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Research Article

Notion of New Structure of Uncertain Sequences Using Δ-Spaces

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The object of this paper is to introduce a new approach of Δ-operator on the theory of uncertainty using sequence spaces. Also, some structural properties will be studied. Moreover, inclusion relations concerning the newly constructed spaces of this paper will be taken care of.

1. Introduction, Background, and Preliminaries

It is a well-known fact that the classical measure, observing nonnegativity and countable additivity, plays a vital role in Science, Mathematics, Engineering, and in particular real life as well. However, the measure employed in real life has no countable additivity in many cases. To taste this structure, researchers proposed different measures like fuzzy measures brought introduced in [1] in 1974. There are many types of uncertainties in real life, such as randomness, fuzziness, and uncertainty which consists of both randomness and fuzziness. To measure fuzzy events, in 1978, Zadeh [2] introduced a possibility measure. A probability measure is used to describe a random event. However, the possibility measure has no self-duality. Thus, B. Liu and Y. Liu [3], in 2002, proposed a self-dual measure, the credibility measure. An axiomatic foundation for credibility theory was given in Liu [4]. Since then, credibility theory has been developed uninterruptedly (see the survey in [5]). In [6], Kwakernaak introduced a fuzzy random variable to describe the phenomenon that fuzziness and randomness simultaneously appear in a system. A hybrid variable was introduced by Liu [7] as a measurable function from a chance space to the set of real numbers. Therefore, a concept of the chance measure was introduced by Li and Liu [8]. Furthermore, some convergence results were discussed (see [9]).

It is obvious that the classical measure, probability measure, credibility measure, and chance measure proposed by Li and Liu [8] are all special cases of uncertain measures. But possibility measure is not an uncertain measure. Thus, the properties of uncertain measure are also applicable to classical measure, probability measure, credibility measure, and chance measure. Since sequence convergence plays a very important role in the fundamental theory of mathematics, there are many convergence concepts in classical measure theory, probability theory, credibility theory, and chance theory, and the relationships between them are discussed.

Let Γ be a nonempty set and $\mathfrak{L}$ a σ-algebra over Γ. Each element $\nu \in \mathfrak{L}$ is called an event. To measure an uncertain event, uncertain measure was introduced as a set function $\mathcal{S}$ satisfying the following axioms:

Axiom 1 (normality): $\mathfrak{M}\{\Gamma\} = 1$

Axiom 2 (monotonicity): $\mathfrak{M}\{\nu_1\} \leq \mathfrak{M}\{\nu_2\}$ whenever $\nu_1 \leq \nu_2$

Axiom 3 (self-duality): $\mathfrak{M}\{\nu\} + \mathfrak{M}\{\nu^c\} = 1$ for any event ν

Axiom 4 (countable subadditivity): for every countable sequence of events $\{\mu_i\}$, we have

$$\mathfrak{M}\{\cup_{i=1}^{\infty}\mu_i\} \leq \sum_{i=1}^{\infty}\mathfrak{M}\{\mu_i\}. \quad (1)$$

Since uncertainty phenomena exist widely in real life, the concept of uncertain variable was introduced by Liu [10] as a measurable function from an uncertainty space $(\Gamma, \mathfrak{M}, \mathfrak{L})$ to the set of real numbers.

As in [10, 11], we have following definitions.

Definition 1. For an uncertain variable, the expected value operator is defined as

$$E[\varsigma] = \int_0^{+\infty} \mathfrak{M}\{\varsigma \geq \mu\}d\mu - \int_{-\infty}^0 \mathfrak{M}\{\varsigma \leq \mu\}d\mu, \tag{2}$$

only if one of the integrals exists.

We call the function $\varphi(\rho)$ to uncertainty distribution of the uncertain variable ς if

$$\varphi(\rho) = \mathfrak{M}\{\tau \in \Gamma | \varsigma(\tau) \leq \rho\}. \tag{3}$$

Definition 2. An uncertain sequence $\{\varsigma_j\}$ is said to be convergent almost surely (a.s.) to an uncertain variable ς if there exists an event ν with $\mathfrak{M}(\nu) = 1$ such that

$$\lim_{j\longrightarrow\infty} \|\varsigma_j(\kappa) - \varsigma(\kappa)\| = 0, \tag{4}$$

for every $\kappa \in \nu$.

Definition 3. An uncertain sequence $\{\varsigma\}$ is said to be convergent in measure to an uncertain variable ς if

$$\lim_{j\longrightarrow\infty} \mathfrak{M}\{\|\varsigma_j(\kappa) - \varsigma(\kappa)\| \geq \varepsilon\} = 0, \tag{5}$$

for every $\varepsilon > 0$.

Definition 4. Let $\varsigma, \varsigma_1, \varsigma_2, \cdots$ be the uncertainty distributions of uncertain variables. Then, the sequence $\{\varsigma_j\}$ is called convergence in mean to ς if

$$\lim_{j\longrightarrow\infty} E[|\varsigma_j(\kappa) - \varsigma(\kappa)|] = 0. \tag{6}$$

Definition 5. Let $\vartheta, \vartheta_1, \vartheta_2, \cdots$ be uncertain variables with finite expected values $\varsigma, \varsigma_1, \varsigma_2, \cdots$, respectively. Then, the sequence $\{\varsigma_j\}$ is called convergence in distribution to ς if $\vartheta_n \longrightarrow \vartheta$ at any continuous point of ϑ.

Definition 6. Let $\vartheta = (\kappa_r)$ be a sequence of natural numbers with $\kappa_0 = 0$, $0 < \kappa_r < \kappa_{r+1}$ and $h_r = \kappa_r - \kappa_{r-1} \longrightarrow \infty$ as $r \longrightarrow \infty$. Then, ϑ is called a lacunary sequence.

The intervals computed by ϑ are abbreviated by $\mathfrak{J}_r = (\kappa_{r-1}, \kappa_r]$, and the quotient κ_r/κ_{r-1} will be symbolized by q_r (see [12]).

Definition 7 see [13]. A sequence space $\mathfrak{E}$ is said to be solid or normal if $(\varsigma_j) \in \mathfrak{E}$ implies $(\beta_j \nu_j) \in \mathfrak{E}$ for all sequence of scalars (β_j) with $|b_j| \leq 1$ for all $j \in \mathbb{N}$.

Lemma 8. *If a sequence space is solid, then it is monotone.*

In [14], the spaces $\mathfrak{T}(\Delta)$ were studied and are defined as follows:

$$\mathfrak{T}(\Delta) = \{\nu = (\nu_i) \in \Lambda : (\Delta\nu_i) \in \mathfrak{T}\}, \tag{7}$$

where $\mathfrak{T} \in \{\ell_\infty, c, \mathfrak{C}_0\}$ and $\Delta\nu_i = \nu_i - \nu_{i-1}$ and were studied further in [15–17] and many others.

Next, for integer $s \geq 0$, the author in [18] had studied the following space:

$$\Delta^s(\mathfrak{T}) = \{\nu = (\nu_k) : (\Delta^s\nu) \in \mathfrak{T}\}, \text{for } \mathfrak{T} = \ell_\infty, c \text{ and } \mathfrak{C}_0, \tag{8}$$

where $\Delta^s\nu_i = \Delta^{s-1}\nu_i - \Delta^{s-1}\nu_{i+1}$ for all $i \in \mathbb{N}$.

Also, let $g = (g_j)$ be any fixed sequence of nonzero complex numbers, and then, as in [19], we have

$$\Delta_g^s(\mathfrak{T}) = \left\{\nu = (\nu_j) \in \Lambda : \left(\Delta_g^s\nu_j\right) \in \mathfrak{T}\right\}, \tag{9}$$

where

$$\Delta_g^s\nu_j = \Delta_g^{s-1}\nu_j - \Delta_g^{s-1}\nu_{j+1} = \sum_{\mu=0}^{s} (-1)^\mu \binom{s}{\mu} g_{j+\mu}\nu_{j+\mu} \forall j \in \mathbb{N}. \tag{10}$$

It is shown that the space $\Delta_g^s(\mathfrak{T})$ is Banach under the norm

$$\|\nu\|_\Delta = \sum_{i=1}^{s} |g_i\nu_i| + \left\|\Delta_g^s\nu\right\|_\infty. \tag{11}$$

Many interesting structures towards this space can be searched in [7, 13, 20–23] and many others.

Inspired by this, in this paper, we interact with Δ-operator to uncertain sequences with the combination of lacunary sequences and syntheses some results in this direction.

2. Main Results

Using the concept of Δ-operator, we deal in this section to introduce some new kind of sequences of uncertain variables.

Following the authors [9, 11, 13, 17, 24–27], we introduce the following new spaces:

$$\left[\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s\right]_0 = \left\{ \varsigma = (\varsigma_i): \lim_{j\longrightarrow\infty} \frac{1}{h_j}\sum_{i\in I_j} \left\| \Delta_g^s \varsigma_j(\nu) \right\| = 0 \right\},$$
$$\left[\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s\right]_c = \left\{ \varsigma = (\varsigma_i): \lim_{j\longrightarrow\infty} \frac{1}{h_j}\sum_{i\in I_j} \left\| \Delta_g^s \varsigma_j(\nu) - \sigma(\nu) \right\| = 0 \right\},$$
$$\left[\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s\right]_\infty = \left\{ \varsigma = (\varsigma_i): \lim_{j\longrightarrow\infty} \frac{1}{h_j}\sum_{i\in I_j} \left\| \Delta_g^s \varsigma_j(\nu) \right\| < \infty \right\}, \tag{12}$$

where $\sigma(\nu) \in (\Gamma, \mathfrak{L}, \mathfrak{S})$.

Theorem 9. *The sets* $[\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s]_0$, $[\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s]_c$, *and* $[\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s]_\infty$ *of complex certain sequences are linear.*

Proof. In order to establish the result, we only consider the case of $[\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s]_0$ and the rest will follow on similar lines. So, let $(u_j), (w_j) \in [\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s]_0$, and then,

$$\lim_{j\longrightarrow\infty} \frac{1}{h_j}\sum_{i\in I_j} \left\| \Delta_g^s u_j(\nu) \right\| = 0,$$
$$\lim_{j\longrightarrow\infty} \frac{1}{h_j}\sum_{i\in I_j} \left\| \Delta_g^s w_j(\nu) \right\| = 0. \tag{13}$$

Now, for any $a, b \in \mathbb{C}$, we have

$$\lim_{j\longrightarrow\infty} \frac{1}{h_j}\sum_{i\in I_j} \left\| \Delta_g^s \left(a u_j(\nu) + b u_j(\nu)\right) \right\| = \lim_{j\longrightarrow\infty} \frac{1}{h_j}\sum_{i\in I_j} \left\| a\Delta_g^s u_j(\nu) + b\Delta_g^s u_j(\nu) \right\| \le |a| \lim_{j\longrightarrow\infty} \frac{1}{h_j}\sum_{i\in I_j} \left\| \Delta_g^s u_j(\nu) \right\| + |b| \lim_{j\longrightarrow\infty} \frac{1}{h_j}\sum_{i\in I_j} \left\| \Delta_g^s w_j(\nu) \right\| \longrightarrow 0, \text{ as } j \longrightarrow \infty. \tag{14}$$

Consequently, $(a u_j(\nu) + b u_j(\nu)) \in [\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s]_0$ and the result follows. □

We state the following result without proof.

Theorem 10. *The sets* $[\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s]_0$, $[\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s]_c$, *and* $[\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s]_\infty$ *of complex certain sequences are normed linear spaces with norm*

$$\|\varsigma(\nu)\|_{\Delta_g^s} = \sum_{k=1}^{n} \|g_k \varsigma_k(\nu)\| + \sup_j \frac{1}{h_j}\sum_{i\in I_j} \left\| \Delta_g^s \varsigma_j(\nu) \right\|. \tag{15}$$

Theorem 11. *For* $\mathfrak{T} = 0, c, \infty$ *and* $s \ge 1$, *we have*

$$\left[\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^{s-1}\right]_{\mathfrak{T}} \subset \left[\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^{s}\right]_{\mathfrak{T}}. \tag{16}$$

The inclusions are sharp.

Proof. In order to establish the result, we only consider the case of $[\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s]_0$ and the rest will follow on similar lines. So, let $\{\varsigma_j\} \in [\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^{s-1}]_0$, and then, we see

$$\lim_{j\longrightarrow\infty} \frac{1}{h_j}\sum_{i\in I_j} \left\| \Delta_g^{s-1} \varsigma_i(\nu) \right\| = 0. \tag{17}$$

We can write

$$\frac{1}{h_j}\sum_{i\in I_j} \left\| \Delta_g^{s} \varsigma_i(\nu) \right\| = \frac{1}{h_j}\sum_{i\in I_j} \left\| \Delta_g^{s-1} \varsigma_i(\nu) - \Delta_g^{s-1} \varsigma_{i+1}(\nu) \right\| \le \left(\frac{1}{h_j}\sum_{i\in I_j} \left\| \Delta_g^{s-1} \varsigma_i(\nu) \right\| - \frac{1}{h_j}\sum_{i\in I_j} \left\| \Delta_g^{s-1} \varsigma_i(\nu) \right\| \right). \tag{18}$$

Using (1) and letting $j \longrightarrow \infty$ in the above inequality, we see

$$\frac{1}{h_j}\sum_{i\in I_j} \left\| \Delta_g^{s} \varsigma_i(\nu) \right\| = 0. \tag{19}$$

Consequently, $\{\varsigma_j\} \in [\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s]_0$.

Now to prove the sharpness, we consider the lacunary sequence $\theta = (2^i)$ and choose the sequence of uncertain variables as $(\varsigma_i) = (i_{s-1})$ and $g_i = 1$ for all $i \in \mathbb{N}$. Then, clearly for all $i \in \mathbb{N}$, we see

$$\Delta_g^s(\varsigma_i) = 0,$$
$$\Delta_g^s \varsigma_i = \sum_{r=0}^{s-1} (-1)^r \binom{s-1}{r} \varsigma_{i+r}. \tag{20}$$

Hence, $(\varsigma_i) \in [\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s]_0$ but not $[\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^{s-1}]_0$, as desired. □

Theorem 12. *For* $\mathfrak{T} = 0, c, \infty$ *and* $s \ge 1$, *the spaces of uncertain sequences* $[\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s]_{\mathfrak{T}}$ *are not symmetric in general.*

Proof. In order to establish the result, we only consider the case of $[\mathfrak{N}_{\theta}^{\mathfrak{U}}, \Delta_g^s]_0$ and the rest will follow on similar lines by choosing $s = 2$ and $g_i = 1$ for all $i \in \mathbb{N}$ and choose a lacunary sequence $\theta = (2^i)$. Choose the uncertainty space $(\Gamma, \mathfrak{L}, \mathfrak{M})$ to be $\{\tau_1, \tau_2, \cdots\}$ with the power set and choose any event $\nu \in \mathfrak{L}$ such that

$$\mathfrak{M}\{\nu\} = \begin{cases} \sup_{\tau_i \in \nu} \frac{i}{2i+1}, & \text{if } \sup_{\tau_i \in \nu} \frac{i}{2i+1} < 0.5, \\ 1 - \sup_{\tau_i \in \nu^c} \frac{i}{2i+1}, & \text{if } \sup_{\tau_i \in \nu^c} \frac{i}{2i+1} < 0.5, \\ 0.5, & \text{if elsewhere.} \end{cases} \tag{21}$$

Further, we define

$$\varsigma_i(\tau_j) = \begin{cases} i, & \text{if } i = j, \\ 0, & \text{if elsewhere.} \end{cases} \tag{22}$$

Then, it is easy to see that $\{\varsigma_j\}$ for $j \in I_j$ and $j = 1, 2, \cdots$ is in $[\mathfrak{N}_\theta^{\mathfrak{U}}, \Delta_g^s]_0$. We now define the rearrangement of $\{\varsigma_j\}$ as $\{\omega_i\}$ defined by

$$\omega_i = (\tau) = \{\varsigma_1, \varsigma_4, \varsigma_9, \varsigma_2, \varsigma_1 0, \cdots\}, \tag{23}$$

which is obviously not in $[\mathfrak{N}_\theta^{\mathfrak{U}}, \Delta_g^s]_0$. This shows that $[\mathfrak{N}_\theta^{\mathfrak{U}}, \Delta_g^s]_0$ is not symmetric in general, as desired. □

In a similar way, we state the following theorem without proof.

Theorem 13. *For $\mathfrak{T} = 0, c, \infty$ and $s \geq 1$, the spaces of uncertain sequences $[\mathfrak{N}_\theta^{\mathfrak{U}}, \Delta_g^s]_{\mathfrak{T}}$ are not monotone in general.*

3. Lacunary Convergence with respect to Mean

In this section, we introduce the convergence notion of Δ-operator of uncertain sequences and compute some relation concerning them.

In this regard, we have following definitions.

Definition 14. An uncertain sequence $\{\varsigma_j\}$ is said to be lacunary strongly convergent almost surely to ς w.r.t. difference sequence if for $\varepsilon > 0$ there exists an event ν with $\mathfrak{M}(\nu) = 1$ such that

$$\lim_{j \longrightarrow \infty} \frac{1}{h_j} \sum_{l \in I_j} \left\| \Delta_g^s \varsigma_l(\kappa) - \sigma(\kappa) \right\| = 0, \tag{24}$$

for every $\kappa \in \nu$.

Definition 15. An uncertain sequence $\{\varsigma_j\}$ is said to be lacunary strongly convergent in measure to ς if

$$\lim_{j \longrightarrow \infty} \mathfrak{M}\left[\left\{\kappa \in \Gamma : \frac{1}{h_j} \sum_{l \in I_j} \left\| \Delta_g^s \varsigma_l(\kappa) - \sigma(\kappa) \right\| > \varepsilon \right\}\right] = 0, \tag{25}$$

for every $\varepsilon > 0$.

Definition 16. Let $\varsigma, \varsigma_1, \varsigma_2, \cdots$ be the uncertainty distributions of uncertain variables. Then, the sequence $\{\varsigma_j\}$ is called convergence in mean to ς if

$$\lim_{j \longrightarrow \infty} E\left[\frac{1}{h_j} \sum_{l \in I_j} \left\| \Delta_g^s \varsigma_l(\kappa) - \sigma(\kappa) \right\| \right] = 0. \tag{26}$$

Definition 17. Let $\vartheta_1, \vartheta_2, \vartheta_3, \cdots$ be uncertain variables with finite expected values $\varsigma, \varsigma_1, \varsigma_2, \cdots$, respectively. Then, the sequence $\{\varsigma_j\}$ is called lacunary strong convergent in distribution to ς w.r.t. difference sequence if

$$\lim_{j \longrightarrow \infty} \frac{1}{h_j} \sum_{l \in I_j} \left\| \Delta_g^s \vartheta_l(\lambda) - \sigma(\lambda) \right\| = 0, \tag{27}$$

for all complex λ at which $\vartheta(\lambda)$ is continuous.

Definition 18. Let $\vartheta_1, \vartheta_2, \vartheta_3, \cdots$ be uncertain variables with finite expected values $\varsigma, \varsigma_1, \varsigma_2, \cdots$, respectively. Then, the sequence $\{\varsigma_j\}$ is called lacunary strong convergent in distribution to ς w.r.t. difference sequence if

$$\lim_{j \longrightarrow \infty} \frac{1}{h_j} \sum_{l \in I_j} \left\| \Delta_g^s \vartheta_l(\lambda) - \sigma(\lambda) \right\| = 0, \tag{28}$$

for all complex λ at which $\vartheta(\lambda)$ is continuous.

Definition 19. Let $\{\vartheta_i\}$ be uncertain sequence and is said to be convergent uniformly almost surely to ς if there exists a sequence of events $\{E_j\}$, $\mathfrak{M}\{E_j\} \longrightarrow 0$ such that $\{\vartheta_i\}$ converges uniformly to ς in $\Gamma - \mathscr{E}_j$ for any fixed $j \in \mathbb{N}$.

Theorem 20. *If the uncertain sequence $\{\varsigma_i\}$ is lacunary strongly convergent in mean to ς w.r.t. difference sequence, then $\{\varsigma_i\}$ lacunary strongly converges in measure to ς.*

Proof. Given $\varepsilon > 0$, we have by Markov's inequality that

$$\begin{aligned} &\lim_{j \longrightarrow \infty} \mathfrak{M}\left[\left\{\kappa \in \Gamma : \frac{1}{h_j} \sum_{i \in I_j} \left\| \Delta_g^s \varsigma_i(\kappa) - \sigma(\kappa) \right\| > \varepsilon \right\}\right] \\ &\leq \lim_{j \longrightarrow \infty} \frac{E\left[(1/h_j) \sum_{i \in I_j} \left\| \Delta_g^s \varsigma_i(\kappa) - \sigma(\kappa) \right\|\right]}{\varepsilon} \longrightarrow 0 \text{ for } i \in I_j. \end{aligned} \tag{29}$$

Thus, $\{\varsigma_i\}$ lacunary strongly converges in measure to L w.r.t. difference sequence.

To prove the converse may not be true, we choose uncertainty space $(\Gamma, \mathfrak{L}, \mathfrak{M})$ to be $\{\tau_1, \tau_2, \cdots\}$ with the power set and choose any event $\nu \in \mathfrak{L}$ such that

$$\mathfrak{M}\{\nu\} = \begin{cases} \sup_{\tau_i \in \nu} \frac{1}{i}, & \text{if } \sup_{\tau_i \in \nu} \frac{1}{i} < 0.5, \\ 1 - \sup_{\tau_i \in \nu^c} \frac{1}{i}, & \text{if } \sup_{\tau_i \in \nu^c} \frac{1}{i} < 0.5, \\ 0.5, & \text{if elsewhere.} \end{cases} \tag{30}$$

Also, set the uncertain variables as

$$\varsigma_i(\tau_r) = \begin{cases} i, & \text{if } r = i, \\ 0, & \text{if elsewhere,} \end{cases} \tag{31}$$

for all $i \in I_j$ and $\varsigma \equiv 0$. Now for $\varepsilon >$, we see

$$\begin{aligned} &\lim_{j \longrightarrow \infty} \mathfrak{M}\left[\left\{\kappa \in \Gamma : \frac{1}{h_j}\sum_{l \in I_j}\left\|\Delta_g^s \varsigma_l(\kappa) - \sigma(\kappa)\right\| > \varepsilon\right\}\right] \\ &= \lim_{j \longrightarrow \infty} \mathfrak{M}\left[\left\{\kappa \in \Gamma : \frac{1}{h_j}\sum_{l \in I_j}\left\|\Delta_g^s \varsigma_l(\kappa)\right\| > \varepsilon\right\}\right] \\ &= \lim_{j \longrightarrow \infty} \mathfrak{M}(\{\kappa_i\}) = \lim_{j \longrightarrow \infty} \frac{1}{i} \longrightarrow 0 \text{ as } i \in I_j. \end{aligned} \tag{32}$$

The sequence $\{\varsigma_r\}$ lacunary strongly converges in measure to ς. However, for each $r \in I_j$, we have the uncertainty distribution of uncertain variable $\|\varsigma_r - \varsigma\| = \|\varsigma_r\|$ that is

$$\vartheta_r(\tau) = \begin{cases} 0, & \text{if } \tau < 0, \\ 1 - \frac{1}{r}, & \text{if } 0 \le \tau < r, \\ 1, & \text{if elsewhere,} \end{cases} \tag{33}$$

$$\begin{aligned} &E\left[\frac{1}{h_j}\sum_{r \in I_j}\left\|\Delta_g^s \varsigma_r(\kappa) - \sigma(\kappa)\right\|\right] = \int_0^{+\infty} \mathfrak{M}\{\varsigma(\tau)\}d\tau - \int_{-\infty}^0 \mathfrak{M}\{\varsigma \le \tau\}d\tau \\ &= \int_0^r 1 - \left(1 - \frac{1}{r}\right)d\tau = 1. \end{aligned} \tag{34}$$

Consequently, $\{\varsigma_r(\tau)$ does not converse in mean to $\varsigma(\tau)$ w.r.t. difference sequence. □

Data Availability

There is no data used in this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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