfr} Y$) whenever $(G(t)/F(t))$ is non-decreasing in $t \in \{t \in \mathbb{R}_+: G(t) > 0\}$. It is also said that $X$ is smaller than $Y$ in the multivariate Laplace transform order (denoted by $X \leq \text{MLt} Y$) if $L_X(s) = E(e^{-s^T X}) \geq L_Y(s) = E(e^{-s^T Y})$, for all $s \in \mathbb{R}_p$ (see, e.g., Shaked and Shanthikumar [16]). Let us consider $X(t) = (X_1 - t_1, \ldots, X_p - t_p, X_1 > t_1, \ldots, X_p > t_p)$ and $Y(t) = (Y_1 - t_1, \ldots, Y_p - t_p, Y_1 > t_1, \ldots, Y_p > t_p)$ as the multivariate residual lifetime vectors associated with $X$ and $Y$, respectively, where $t = (t_1, \ldots, t_p)$ is a vector of time points in $\mathbb{R}_p$. We can also consider the marginal conditional residual lifetimes $X_i(t) = (X_i - t_i | X_1 > t_1, \ldots, X_p > t_p)$ and $Y_i(t) = (Y_i - t_i | Y_1 > t_1, \ldots, Y_p > t_p)$ for every $i = 1, \ldots, p$. The result of Theorem 1 may be extended to the case where $X \leq \text{wfr} Y$ in the multivariate case, is equivalent to $X_i(t) \leq Y_i(t), i = 1, \ldots, p$, for all $t \in \mathbb{R}_p$ where $p > 2$. To develop the result of Theorem 2, we can also study whether $X \leq \text{wfr} Y$ in the multivariate setting, is equivalent to $X(t) \leq \text{MLt} Y(t)$, for all $t \in \mathbb{R}_p$ where $p > 2$. What success can be achieved depends on whether the techniques used in the bivariate setting can also be used in the multivariate case. The author is confident that the developments described above are feasible and can be presented in a future research work.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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References


