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Research Article

Degenerate r-Bell Polynomials Arising from Degenerate Normal Ordering

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Recently, Kim-Kim introduced the degenerate r-Bell polynomials and investigated some results which are derived from umbral calculus. The aim of this paper is to study some properties of the degenerate r-Bell polynomials and numbers via boson operators. In particular, we obtain two expressions for the generating function of the degenerate r-Bell polynomials in $|z|^2$, and a recurrence relation and Dobinski-like formula for the degenerate r-Bell numbers. These are derived from the degenerate normal ordering of a degenerate integral power of the number operator in terms of boson operators where the degenerate r-Stirling numbers of the second kind appear as the coefficients.

1. Introduction and Preliminaries

It turns out that it is fascinating and fruitful to study various degenerate versions of some special polynomials and numbers (see [1–8] and the references therein), which has its origin in the work of Carlitz [9]. They have been explored by using various different methods and led to the introduction of degenerate gamma functions and degenerate umbral calculus (see [8, 10]).

The aim of this paper is to study some properties of the degenerate r-Bell polynomials and numbers via boson operators. In particular, we obtain two expressions for the generating function of the degenerate r-Bell polynomials in $|z|^2$, and a recurrence relation and Dobinski-like formula for the degenerate r-Bell numbers. These are derived from the degenerate normal orderings of a degenerate integral power of the number operator in terms of boson operators where the degenerate r-Stirling numbers of the second kind appear as the coefficients.

In more detail, the outline of this paper is as follows: in Section 1, we recall the degenerate exponentials, the degenerate r-Stirling numbers of the second kind, and the degenerate r-Bell polynomials. We remind the reader of the boson operators, the number operators, the normal ordering of an integral power of the number operator in terms of boson operators, and the degenerate normal ordering of a degenerate integral power of the number operator in terms of boson operators. Section 2 is the main result of this paper. In Theorem 1, we state that the degenerate normal orderings of the 'degenerate integral powers of the number operator' $(\hat{n}+r)_{n,\lambda}$ and $(\hat{n})_{n-r,\lambda}(a^{\dagger})^{r}a^{r}$ in terms of the boson operators with the coefficients given by the degenerate r-Stirling numbers of the second kind. Let $f(t)=\langle z|e_{\lambda}^{r+\hat{n}}(t)|z\rangle$. We show that $\langle z|(\hat{n}+r)_{n,\lambda}|z\rangle$ is equal to $\phi_{k,\lambda}^{(r)}(|z|^2)$ in Theorem 2 and hence that $f(t)$ is the generating function of that in Theorem 3, where $\phi_{k,\lambda}^{(r)}(x)$ is the degenerate r-Bell polynomial. We derive a differential equation for $f(t)$ in Theorem 4 and thereby get another expression for $f(t)$ in Theorem 5, which in turn yields an explicit generating function of $\phi_{k,\lambda}^{(r)}(|z|^2)$. In Theorem 6, we obtain a recurrence relation for the degenerate r-Bell numbers $\phi_{n,\lambda}^{(r)}=\phi_{n,\lambda}^{(r)}(1)$. Finally, we get a Dobinski-like formula for the degenerate r-Bell numbers from the representation of the coherent state in terms of the number operator in Theorem 8. We conclude the paper in Section 3. For the rest of this section, we recall the facts that are needed throughout this paper.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponentials are defined by

$$e_\lambda^x(t) = (1+\lambda t)^{(x/\lambda)} = \sum_{k=0}^{\infty} (x)_{k,\lambda} \frac{t^k}{k!}, e_\lambda(t) = e_\lambda^1(t), \quad (1)$$

where (see [1–5, 8, 9])

$$(x)_{0,\lambda} = 1, (x)_{k,\lambda} = x(x-\lambda)\cdots(x-(k-1)\lambda), (k \geq 1). \quad (2)$$

For $r \in \mathbb{Z}$ with $r \geq 0$, the degenerate r-Stirling numbers of the second kind are defined by Kim-Kim as (see [6–8]),

$$(x+r)_{n,\lambda} = \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} (x)_k, (n \geq 0), \quad (3)$$

where $(x)_0 = 1, (x)_n = x(x-1)\cdots(x-n+1), (n \geq 1)$.

From equation (3), we note that (see [7, 8])

$$\frac{1}{k!}(e_\lambda(t)-1)^k e_\lambda^r(t) = \sum_{n=k}^{\infty} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} \frac{t^n}{n!}, (k \geq 0). \quad (4)$$

Note that $\lim_{\lambda \longrightarrow 0} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} = \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r$ are the r-Stirling numbers of the second kind. It is well known that the r-Stirling number of the second kind $\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r$ counts the number of partitions of the set $[n] = \{1, 2, \cdots, n\}$ into k nonempty disjoint subsets in such a way that the numbers $1, 2, 3, \cdots, r$ are in distinct subsets.

The degenerate r-Bell polynomials $\phi_{n,\lambda}^{(r)}(x)$ are defined by

$$e_\lambda^r(t) e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \phi_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}. \quad (5)$$

From equations (4) and (5), we note that (see [8])

$$\phi_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} x^k, (n \geq 0). \quad (6)$$

The boson operators $a^\dagger$ and a satisfy the following commutation relation: (see [11–13])

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1. \quad (7)$$

The number states $|m\rangle, m = 0, 1, 2, \cdots,$ are defined as (see [3–5, 11])

$$a|m\rangle = \sqrt{m}|m-1\rangle, a^\dagger|m\rangle = \sqrt{m+1}|m+1\rangle. \quad (8)$$

The number operator is defined by (see [3–5, 12])

$$\hat{n}|k\rangle = k|k\rangle, (k \geq 0). \quad (9)$$

By equations (8) and (9), we get $\hat{n} = a^\dagger a$.

Thus, we note

$$[a, \hat{n}] = a\hat{n} - \hat{n}a = a, [\hat{n}, a^\dagger] = \hat{n}a^\dagger - a^\dagger\hat{n} = a^\dagger. \quad (10)$$

The normal ordering of an integral power of the number operator $\hat{n} = a^\dagger a$ in terms of boson operators a and $a^\dagger$ can be written in the form (see [3, 4, 11–13])

$$(a^\dagger a)^k = \sum_{l=0}^{k} S_2(k,l)(a^\dagger)^l a^l, \quad (11)$$

where $S_2(k,l)$ are the Stirling numbers of the second kind defined by $x^n = \sum_{k=0}^n S_2(n,k)(x)_k$, (see [10, 14, 15]).

The degenerate normal ordering of a degenerate integral power of the number operator $\hat{n}$ in terms of boson operators a and $a^\dagger$ is given by (see [3–5])

$$(a^\dagger a)_{k,\lambda} = \sum_{l=0}^{k} S_{2,\lambda}(k,l)(a^\dagger)^l a^l, \quad (12)$$

where $S_{2,\lambda}(n,k)$ are the degenerate Stirling numbers of the second kind defined by (see [1])

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k)(x)_k, (n \geq 0). \quad (13)$$

2. Degenerate r-Bell Polynomials Arising from Degenerate Normal Ordering

We recall that the coherent states $|z\rangle, z \in \mathbb{C}$, satisfy $a|z\rangle = z|z\rangle, \langle z|z\rangle = 1$. Note that $\langle z|a^\dagger = \langle z|\bar{z}$. For the coherent state $|z\rangle$, we have (see [3–5, 11, 13])

$$|z\rangle = e^{(-(|z|^2/2))} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle. \quad (14)$$

By equation (14), we get

$$\langle x|y\rangle = e^{(-(1/2))(|x|^2+|y|^2)+\bar{x}y}, (x, y \in \mathbb{C}). \quad (15)$$

It is easy to show that

$$\frac{d}{dx} x = 1 + x\frac{d}{dx}. \quad (16)$$

We recall that the standard boson commutation relation $[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$ can be considered, in a suitable space of functions f, by letting $a = (d/dx)$ and $a^\dagger = x$.

Now we observe that

$$\left(x\frac{d}{dx} + r\right)_{n,\lambda} f(x) = \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} x^k \left(\frac{d}{dx}\right)^k f(x), \quad (17)$$

$$\left(x\frac{d}{dx}\right)_{n,\lambda} x^r f(x) = \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} x^{k+r} \left(\frac{d}{dx}\right)^k f(x), \quad (18)$$

where $n, r \in \mathbb{Z}$ with $n, r \geq 0$.

The equations (17) and (18) can be represented, respectively, by the degenerate normal orderings of the degenerate n-th powers in equations (19) and (20) of the number operator $\widehat{n} = a^\dagger$ in terms of boson operators a and $a^\dagger$.

$$(\widehat{n} + r)_{n,\lambda} = \left(a^\dagger a + r\right)_{n,\lambda} = \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} \left(a^\dagger\right)^k a^k, \quad (19)$$

$$(\widehat{n})_{n,\lambda} \left(a^\dagger\right)^r = \left(a^\dagger a\right)_{n,\lambda} \left(a^\dagger\right)^r = \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} \left(a^\dagger\right)^{k+r} a^k. \quad (20)$$

Therefore, from equations (19) and (20), we obtain the following theorem:

Theorem 1. *For $n \geq 0$, we have*

$$(\widehat{n} + r)_{n,\lambda} = \left(a^\dagger a + r\right)_{n,\lambda} = \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} \left(a^\dagger\right)^k a^k, \quad (21)$$

and, for $n \geq r$, we have

$$(\widehat{n})_{n-r,\lambda} \left(a^\dagger\right)^r a^r = \left(a^\dagger a\right)_{n-r,\lambda} \left(a^\dagger\right)^r a^r = \sum_{k=r}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{r,\lambda} \left(a^\dagger\right)^k a^k. \quad (22)$$

Let $m = 0, 1, 2, \cdots$. Then, by equation (8), we get

$$\left(a^\dagger a + r\right)_{n,\lambda} |m\rangle = (\widehat{n} + r)_{n,\lambda} |m\rangle = (m + r)_{n,\lambda} |m\rangle, \quad (23)$$

$$\begin{aligned} (\widehat{n} + r)_{n,\lambda} |m\rangle &= \left(a^\dagger a + r\right)_{n,\lambda} |m\rangle \\ &= \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} \left(a^\dagger\right)^k a^k |m\rangle \\ &= \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} (m)_k |m\rangle. \end{aligned} \quad (24)$$

Thus, by equations (23) and (24), we get

$$(m + r)_{n,\lambda} = \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} (m)_k, \, (n \geq 0). \quad (25)$$

This is the classical expression for the degenerate n-th power of $m + r$ in terms of the falling factorial $(m)_k$. This shows that equation (3) holds for all nonnegative integers $x = m = 0, 1, 2, \cdots$, which in turn implies equation (3) itself holds true.

From equation (25), we note that

$$\begin{aligned} \langle z|(\widehat{n} + r)_{n,\lambda}|z\rangle &= \langle z|\left(a^\dagger a + r\right)_{n,\lambda}|z\rangle \\ &= \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} \langle z|\left(a^\dagger\right)^k a^k|z\rangle \\ &= \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} (\overline{z})^k z^k \langle z|z\rangle \\ &= \sum_{k=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_{r,\lambda} |z|^{2k} = \phi_{n,\lambda}^{(r)}\left(|z|^2\right). \end{aligned} \quad (26)$$

Therefore, by equation (26), we obtain the following theorem:

Theorem 2. *For $n \geq 0$, we have*

$$\langle z|(\widehat{n} + r)_{n,\lambda}|z\rangle = \langle z|\left(a^\dagger a + r\right)_{n,\lambda}|z\rangle = \phi_{n,\lambda}^{(r)}\left(|z|^2\right). \quad (27)$$

Let us take $f(t) = \langle z|e_\lambda^{r+\widehat{n}}(t)|z\rangle$. Then, by equation (26), we get

$$\begin{aligned} f(t) = \langle z|e_\lambda^{r+\widehat{n}}(t)|z\rangle &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \langle z|(\widehat{n} + r)_{k,\lambda}|z\rangle \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \langle z|\left(a^\dagger a + r\right)_{k,\lambda}|z\rangle \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{l=0}^{k} \left\{ \begin{matrix} k+r \\ l+r \end{matrix} \right\}_{r,\lambda} |z|^{2k} = \sum_{k=0}^{\infty} \phi_{k,\lambda}^{(r)}\left(|z|^2\right) \frac{t^k}{k!}. \end{aligned} \quad (28)$$

Indeed, equation (28) says that $f(t)$ is the generating function of the degenerate r-Bell polynomials which are considered by Kim-Kim.

Therefore, by equation (28), we obtain the following theorem:

Theorem 3. *The generating function of degenerate r-Bell polynomials is given by*

$$\langle z|e_\lambda^{r+\widehat{n}}(t)|z\rangle = \langle z|e_\lambda^{r+a^\dagger a}(t)|z\rangle = \sum_{n=0}^{\infty} \phi_{n,\lambda}^{(r)}\left(|z|^2\right) \frac{t^n}{n!}. \quad (29)$$

Now, we observe that

$$\begin{aligned}
&\hat{n}(\hat{n}+r-\lambda)_{k,\lambda}\\
&\quad= a^{\dagger}a\left(a^{\dagger}a+r-\lambda\right)_{k,\lambda}\\
&\quad= a^{\dagger}a\left(a^{\dagger}a+r-\lambda\right)\left(a^{\dagger}a+r-2\lambda\right)\cdots\left(a^{\dagger}a+r-k\lambda\right)\\
&\quad= a^{\dagger}\left(aa^{\dagger}+r-\lambda\right)a\left(a^{\dagger}a+r-2\lambda\right)\cdots\left(a^{\dagger}a+r-k\lambda\right)\\
&\quad= a^{\dagger}\left(a^{\dagger}a+1+r-\lambda\right)a\left(a^{\dagger}a+r-2\lambda\right)\cdots\left(a^{\dagger}a+r-k\lambda\right)\\
&\quad= \cdots = a^{\dagger}\left(a^{\dagger}a+1+r-\lambda\right)\\
&\qquad\cdot\left(a^{\dagger}a+1+r-2\lambda\right)\cdots\left(a^{\dagger}a+1+r-k\lambda\right)a\\
&\quad= a^{\dagger}\left(a^{\dagger}a+1+r-\lambda\right)_{k,\lambda}a\\
&\quad= a^{\dagger}(\hat{n}+1+r-\lambda)_{k,\lambda}a,\ (k\geq 0).
\end{aligned} \tag{30}$$

From equation (30), we note that

$$\hat{n}e_{\lambda}^{\hat{n}+r-\lambda}(t) = e_{\lambda}^{\hat{n}+r-\lambda}(t)\hat{n} = a^{\dagger}e_{\lambda}^{aa^{\dagger}+r-\lambda}(t)a = a^{\dagger}e_{\lambda}^{\hat{n}+1+r-\lambda}(t)a. \tag{31}$$

By equations (28) and (31), we get

$$\begin{aligned}
\frac{\partial f(t)}{\partial t} &= \frac{\partial}{\partial t}\langle z|e_{\lambda}^{a^{\dagger}a+r}(t)|z\rangle = \langle z|(\hat{n}+r)e_{\lambda}^{\hat{n}+r-\lambda}(t)|z\rangle\\
&= \langle z|a^{\dagger}e_{\lambda}^{a^{\dagger}a+r+1-\lambda}(t)a|z\rangle + r\langle z|e_{\lambda}^{a^{\dagger}a+r-\lambda}(t)|z\rangle\\
&= e_{\lambda}^{1-\lambda}(t)\bar{z}z\langle z|e_{\lambda}^{a^{\dagger}a+r}(t)|z\rangle + r\langle z|e_{\lambda}^{a^{\dagger}a+r}(t)|z\rangle e_{\lambda}^{-\lambda}(t)\\
&= e_{\lambda}^{1-\lambda}(t)|z|^2 f(t) + re_{\lambda}^{-\lambda}(t)f(t).
\end{aligned} \tag{32}$$

Therefore, by equation (32), we obtain the following theorem:

Theorem 4. *Let $f(t) = \langle z|e_{\lambda}^{r+\hat{n}}(t)|z\rangle$. Then we have*

$$\frac{f'(t)}{f(t)} = |z|^2 e_{\lambda}^{1-\lambda}(t) + re_{\lambda}^{-\lambda}(t). \tag{33}$$

Note that $f(0) = 1$. From Theorem 4, we have

$$\begin{aligned}
\log f(t) &= \int_0^t \frac{f'(t)}{f(t)}dt\\
&= |z|^2\int_0^t e_{\lambda}^{1-\lambda}(t)dt + r\int_0^t e_{\lambda}^{-\lambda}(t)dt\\
&= |z|^2(e_{\lambda}(t)-1) + r\frac{1}{\lambda}\log(1+\lambda t).
\end{aligned} \tag{34}$$

Thus, by equation (34), we get

$$f(t) = e^{|z|^2(e_{\lambda}(t)-1)+r(1/\lambda)\log(1+\lambda t)} = e_{\lambda}^{r}(t)e^{|z|^2(e_{\lambda}(t)-1)}. \tag{35}$$

Therefore, by equation (35), we obtain the following theorem:

Theorem 5. *Let $f(t) = \langle z|e_{\lambda}^{r+\hat{n}}(t)|z\rangle = \langle z|e_{\lambda}^{r+a^{\dagger}a}|z\rangle$. Then we have*

$$f(t) = e_{\lambda}^{r}(t)e^{|z|^2(e_{\lambda}(t)-1)}. \tag{36}$$

From Theorems 3 and 5, we have

$$e_{\lambda}^{r}(t)e^{|z|^2(e_{\lambda}(t)-1)} = \sum_{n=0}^{\infty}\phi_{n,\lambda}^{(r)}\left(|z|^2\right)\frac{t^n}{n!}. \tag{37}$$

That is, by equation (35), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\langle z|(r+\hat{n})_{n,\lambda}|z\rangle\frac{t^n}{n!} &= \langle z|e_{\lambda}^{r+\hat{n}}(t)|z\rangle = f(t)\\
&= \sum_{k=0}^{\infty}|z|^{2k}\frac{1}{k!}(e_{\lambda}(t)-1)^k e_{\lambda}^{r}(t)\\
&= \sum_{k=0}^{\infty}|z|^{2k}\sum_{n=k}^{\infty}\left\{\begin{matrix} n+r\\ k+r\end{matrix}\right\}_{r,\lambda}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}|z|^{2k}\left\{\begin{matrix} n+r\\ k+r\end{matrix}\right\}_{r,\lambda}\right)\frac{t^n}{n!}.
\end{aligned} \tag{38}$$

Comparing the coefficients on both sides of equation (38), we have

$$\langle z|(r+\hat{n})_{n,\lambda}|z\rangle = \langle z|\left(r+a^{\dagger}a\right)_{n,\lambda}|z\rangle = \sum_{k=0}^{n}\left\{\begin{matrix} n+r\\ k+r\end{matrix}\right\}_{r,\lambda}|z|^{2k}. \tag{39}$$

By Theorems 3 and 5, we get

$$f(t) = \langle z|e_{\lambda}^{r+a^{\dagger}a}(t)|z\rangle = \sum_{n=0}^{\infty}\phi_{n,\lambda}^{(r)}\left(|z|^2\right)\frac{t^n}{n!}. \tag{40}$$

Differentiating equation (40) with respect to t, we have

$$\frac{\partial f(t)}{\partial t} = \sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!}\phi_{k,\lambda}^{(r)}\left(|z|^2\right) = \sum_{k=0}^{\infty}\phi_{k+1,\lambda}^{(r)}\left(|z|^2\right)\frac{t^k}{k!}. \tag{41}$$

On the other hand, by equation (32), we get

$$\begin{aligned}\frac{\partial f(t)}{\partial t} &= e_\lambda^{1-\lambda}(t)|z|^2 f(t) + re_\lambda^{-\lambda}(t)f(t)\\ &= e_\lambda^{1-\lambda}(t)|z|^2\sum_{k=0}^{\infty}\frac{t^k}{k!}\phi_{k,\lambda}^{(r)}\left(|z|^2\right) + \mathrm{r}e_\lambda^{-\lambda}(t)\sum_{k=0}^{\infty}\phi_{k,\lambda}^{(r)}\left(|z|^2\right)\frac{t^k}{k!}\\ &= |z|^2\sum_{l=0}^{\infty}(1-\lambda)_{l,\lambda}\frac{t^l}{l!}\sum_{k=0}^{\infty}\frac{t^k}{k!}\phi_{k,\lambda}^{(r)}\left(|z|^2\right)\\ &\quad + r\sum_{l=0}^{\infty}(-\lambda)_{l,\lambda}\frac{t^l}{l!}\sum_{k=0}^{\infty}\phi_{k,\lambda}^{(r)}\left(|z|^2\right)\frac{t^k}{k!}\\ &= \sum_{n=0}^{\infty}\left(|z|^2\sum_{k=0}^{n}\binom{n}{k}(1-\lambda)_{n-k,\lambda}\phi_{k,\lambda}^{(r)}\left(|z|^2\right)\right.\\ &\quad\left. + r\sum_{k=0}^{n}\binom{n}{k}(-\lambda)_{n-k,\lambda}\phi_{k,\lambda}^{(r)}\left(|z|^2\right)\right)\frac{t^n}{n!}.\end{aligned} \tag{42}$$

Therefore, by equations (41) and (42), we obtain the following theorem:

Theorem 6. *For $n \geq 0$, we have*

$$\phi_{n+1,\lambda}^{(r)}\left(|z|^2\right) = \sum_{k=0}^{n}\binom{n}{k}\phi_{k,\lambda}^{(r)}\left(|z|^2\right)\left(|z|^2(1-\lambda)_{n-k,\lambda} + r(-\lambda)_{n-k,\lambda}\right). \tag{43}$$

When $|z| = 1$ in Theorem 6, we have

$$\phi_{n+1,\lambda}^{(r)} = \sum_{k=0}^{n}\binom{n}{k}\phi_{k,\lambda}^{(r)}\left((1-\lambda)_{n-k,\lambda} + r(-\lambda)_{n-k,\lambda}\right), \tag{44}$$

where $\phi_{n,\lambda}^{(r)}(1) = \phi_{n,\lambda}^{(r)}$ are called the degenerate r-Bell numbers.

Remark 7. We can determine the degenerate r-Bell numbers by using the recurrence relation in the above Theorem 6. The first few of them are as follows:

$$\begin{aligned}\phi_{0,\lambda}^{(r)} &= 1,\\ \phi_{1,\lambda}^{(r)} &= 1 + r,\\ \phi_{2,\lambda}^{(r)} &= 2 + 2r + r^2 - (1+r)\lambda,\\ \phi_{3,\lambda}^{(r)} &= 5 + r(6 + r(3+r)) - 6\lambda - 3r(2+r)\lambda + 2(1+r)\lambda^2,\\ \phi_{4,\lambda}^{(r)} &= 15 + r(20 + r(12 + r(4+r))) - 30\lambda - 6r(6 + r(3+r))\lambda\\ &\quad + 11(2 + r(2+r))\lambda^2 - 6(1+r)\lambda^3,\\ \phi_{5,\lambda}^{(r)} &= 52 + r(75 + r(50 + r(20 + r(5+r)))) - 150\lambda\\ &\quad - 10r(20 + r(12 + r(4+r)))\lambda\\ &\quad + 35(5 + r(6 + r(3+r)))\lambda^2 - 50(2 + r(2+r))\lambda^3\\ &\quad + 24(1+r)\lambda^4,\\ \phi_{6,\lambda}^{(r)} &= 203 + r(312 + r(225 + r(100 + r(30 + r(6+r)))))\\ &\quad - 780\lambda - 15r(75 + r(50 + r(20 + r(5+r))))\lambda\\ &\quad + 85(15 + r(20 + r(12 + r(4+r))))\lambda^2\\ &\quad - 225(5 + r(6 + r(3+r)))\lambda^3 + 274(2 + r(2+r))\lambda^4\\ &\quad - 120(1+r)\lambda^5.\end{aligned} \tag{45}$$

Evaluating the left hand side of Theorem 2 by using the representation of the coherent state in terms of the number operator in equation (14), we have

$$\begin{aligned}&\left\langle z\middle|\left(a^\dagger a + r\right)_{k,\lambda}\middle|z\right\rangle\\ &= e^{\left(-\left(|z|^2/2\right)\right)}e^{\left(-\left(|z|^2/2\right)\right)}\sum_{m,n=0}^{\infty}\frac{|\bar{z}|^m z^n}{\sqrt{m!}\sqrt{n!}}(n+r)_{k,\lambda}\langle m|n\rangle\\ &= e^{-|z|^2}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!}(n+r)_{k,\lambda}.\end{aligned} \tag{46}$$

By Theorem 2 and equation (46), we get

$$\begin{aligned}\phi_{k,\lambda}^{(r)}\left(|z|^2\right) &= \langle \mathrm{z}|\left(a^\dagger a + r\right)_{k,\lambda}|\mathrm{z}\rangle = \sum_{l=0}^{k}|z|^{2l}\left\{\begin{matrix}k+r\\ l+r\end{matrix}\right\}_{r,\lambda}\\ &= e^{-|z|^2}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!}(n+r)_{k,\lambda}, (k \geq 0).\end{aligned} \tag{47}$$

In particular, by letting $|z| = 1$, we get the Dobinski-like formula.

Theorem 8. *The following Dobinski-like formula holds true:*

$$\phi_{k,\lambda}^{(r)} = \frac{1}{e}\sum_{n=0}^{\infty}\frac{1}{n!}(n+r)_{k,\lambda}, (k, r \in \mathbb{N} \cup \{0\}). \tag{48}$$

3. Conclusion

In recent years, various degenerate versions of many special numbers and polynomials have been explored by employing diverse tools such as generating functions, combinatorial methods, umbral calculus, p-adic analysis, differential equations, probability theory, operator theory, and analytic number theory.

In this paper, we studied the degenerate r-Bell polynomials and numbers via boson operators in quantum physics. To do that, we first recalled the degenerate normal ordering of a degenerate integral power of the number operator in terms of boson operators. Here the degenerate r-Stirling numbers of the second kind appear as the coefficients. From the degenerate normal ordering, we derived two expressions for the generating function of the degenerate r-Bell polynomials in $|z|^2$, and a recurrence relation and Dobinski-like formula for the degenerate r-Bell numbers.

It is one of our future research projects to continue to study many different versions of some special numbers and polynomials and to find their applications in physics, science and engineering as well as in mathematics.

Data Availability

No data were used to support this paper.

Disclosure

An earlier version of the paper has been presented as preprint in the following link: https://arxiv.org/pdf/2208.05465.pdf.

Conflicts of Interest

All authors declare no conflict of interest.

Authors' Contributions

TK and DSK conceived of the framework and structured the whole paper; DSK and TK wrote the whole paper. HKK checked the results of the paper and typed the paper. HKK paid for the charge of publication fee of this paper. All authors read and agreed with the published version of the manuscript.

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