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Research Article

On q-Convex Functions Defined by the q-Ruscheweyh Derivative Operator in Conic Regions

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The core objective of this article is to introduce and investigate a new class $\beta - \mathrm{UCV}_q^{\lambda}[A, B]$ of convex functions associated with the conic domain defined by the Ruscheweyh q-differential operator. Many interesting properties such as sufficiency criteria, coefficient bounds, partial sums, and radius of convexity of order α for the functions of the said class are investigated here.

1. Introduction

Quantum calculus has emerged as one of the most vibrant areas of research in recent years. Researchers have discussed and found its applications in numerous dimensions, such as hypergeometric series, complex analysis, and applied physics. It has developed techniques to be used in q-calculus, time scales, partitions, and continued fractions. Jackson, for the first time, in the beginning of the 20th century, introduced quantum calculus, where he developed and standardized it. For more details about quantum calculus, see [1–14]. To make a good pace and understanding of the results presented in this article, we are going to give below some primary definitions and relevant details of quantum calculus. Suppose $\mathcal{F}$ represents the class of holomorphic functions of type

$$y(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{1}$$

in open unit disk $\mathcal{E} = \{z\colon z \in \mathbb{C} \text{ and } |z| < 1\}$ and normalized by the conditions $y'(0) = 1$ and $y(0) = 0$. Moreover, $\mathcal{S}$ represents the class of all functions in $\mathcal{F}$ which are univalent in $\mathcal{E}$; see [15].

A domain $\mathcal{D}$ is starlike with respect to a point $z_0 \in \mathcal{D}$ if all possible lines which are confined by two points, connecting z_0 to any other point, lie entirely within $\mathcal{D}$. Correspondingly, a domain $\mathcal{E}$ is convex if all possible lines which are obtained by connecting any two points in $\mathcal{D}$ lie thoroughly within $\mathcal{D}$. More clearly, we can say that if the domain is starlike with respect to each of its points in $\mathcal{D}$, then it is convex. If $y(\mathcal{E})$ is starlike for $y \in \mathcal{S}$ with respect to the origin, then it is called a starlike function, whereas if $y(\mathcal{E})$ is convex, then it is called a convex function. The class of all convex functions is represented by C, and the class of all starlike functions is represented by S^*. Analytically, these are defined as follows:

$$\begin{aligned} S^*\colon &= \left\{ y \in \mathcal{S}\colon \Re\left\{\frac{zy'(z)}{y(z)}\right\} > 0, \quad z \in \mathcal{E} \right\}, \\ C\colon &= \left\{ y \in \mathcal{S}\colon \Re\left\{1 + \frac{zy''(z)}{y'(z)}\right\} > 0, \quad z \in \mathcal{E} \right\}. \end{aligned} \tag{2}$$

For $\alpha \in [0,1)$, suppose that $S^*(\alpha)$ and $C(\alpha)$ are subclasses of $\mathcal{S}$ consisting of α-starlike functions and α-convex functions, respectively, defined analytically as follows:

$$\begin{aligned} S^*(\alpha) &:= \left\{ y \in \mathcal{S}: \Re\left\{\frac{zy'(z)}{y(z)}\right\} > \alpha, \quad z \in \mathscr{E} \right\}, \\ C(\alpha) &:= \left\{ y \in \mathcal{S}: \Re\left\{1 + \frac{zy''(z)}{y'(z)}\right\} > \alpha, \quad z \in \mathscr{E} \right\}. \end{aligned} \tag{3}$$

For $\alpha = 0$, the class $S^*(\alpha) \Rightarrow S^*$ and the class $C(\alpha) \Rightarrow C$. Moreover, the following two classes are closely related with their functions defined, respectively.

$$\begin{aligned} S^*_\alpha &:= \left\{ y \in \mathcal{S}: \left|\frac{zy'(z)}{y(z)} - 1\right| < 1 - \alpha, \quad z \in \mathscr{E} \right\}, \\ C_\alpha &:= \left\{ y \in \mathcal{S}: \left|\frac{zy''(z)}{y'(z)}\right| < 1 - \alpha, \quad z \in \mathscr{E} \right\}. \end{aligned} \tag{4}$$

Note that $S^*_\alpha \subseteq S^*(\alpha)$ and $C_\alpha \subseteq C(\alpha)$. The k^{th} partial sum of the function y, denoted by y_k, is the polynomial, defined by

$$y_k(z) = z + \sum_{n=2}^{k} b_n z^n. \tag{5}$$

Generally, lower bounds on ratios such as $\Re\{y(z)/y_k(z)\}$ or $\Re\{y_k(z)/y(z)\}$ have been found to be sharp only when $k = 1$, but Silverman determined sharpness $\forall n \in \mathbb{N}$; see [16, 17]. He investigated that lower bounds are strictly increasing functions of k. In the present article, by using Silverman's technique [16], we will find the function's ratio having Taylor series (1) to its sequence of partial sums $y_k(z) = z + \sum_{n=2}^{k} b_n z^n$ when the coefficients of y are sufficiently small to fulfill the necessary and sufficient condition. In more details to clarify, we will find sharp lower bounds for $y(z)/y_k(z)$, $y'(z)/y'_k(z)$, $y_k(z)/y(z)$, and $y'_k(z)/y'(z)$. Indeed, we will use the familiar result, i.e., $\Re\{w(z) - 1/w(z) + 1\} > 0$, $z \in \mathscr{E}$, if and only if $w(z) = \sum_{n=1}^{\infty} c_n z^n$ satisfies $|w(z)| \leq |z|$. Unless otherwise stated, we will presume that y has form (1) and that its sequence of partial sums is represented by (5).

For $\alpha \in [0,1)$, Ravichandran gave the sharp radius of starlike and convex functions of order α with form (1) whose Taylor series coefficients b_n satisfy the conditions $|b_2| = 2\,d$, $d \in [0,1]$, and $|b_n| \leq n$; M or M/n $(M > 0)$ for $n \geq 3$.

Consider that y_1 and y_2 are holomorphic functions in $\mathscr{E}$ with $w(0) = 0$ and $|w(z) \leq 1|$, $\forall z \in \mathscr{E}$, so that $y_1(z) = y_2(w(z))$; y_1 will be subordinated by y_2 and denoted by $y_1 \prec y_2$. If y_2 is holomorphic, then $y_1 \prec y_2$ iff $y_1(0) = y_2(0)$ and $y_1(\mathscr{E}) \subseteq y_2(\mathscr{E})$.

For two holomorphic functions

$$y_1(z) = \sum_{k=0}^{\infty} a_k z^k \text{ and } y_2(z) = \sum_{k=0}^{\infty} b_k z^k \ (z \in \mathrm{E}), \tag{6}$$

the Hadamard product of $y_1(z)$ and $y_2(z)$ is defined as

$$y_1(z) * y_2(z) = \sum_{k=0}^{\infty} a_k b_k z^k. \tag{7}$$

We will define some notations and concepts of quantum calculus which are to be used in this article. All results can be found in [2, 3, 18]. For $n \in \mathbb{N}$, $0 < q < 1$, we see the classical q-theory begins with the q-extension of the positive numbers. The expression

$$\lim_{q \longrightarrow 1} \frac{1 - q^n}{1 - q} = n. \tag{8}$$

proposes that we define the q-generalization of n, which is also called the q-bracket of n, given as

$$[n, q] = [n]_q = \frac{1 - q^n}{1 - q}, \tag{9}$$

and the q-generalization of the factorial which is called q-factorial given by

$$[n]_q! = \begin{cases} [n]_q [n-1]_q \ldots [1]_q, & n = 1, 2, \ldots, \\ 1, & n = 0. \end{cases} \tag{10}$$

The q-difference operator for $y \in \mathscr{F}$ is defined as

$$\partial_q y(z) = \frac{y(qz) - y(z)}{z(q-1)}, \quad (z \in \mathrm{E}), \tag{11}$$

and we can see that, for $n \in \mathbb{N}$ and $z \in \mathrm{E}$,

$$\begin{aligned} \partial_q z^n &= [n]_q z^{n-1}, \\ \partial_q \left\{ \sum_{n-1}^{\infty} b_n z^n \right\} &= \sum_{n-1}^{\infty} [n]_q b_n z^{n-1}. \end{aligned} \tag{12}$$

For $y(z) \in \mathscr{F}$, the q-analogue of the Ruscheweyh differential operator is defined as

$$\begin{aligned} R_q^\lambda y(z) &= \varphi(q, \lambda + 1; z) * y(z) \\ &= z + \sum_{n=2}^{\infty} \psi_{n-1} b_n z^n, \ (z \in \mathrm{E} \text{ and } \lambda > -1), \end{aligned} \tag{13}$$

where

$$\varphi(q, \lambda + 1; z) = z + \sum_{n=2}^{\infty} \psi_{n-1} z^n, \tag{14}$$

and

$$\psi_{n-1} = \frac{\Gamma_q(\lambda + n)}{[n-1]_q! \Gamma_q(\lambda + 1)} = \frac{[\lambda + 1, q]_{n-1}}{[n-1]_q!}, \quad (\psi_0 = 1), \tag{15}$$

where $[\lambda + 1, q]_{n-1}$ is a Pochhammer symbol, which is defined as follows:

$$[n,q]_m = \begin{cases} 1, & n = 0, \\ [n,q][n+1,q][n+2,q][n+3,q]\ldots[m+n-1,q], & n \in \mathbb{N}. \end{cases} \quad (16)$$

From (13), it is clear that

$$\begin{aligned} &R_q^0 y(z) = y(z) \text{ and } R_q^1 y(z) = z\partial_q y(z), \\ &R_q^m y(z) = \frac{z\partial_q^m\left(z^{m-1}y(z)\right)}{[m]_q!}, \quad (m \in \mathbb{N}), \\ &\lim_{q\longrightarrow 1^-} \varphi(q,\lambda+1;z) = \frac{z}{(1-z)^{\lambda+1}}, \\ &\lim_{q\longrightarrow 1^-} R_q^\lambda y(z) = y(z) * \frac{z}{(1-z)^{\lambda+1}}. \end{aligned} \quad (17)$$

It follows that $q \longrightarrow 1^-$, and the Ruscheweyh q-differential operator converts into the Ruscheweyh differential operator $D^\delta(y(z))$; for more details, see [19]. Using (13),

$$z\partial R_q^\lambda y(z) = \left(1 + \frac{[\lambda]_q}{q^\lambda}\right) R_q^{\lambda+1} y(z) - \frac{[\lambda]_q}{q^\lambda} R_q^\lambda y(z). \quad (18)$$

If $q \longrightarrow 1^-$, then

$$z\left(R^\lambda y(z)\right)' = (1+\lambda) R^{\lambda+1} y(z) - \lambda R^\lambda y(z). \quad (19)$$

Definition 1. The function $p(z)$ will lie in the class $\beta - P_q[A,B]$ if and only if

$$p(z) \prec \frac{(A(1+q)+(3-q))\widetilde{p}_\beta(z) - (A(1+q)-(3-q))}{(B(1+q)+(3-q))\widetilde{p}_\beta(z) - (B(1+q)-(3-q))}, \quad \beta \geq 0, \quad (20)$$

where

$$\widetilde{p}_\beta(z) = \begin{cases} \dfrac{1+z}{1-z}, & \beta = 0, \\ 1 + \dfrac{2}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & \beta = 1, \\ 1 + \dfrac{2}{1-\beta^2}\sinh^2\left[\left(\dfrac{2}{\pi}\arccos\beta\right)\arctan\, y\sqrt{z}\right], & 0 < \beta < 1, \\ 1 + \dfrac{1}{\beta^2-1}\sin\left(\dfrac{\pi}{2R(n)}\displaystyle\int_0^{u(z)/\sqrt{t}} \dfrac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}\,dx\right) + \dfrac{1}{\beta^2-1}, & \beta > 1. \end{cases} \quad (21)$$

For more details, see [20–24]. If $\widetilde{p}_\beta(z) = 1 + \delta_\beta z + \cdots$, then it is shown in [25] that, from (46), one can have

$$\delta_\beta = \begin{cases} \dfrac{8(\arccos\beta)^2}{\pi^2(1-\beta^2)}, & 0 \le \beta < 1, \\[2ex] \dfrac{8}{\pi^2}, & \beta = 1, \\[2ex] \dfrac{\pi^2}{4(\beta^2-1)\sqrt{t}\,(1+t)R^2(t)}, & \beta > 1. \end{cases} \tag{22}$$

Definition 2. A function $y(z) \in \mathcal{F}$ will lie in the class $\beta - \mathrm{UCV}_q[A,B]$, $\beta \ge 0$, $-1 \le B < A \le 1$, if and only if

$$\Re\left[\frac{(B(1+q)-(3-q))D_q(zD_q y(z))/D_q y(z) - (A(1+q)-(3-q))}{(B(1+q)+(3-q))D_q(zD_q y(z))/D_q y(z) - (A(1+q)+(3-q))}\right] > \beta\left|\frac{(B(1+q)-(3-q))D_q(zD_q y(z))/D_q y(z) - (A(1+q)-(3-q))}{(B(1+q)+(3-q))D_q(zD_q y(z))/D_q y(z) - (A(1+q)+(3-q))} - 1\right|, \tag{23}$$

or equivalently,

$$\frac{D_q(zD_q y(z))}{D_q y(z)} \in \beta - P_q[A,B]. \tag{24}$$

For more details about the above classes and conic domain, we refer the readers to [20, 25–28]. Using the q-Ruscheweyh differential operator, we now define the following more general class $\beta - \mathrm{UCV}_q^\lambda[A,B]$ of functions associated with the conic domain defined by Janowski functions.

Definition 3. A function $y(z) \in \mathcal{F}$ will lie in the class $\beta - \mathrm{UCV}_q^\lambda[A,B]$, $\beta \ge 0, -1 \le B < A \le 1$, if and only if

$$\Re\left[\frac{(B(1+q)-(3-q))\partial_q(z\partial_q R_q^\lambda y(z))/\partial_q R_q^\lambda y(z) - (A(1+q)-(3-q))}{(B(1+q)+(3-q))\partial_q(z\partial_q R_q^\lambda y(z))/\partial_q R_q^\lambda y(z) - (A(1+q)+(3-q))}\right] > \beta\left|\frac{(B(1+q)-(3-q))\partial_q(z\partial_q R_q^\lambda y(z))/\partial_q R_q^\lambda y(z) - (A(1+q)-(3-q))}{(B(1+q)+(3-q))\partial_q(z\partial_q R_q^\lambda y(z))/\partial_q R_q^\lambda y(z) - (A(1+q)+(3-q))} - 1\right|, \tag{25}$$

or equivalently,

$$\frac{\partial_q(z\partial_q R_q^\lambda y(z))}{\partial_q R_q^\lambda y(z)} \in \beta - P[A,B]. \tag{26}$$

The above defined class $\beta - \mathrm{UCV}_q^\lambda[A,B]$ generalizes many known classes which can be obtained by setting suitable particular values to the parameters as follows.

Special cases:

(1) $\beta - \mathrm{UCV}_{1^-}^0[A,B] = \beta - \mathrm{UCV}[A,B]$, the well-known class of β-uniformly Janowski convex functions, introduced by Noor and Malik [27]

(2) $0 - \mathrm{UCV}_{1^-}^0[A,B] = C[A,B]$, the well-known class of Janowski convex functions, introduced by Janowski [20]

(3) $\beta - \mathrm{UCV}_{1^-}^0[1-2\alpha,-1] = \mathrm{KD}(\beta,\alpha)$, see [29]

(4) $0 - \mathrm{UCV}_{1^-}^0[1-2\alpha,-1] = C(\alpha)$, see [15]

Lemma 1 (see [30]). *Let $g(z) = 1 + \sum_{n=1}^\infty c_n z^n$ be subordinate to $G(z) = 1 + \sum_{n=1}^\infty C_n z^n$. If $G(z)$ is holomorphic in $\mathscr{E}$ and $G(\mathscr{E})$ is convex, then*

$$|c_n| \le |C_1|, \quad n \ge 1. \tag{27}$$

2. Main Results

Theorem 1. *A function $y(z) \in \mathcal{F}$ with form (1) will lie in class $\beta - \mathrm{UCV}_q^\lambda[A,B]$, $\beta \ge 0$, $-1 \le B < A \le 1$, if it satisfies the condition*

$$\sum_{n=2}^\infty \frac{\mathbb{E}_n}{\varepsilon}|b_n| < 1, \tag{28}$$

where

$$\mathbb{E}_n = [n]_q \left\{ 2(3-q)(\beta+1) q [n-1]_q + \left| (B(1+q)+(3-q))[n]_q - (A(1+q)+(3-q)) \right| \right\} \psi_{n-1}, \tag{29}$$

and

$$\varepsilon = (1+q)|B-A|. \tag{30}$$

Proof. Suppose that (28) holds; then, it is enough to show that

$$\beta \left| \frac{(B(1+q)-(3-q))\partial_q\left(z\partial_q R_q^\lambda y(z)\right)/\partial_q R_q^\lambda y(z) - (A(1+q)-(3-q))}{(B(1+q)+(3-q))\partial_q\left(z\partial_q R_q^\lambda y(z)\right)/\partial_q R_q^\lambda y(z) - (A(1+q)+(3-q))} - 1 \right| - \Re\left[\frac{(B(1+q)-(3-q))\partial_q\left(z\partial_q R_q^\lambda y(z)\right)/\partial_q R_q^\lambda y(z) - (A(1+q)-(3-q))}{(B(1+q)+(3-q))\partial_q\left(z\partial_q R_q^\lambda y(z)\right)/\partial_q R_q^\lambda y(z) - (A(1+q)+(3-q))} - 1 \right] < 1. \tag{31}$$

We consider

$$\begin{aligned}
&\beta \left| \frac{(B(1+q)-(3-q))\partial_q\left(z\partial_q R_q^\lambda y(z)\right)/\partial_q R_q^\lambda y(z) - (A(1+q)-(3-q))}{(B(1+q)+(3-q))\partial_q\left(z\partial_q R_q^\lambda y(z)\right)/\partial_q R_q^\lambda y(z) - (A(1+q)+(3-q))} - 1 \right| \\
&\quad - \Re\left[\frac{(B(1+q)-(3-q))\partial_q\left(z\partial_q R_q^\lambda y(z)\right)/\partial_q R_q^\lambda y(z) - (A(1+q)-(3-q))}{(B(1+q)+(3-q))\partial_q\left(z\partial_q R_q^\lambda y(z)\right)/\partial_q R_q^\lambda y(z) - (A(1+q)+(3-q))} - 1 \right] \\
&\le (\beta+1) \left| \frac{(B(1+q)-(3-q))\partial_q\left(z\partial_q R_q^\lambda y(z)\right) - (A(1+q)-(3-q))\partial_q R_q^\lambda y(z)}{(B(1+q)+(3-q))\partial_q\left(z\partial_q R_q^\lambda y(z)\right) - (A(1+q)+(3-q))\partial_q R_q^\lambda y(z)} - 1 \right| \\
&= 2(3-q)(\beta+1) \left| \frac{\partial_q R_q^\lambda y(z) - \partial_q\left(z\partial_q R_q^\lambda y(z)\right)}{(B(1+q)+(3-q))\partial_q\left(z\partial_q R_q^\lambda y(z)\right) - (A(1+q)+(3-q))\partial_q R_q^\lambda y(z)} \right| \\
&= 2(3-q)(\beta+1) \left| \frac{\sum_{n=2}^{\infty}\left(1-[n]_q\right)\psi_{n-1}[n]_q b_n z^n}{z(B-A)(1+q) + \sum_{n=2}^{\infty}\begin{pmatrix} (B(1+q)+(3-q))[n]_q \\ -(A(1+q)+(3-q)) \end{pmatrix}\psi_{n-1}[n]_q b_n z^n} \right| \\
&\le \frac{2(3-q)(\beta+1)\sum_{n=2}^{\infty} q[n-1]_q [n]_q |b_n|}{(1+q)|B-A| 1/\psi_{n-1} - \sum_{n=2}^{\infty}\left|(B(1+q)+(3-q))[n]_q - (A(1+q)+(3-q))\right| [n]_q |b_n|}.
\end{aligned} \tag{32}$$

The last expression is bounded above by 1 if

$$2(3-q)(\beta+1)\sum_{n=2}^{\infty} q[n-1]_q[n]_q|b_n| < (1+q)|B-A|\frac{1}{\psi_{n-1}} - \sum_{n=2}^{\infty}\left|(B(1+q)+(3-q))[n]_q - (A(1+q)+(3-q))\right|[n]_q|b_n|, \tag{33}$$

which reduces to

$$\sum_{n=2}^{\infty}[n]_q\left\{2(3-q)(\beta+1)q[n-1]_q + \left|\begin{matrix}(B(1+q)+(3-q))[n]_q \\ -(A(1+q)+(3-q))\end{matrix}\right|\right\}\psi_{n-1}|b_n| < (1+q)|B-A|. \tag{34}$$

This finalizes the proof. □ □

For $q \longrightarrow 1^-$ and $\lambda = 0$, we have the following known result, proved in [27].

Corollary 1. *A function $y(z) \in \mathcal{F}$ with form (1) will lie in class $\beta - UCV[A,B]$, $\beta \geq 0$, $-1 \leq B < A \leq 1$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} n\{2(\beta+1)(n-1) + |n(B+1)-(A+1)|\}|b_n| < |B-A|. \tag{35}$$

For $q \longrightarrow 1^-$, $\lambda = 0$, and $A = 1-2\alpha$ and $B = -1$, we have the following known result, proved in [29].

Corollary 2. *A function $y(z) \in \mathcal{F}$ with form (1) will lie in class $KD(\beta,\alpha)$, $\beta \geq 0$, $0 \leq \alpha < 1$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} n\{n(\beta+1)-(\beta+\alpha)\}|b_n| < (1-\alpha). \tag{36}$$

Theorem 2. *Let $y(z) \in \beta - UCV_q^{\lambda}[A,B]$, $\beta \geq 0$, $-1 \leq B < A \leq 1$, and be of form (1); then, for $n \geq 2$,*

$$|b_n| \leq \frac{1}{[n]_q}\prod_{j=0}^{n-2}\frac{|(A-B)(q+1)\delta_\beta\psi_j - 4Bq[j]_q\psi_j|}{4q[j+1]_q\psi_{j+1}}, \tag{37}$$

where ψ is defined by (15).

Proof. By the definition for $y(z) \in \beta - \mathrm{UCV}_q^{\lambda}[A,B]$, we have

$$\frac{\partial_q\left(z\partial_q R_q^{\lambda} y(z)\right)}{\partial_q R_q^{\lambda} y(z)} = p(z), \tag{38}$$

where

$$p(z) \prec \frac{(A(1+q)+(3-q))\widetilde{p}_\beta(z) - (A(1+q)-(3-q))}{(B(1+q)+(3-q))\widetilde{p}_\beta(z) - (B(1+q)-(3-q))}. \tag{39}$$

If $\widetilde{p}_\beta(z) = 1 + \delta_\beta z + \ldots$, then

$$\frac{(A(1+q)+(3-q))\widetilde{p}_\beta(z) - (A(1+q)-(3-q))}{(B(1+q)+(3-q))\widetilde{p}_\beta(z) - (B(1+q)-(3-q))} = 1 + \frac{1}{4}(A-B)(q+1)\delta_\beta + \frac{1}{4}\left[\left(-\frac{1}{4}Aq - \frac{1}{4}A + \frac{1}{4}Bq + \frac{1}{4}B\right)((B+1)(1+q)+2-2q)\right]\delta_\beta^2 + \cdots. \tag{40}$$

Now, if $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, then by (27) and (39), we have

$$|c_n| \leq \frac{1}{4}(A-B)(q+1)|\delta_\beta|, \quad n \geq 1. \tag{41}$$

Now, from (38), we have

$$\partial_q\left(z\partial_q R_q^{\lambda} y(z)\right) = p(z)\partial_q R_q^{\lambda} y(z). \tag{42}$$

Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, and using the Cauchy product formula, we obtain

$$\begin{aligned}1+\sum_{n=2}^{\infty}[n]_q[n]_q\psi_{n-1}b_nz^{n-1} &= \left(1+\sum_{n=1}^{\infty}c_nz^n\right)\left(1+\sum_{n=2}^{\infty}[n]_q\psi_{n-1}b_nz^{n-1}\right)\\ \sum_{n=2}^{\infty}[n]_q[n]_q\psi_{n-1}b_nz^{n-1} &= \sum_{n=2}^{\infty}c_{n-1}z^{n-1}+\sum_{n=2}^{\infty}[n]_q\psi_{n-1}b_nz^{n-1}+\left(\sum_{n=2}^{\infty}[n]_q\psi_{n-1}b_nz^{n-1}\right).\left(\sum_{n=2}^{\infty}c_{n-1}z^{n-1}\right).\end{aligned} \tag{43}$$

This implies that

$$\sum_{n=2}^{\infty}\left([n]_q-1\right)[n]_q\psi_{n-1}b_nz^{n-1}=\left(\sum_{n=2}^{\infty}c_{n-1}+\sum_{n=2}^{\infty}\sum_{j=2}^{n}[j]_q\psi_{j-1}b_jc_{n-j}\right)z^{n-1}. \tag{44}$$

Comparison of coefficients of z^{n-1} gives us

$$\left([n]_q-1\right)[n]_q\psi_{n-1}b_n=\left(c_{n-1}+\sum_{j=2}^{n}[j,q]\psi_{j-1}b_jc_{n-j}\right)(b_1=1), \tag{45}$$

or

$$b_n=\frac{1}{\left([n]_q-1\right)[n]_q\psi_{n-1}}\left(c_{n-1}+\sum_{j=2}^{n}[j]_q\psi_{j-1}b_jc_{n-j}\right). \tag{46}$$

Using (41), we have

$$\left|b_n\right|\leq\frac{(A-B)(q+1)\left|\delta_\beta\right|}{4q[n-1]_q[n]_q\psi_{n-1}}\left(1+\sum_{j=2}^{n-1}[j]_q\psi_{j-1}\left|b_j\right|\right), \tag{47}$$

or

$$\left|b_n\right|\leq\frac{(A-B)(q+1)\left|\delta_\beta\right|}{4q[n-1]_q[n]_q\psi_{n-1}}\left(\sum_{j=1}^{n-1}[j]_q\psi_{j-1}\left|b_j\right|\right). \tag{48}$$

Now, we prove that

$$\frac{(A-B)(q+1)\left|\delta_\beta\right|}{4q[n-1]_q[n]_q\psi_{n-1}}\left(\sum_{j=1}^{n-1}[j]_q\psi_{j-1}\left|b_j\right|\right)\leq\frac{1}{[n]_q}\prod_{j=0}^{n-2}\frac{\left|(A-B)(q+1)\delta_\beta\psi_j-4Bq[n]_q\psi_j\right|}{4q[j+1]_q\psi_{j+1}}. \tag{49}$$

For this, we use the induction technique. For $n=2$, we have from (46),

$$\left|b_2\right|\leq\frac{\left|\delta_\beta\right|(q+1)(A-B)}{4q[1]_q[2]_q\psi_1}\sum_{j=1}^{2-1}[j]_q\psi_{j-1}\left|b_j\right|, \tag{50}$$

or

$$\left|b_2\right|\leq\frac{\left|\delta_\beta\right|(A-B)(q+1)}{4q[1]_q[2]_q\psi_1},\quad \psi_0=1. \tag{51}$$

For $n=3$, we have from (46),

$$\begin{aligned}\left|b_3\right| &\leq\frac{\left|\delta_\beta\right|(A-B)(q+1)}{4q[2]_q[3]_q\psi_2}\sum_{j=1}^{2}[j]_q\psi_{j-1}\left|b_j\right|\\ &=\frac{\left|\delta_\beta\right|(A-B)(q+1)}{4q[2]_q[3]_q\psi_2}\left([1]_q\psi_0\left|b_1\right|+[2]_q\psi_1\left|b_2\right|\right)\\ &\leq\frac{\left|\delta_\beta\right|(A-B)(q+1)}{4q[2]_q[3]_q\psi_2}\left(1+\frac{(A-B)(q+1)\left|\delta_\beta\right|}{4q[1]_q}\right).\end{aligned} \tag{52}$$

From (37), we have

$$\begin{aligned}\left|b_3\right| &\leq \frac{1}{[3]_q}\prod_{j=0}^{1}\frac{\left|(A-B)(q+1)\delta_\beta\psi_j - 4Bq[j]_q\psi_j\right|}{4q[j+1]_q\psi_{j+1}}\\ &= \frac{1}{[3]_q}\frac{(A-B)(q+1)\left|\delta_\beta\right|}{4q[1]_q\psi_1}\left(\frac{(A-B)(q+1)\left|\delta_\beta\right|\psi_1 + 4q[1]_q\psi_1}{4q[2]_q\psi_2}\right)\\ &= \frac{\left|\delta_\beta\right|(A-B)(q+1)}{4q[2]_q[3]_q\psi_2}\left(1+\frac{(A-B)(q+1)\left|\delta_\beta\right|}{4q[1]_q}\right).\end{aligned} \tag{53}$$

Let the assumption be true for $n = m+1$. From (46), we have

$$\left|b_m\right| \leq \frac{\left|\delta_\beta\right|(A-B)(q+1)}{4q[m-1]_q[m]_q\psi_{m-1}}\left(\sum_{j=1}^{m-1}[j]_q\psi_{j-1}\left|b_j\right|\right). \tag{54}$$

From (37), we have

$$\left|b_m\right| \leq \frac{1}{[m]_q}\prod_{j=0}^{m-2}\frac{\left|(A-B)(q+1)\delta_\beta\psi_j - 4Bq[j]_q\psi_j\right|}{4q[j+1]_q\psi_{j+1}}. \tag{55}$$

By the induction hypothesis,

$$\frac{1}{[m]_q}\prod_{j=0}^{m-2}\frac{\left|(A-B)(q+1)\delta_\beta\psi_j - 4Bq[j]_q\psi_j\right|}{4q[j+1]_q\psi_{j+1}} \geq \frac{\left|\delta_\beta\right|(A-B)(q+1)}{4q[m-1]_q[m]_q\psi_{m-1}}\sum_{j=1}^{m-1}[j]_q\psi_{j-1}\left|b_j\right|. \tag{56}$$

Multiplying both sides by $1/[m]_q(A-B)(q+1)|\delta_\beta|\,\psi_{m-1} + 4q[m-1]_q\psi_{m-1}/4q[m]_q\psi_m$, we have

$$\begin{aligned}&\frac{1}{[m]_q}\prod_{j=0}^{m-2}\frac{\left|(A-B)(q+1)\delta_\beta\psi_j - 4Bq[j]_q\psi_j\right|}{4q[j+1]_q\psi_{j+1}}\\ &\geq \left(\frac{1}{[m]_q}\frac{(A-B)(q+1)\left|\delta_\beta\right|\psi_{m-1} + 4q[m-1]_q\psi_{m-1}}{4q[m]_q\psi_m}\right)\times\\ &\cdot\left(\frac{\left|\delta_\beta\right|(A-B)(q+1)}{4q[m-1]_q\psi_{m-1}}\sum_{j=1}^{m-1}\psi_{j-1}\left|b_j\right|\right)\\ &= \frac{\left|\delta_\beta\right|(A-B)(q+1)}{4q[m]_q\psi_m}\times\\ &\cdot\left(\psi_{m-1}\frac{\left|\delta_\beta\right|(A-B)(q+1)}{4q[m-1]_q[m]_q\psi_{m-1}}\sum_{j=1}^{m-1}\psi_{j-1}\left|b_j\right| + \frac{1}{[m]_q}\sum_{j=1}^{m-1}\psi_{j-1}\left|b_j\right|\right)\\ &\geq \frac{\left|\delta_\beta\right|(A-B)(q+1)}{4q[m]_q\psi_m}\left(\psi_{m-1}\left|b_m\right| + \frac{1}{[m]_q}\sum_{j=1}^{m-1}\psi_{j-1}\left|b_j\right|\right)\\ &= \frac{\left|\delta_\beta\right|(A-B)(q+1)}{4q[m]_q[m]_q\psi_m}\sum_{j=1}^{m}\psi_{j-1}\left|b_j\right|.\end{aligned} \tag{57}$$

That is,

$$\frac{\left|\delta_{\beta}\right|(A-B)(q+1)}{4q[m-1]_q[m]_q\psi_{m-1}}\sum_{j=1}^{m-1}\psi_{j-1}\left|b_j\right|\leq\frac{1}{[m]_q}\prod_{j=0}^{m-2}\frac{\left|(A-B)(q+1)\delta_{\beta}\psi_j-4Bq[j]_q\psi_j\right|}{4q[j+1]_q\psi_{j+1}}. \tag{58}$$

Hence, the consequence is true for $n=m+1$. Therefore, using mathematical induction, we have proved that (37) is true $\forall n$, $n\geq 2$. □ □

For $q\longrightarrow 1^-$ and $\lambda=0$, we have the following known result, proved in [27].

Corollary 3. *Let $y(z)\in\beta-UCV[A,B]$, $\beta\geq 0$, $-1\leq B<A\leq 1$, and be of form (1); then, for $n\geq 2$,*

$$\left|b_n\right|\leq\frac{1}{n}\prod_{j=0}^{n-2}\frac{\left|(A-B)\delta_{\beta}-2Bj\right|}{2(j+1)}. \tag{59}$$

For $q\longrightarrow 1^-$, $\lambda=0$, and $A=1-2\alpha$ and $B=-1$, we have the following known result, proved in [29].

Corollary 4. *Let $y(z)\in KD(\beta,\alpha)$, $\beta\geq 0$, $0\leq\alpha<1$, and be of form (1); then, for $n\geq 2$,*

$$\left|b_n\right|\leq\frac{1}{n}\prod_{j=0}^{n-2}\frac{\left|(1-\alpha)\delta_{\beta}+j\right|}{(j+1)}. \tag{60}$$

Using the already proven results of Silverman [16] and Silvia [17] on partial sums of holomorphic functions, we will find the fraction of (1) to its sequence of partial sums $y_k(z)=z+\sum_{n=2}^{k}b_nz^n$ when the function $y(z)$ has coefficients small enough to satisfy condition (28). We will investigate sharp lower bounds for $\Re\{y(z)/y_k(z)\}$, $\Re\{y'(z)/y_k'(z)\}$, $\Re\{y_k(z)/y(z)\}$, and $\Re\{y_k'(z)/y'(z)\}$ in the class $\beta-\mathrm{UCV}_q^{\lambda}[A,B]$.

Theorem 3. *If $y(z)\in\beta-UCV_q^{\lambda}[A,B]$, then*

$$\Re\left\{\frac{y(z)}{y_k(z)}\right\}\geq 1-\frac{\varepsilon}{\mathbb{E}_{k+1}}, \tag{61}$$

where $\mathbb{E}_{k+1}$ is defined by (29) and $\varepsilon=(1+q)|B-A|$. The extremal function

$$y(z)=z+\frac{\varepsilon}{\mathbb{E}_{k+1}}z^{k+1}. \tag{62}$$

gives the sharp result.

Proof. Define a function $w(z)$:

$$w(z)=\frac{\mathbb{E}_{k+1}}{\varepsilon}.\left[\frac{y(z)}{y_k(z)}-\left(1-\frac{\varepsilon}{\mathbb{E}_{k+1}}\right)\right], \tag{63}$$

and this will reduce to

$$\begin{aligned}&=\frac{\mathbb{E}_{k+1}\left(1+\sum_{n=2}^{\infty}b_nz^{n-1}\right)}{\varepsilon\left(1+\sum_{n=2}^{k}b_nz^{n-1}\right)}-\frac{\mathbb{E}_{k+1}}{\varepsilon}+1\\ w(z)&=\frac{1+\sum_{n=2}^{k}b_nz^{n-1}+\mathbb{E}_{k+1}/\varepsilon\sum_{n=k+1}^{\infty}b_nz^{n-1}}{1+\sum_{n=2}^{k}b_nz^{n-1}}.\end{aligned} \tag{64}$$

We have

$$\left|\frac{w(z)-1}{w(z)+1}\right|\leq\frac{\mathbb{E}_{k+1}/\varepsilon\sum_{n=k+1}^{\infty}\left|b_n\right|}{2-2\sum_{n=2}^{k}\left|b_n\right|-\mathbb{E}_{k+1}/\varepsilon\sum_{n=k+1}^{\infty}\left|b_n\right|}. \tag{65}$$

Now,

$$\left|\frac{w(z)-1}{w(z)+1}\right|\leq 1 \tag{66}$$

if

$$\sum_{n=2}^{k}\left|b_n\right|+\frac{\mathbb{E}_{k+1}}{\varepsilon}\sum_{n=k+1}^{\infty}\left|b_n\right|\leq 1. \tag{67}$$

It is sufficient to show that the left hand side of (28) is bounded above by $\sum_{n=2}^{\infty}\mathbb{E}_n/\varepsilon|b_n|$ if

$$\sum_{n=2}^{k}\left|b_n\right|+\frac{\mathbb{E}_{k+1}}{\varepsilon}\sum_{n=k+1}^{\infty}\left|b_n\right|\leq\sum_{n=2}^{\infty}\frac{\mathbb{E}_n}{\varepsilon}\left|b_n\right|. \tag{68}$$

This leads to the following expression:

$$\sum_{n=2}^{k}\left(\frac{\mathbb{E}_n-\varepsilon}{\varepsilon}\right)\left|b_n\right|+\left(\frac{\mathbb{E}_n-\mathbb{E}_{k+1}}{\varepsilon}\right)\sum_{n=k+1}^{\infty}\left|b_n\right|\geq 0. \tag{69}$$

To ensure that the function defined by (62) gives the sharp outcome, we note that, for $z=re^{i\pi/n}$,

$$\begin{aligned}\frac{y(z)}{y_k(z)}&=1+\frac{\varepsilon}{\mathbb{E}_{k+1}}z^n\\&=1+\frac{\varepsilon}{\mathbb{E}_{k+1}}r^ne^{\frac{i\pi}{n}}\\&=1+\frac{\varepsilon r^n}{\mathbb{E}_{k+1}}\left(\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)\\&=1-\frac{\varepsilon r^n}{\mathbb{E}_{k+1}}\\ \frac{y(z)}{y_k(z)}&=\frac{\mathbb{E}_{k+1}-\varepsilon}{\mathbb{E}_{k+1}}\text{ when } r\longrightarrow 1.\end{aligned} \tag{70}$$

□

Theorem 4. *If $y(z) \in \beta - UCV_q^{\lambda}[A, B]$, then*

$$\Re\left\{\frac{y_k(z)}{y(z)}\right\} \geq \frac{\mathbb{E}_{k+1}}{\mathbb{E}_{k+1} + \varepsilon}, \tag{71}$$

where $\mathbb{E}_{k+1}$ is defined by (29) and $\varepsilon = (1+q)|B - A|$. The result (71) is sharp with the function given by (62).

Proof. Define the function $w(z)$:

$$\begin{aligned} w(z) &= \frac{\mathbb{E}_{k+1} + \varepsilon}{\varepsilon} \cdot \left[\frac{y_k(z)}{y(z)} - \frac{\mathbb{E}_{k+1}}{\mathbb{E}_{k+1} + \varepsilon}\right] \\ &= \frac{1 + \sum_{n=2}^{k} b_n z^{n-1} - \mathbb{E}_{k+1}/\varepsilon \sum_{n=k+1}^{\infty} b_n z^{n-1}}{1 + \sum_{n=2}^{\infty} b_n z^{n-1}}. \end{aligned} \tag{72}$$

This will become

$$\begin{aligned} \frac{w(z) - 1}{w(z) + 1} &= \frac{\sum_{n=2}^{k} b_n z^{n-1} - \sum_{n=2}^{\infty} b_n z^{n-1} - \mathbb{E}_{k+1}/\varepsilon \sum_{n=k+1}^{\infty} b_n z^{n-1}}{2 + \sum_{n=2}^{k} b_n z^{n-1} + \sum_{n=2}^{\infty} b_n z^{n-1} - \mathbb{E}_{k+1}/\varepsilon \sum_{n=k+1}^{\infty} |b_n| z^{k-1}} \\ &= \frac{-(1 + \mathbb{E}_{k+1}/\varepsilon) \sum_{n=k+1}^{\infty} b_n z^{n-1}}{2 + 2\sum_{n=2}^{k} b_n z^{n-1} + (1 - \mathbb{E}_{k+1}/\varepsilon) \sum_{n=k+1}^{\infty} |b_n| z^{k-1}}. \end{aligned} \tag{73}$$

This implies that

$$\left|\frac{w(z) - 1}{w(z) + 1}\right| \leq \frac{(1 + \mathbb{E}_{k+1}/\varepsilon) \sum_{n=k+1}^{\infty} |b_n|}{2 - 2\sum_{n=2}^{k} |b_n| - (1 - \mathbb{E}_{k+1}/\varepsilon) \sum_{n=k+1}^{\infty} |b_n|}. \tag{74}$$

Now,

$$\left|\frac{w(z) - 1}{w(z) + 1}\right| \leq 1 \tag{75}$$

if

$$\sum_{n=2}^{k} |b_n| + \sum_{n=k+1}^{\infty} |b_n| \leq 1. \tag{76}$$

It would be enough to show that the left side of (28) is bounded above by $\sum_{n=2}^{\infty} \mathbb{E}_n/\varepsilon |b_n|$ if

$$\sum_{n=2}^{k} |b_n| + \sum_{n=k+1}^{\infty} |b_n| \leq \sum_{n=2}^{\infty} \frac{\mathbb{E}_n}{\varepsilon} |b_n|, \tag{77}$$

which leads to the following expression:

$$\sum_{n=2}^{k} \left(\frac{\mathbb{E}_n}{\varepsilon} - 1\right) |b_n| + \sum_{n=k+1}^{\infty} \left(\frac{\mathbb{E}_n}{\varepsilon} - 1\right) |b_n| \geq 0 \tag{78}$$

or

$$\sum_{n=2}^{\infty} \left(\frac{\mathbb{E}_n}{\varepsilon} - 1\right) |b_n| \geq 0. \tag{79}$$

Consequently, the equality holds for the extreme function $y(z)$ given by (62). □ □

We now turn to fractions related to the derivatives.

Theorem 5. *If $y(z) \in \beta - UCV_q^{\lambda}[A, B]$, then*

$$\Re\left\{\frac{y'(z)}{y_k'(z)}\right\} \geq \frac{\mathbb{E}_{k+1} - \varepsilon(k+1)}{\mathbb{E}_{k+1}}, \tag{80}$$

where $\mathbb{E}_{k+1}$ is defined by (29) and $\varepsilon = (1+q)|B - A|$. The result (80) is sharp with the function given by (62).

Proof. Define the function $w(z)$:

$$\begin{aligned} w(z) &= \frac{\mathbb{E}_{k+1}}{\varepsilon(k+1)} \cdot \left[\frac{y'(z)}{y_k'(z)} - \frac{\mathbb{E}_{k+1} - \varepsilon(k+1)}{\mathbb{E}_{k+1}}\right] \\ &= \frac{\mathbb{E}_{k+1}\left(1 + \sum_{n=2}^{\infty} n b_n z^{n-1}\right)}{\varepsilon(k+1)\left(1 + \sum_{n=2}^{k} n b_n z^{n-1}\right)} - \frac{(\mathbb{E}_{k+1} - \varepsilon(k+1))}{\varepsilon(k+1)}, \end{aligned} \tag{81}$$

and this will reduce to

$$w(z) = \frac{1 + \sum_{n=2}^{k} n b_n z^{n-1} + \mathbb{E}_{k+1}/\varepsilon(k+1) \sum_{n=k+1}^{\infty} n b_n z^{n-1}}{1 + \sum_{n=2}^{k} n b_n z^{n-1}}. \tag{82}$$

Now, we have

$$\frac{w(z) - 1}{w(z) + 1} = \frac{\mathbb{E}_{k+1}/\varepsilon(k+1) \sum_{n=k+1}^{\infty} n b_n z^{n-1}}{2 + 2\sum_{n=2}^{k} n b_n z^{n-1} + \mathbb{E}_{k+1}/\varepsilon(k+1) \sum_{n=k+1}^{\infty} n b_n z^{n-1}}. \tag{83}$$

This implies that

$$\left|\frac{w(z)-1}{w(z)+1}\right| \leq \frac{\mathbb{E}_{k+1}/\varepsilon(k+1)\sum_{n=k+1}^{\infty} n|b_n|}{2-2\sum_{n=2}^{k} n|b_n| - \mathbb{E}_{k+1}/\varepsilon(k+1)\sum_{n=k+1}^{\infty} n|b_n|}. \tag{84}$$

Now,

$$\left|\frac{w(z)-1}{w(z)+1}\right| \leq 1 \tag{85}$$

if

$$\sum_{n=2}^{k} n|b_n| + \frac{\mathbb{E}_{k+1}}{\varepsilon(k+1)} \sum_{k=n+1}^{\infty} n|b_n| \leq 1. \tag{86}$$

It would be enough to show that the left side of (28) is bounded above by $\sum_{n=2}^{\infty} \mathbb{E}_n/\varepsilon|b_n|$ if

$$\sum_{n=2}^{k} n|b_n| + \frac{\mathbb{E}_{k+1}}{\varepsilon(k+1)} \sum_{n=k+1}^{\infty} n|b_n| \leq \sum_{n=2}^{\infty} \frac{\mathbb{E}_k}{\varepsilon}|b_n|, \tag{87}$$

which leads to the following expression:

$$\sum_{n=2}^{k}\left(\frac{\mathbb{E}_n}{\varepsilon} - n\right)|b_n| + \sum_{n=k+1}^{\infty}\left(\frac{\mathbb{E}_n}{\varepsilon} - \frac{n\mathbb{E}_{k+1}}{\varepsilon(k+1)}\right)|b_n| \geq 0. \tag{88}$$

The result (80) is sharp with respect to the function given by (62). □ □

Theorem 6. *If $y(z) \in \beta - UCV_q^{\lambda}[A,B]$, then*

$$\Re\left\{\frac{y_k'(z)}{y'(z)}\right\} \geq \frac{\mathbb{E}_{k+1}}{\varepsilon(k+1)+\mathbb{E}_{k+1}}, \tag{89}$$

where $\mathbb{E}_{k+1}$ is defined by (29) and $\varepsilon = (1+q)|B-A|$. The result (89) is sharp with respect to the function given by (62).

Proof. Define the function $w(z)$:

$$\begin{aligned} w(z) &= \frac{\varepsilon(k+1)+\mathbb{E}_{k+1}}{\varepsilon(k+1)} \cdot \left[\frac{y_k'(z)}{y'(z)} - \frac{\mathbb{E}_{k+1}}{\varepsilon(k+1)+\mathbb{E}_{k+1}}\right] \\ &= \frac{\left(\varepsilon(k+1)+\mathbb{E}_{k+1}\right)\left(1+\sum_{n=2}^{k} n b_n z^{n-1}\right)}{\varepsilon(k+1)\left(1+\sum_{n=2}^{\infty} n b_n z^{n-1}\right)} - \frac{\mathbb{E}_{k+1}}{\varepsilon(k+1)}. \end{aligned} \tag{90}$$

This will become

$$w(z) = \frac{1+\sum_{n=2}^{k} n b_n z^{n-1} - \mathbb{E}_{k+1}/\varepsilon(k+1)\sum_{n=k+1}^{\infty} n b_n z^{n-1}}{\left(1+\sum_{n=2}^{\infty} n b_n z^{n-1}\right)}. \tag{91}$$

This leads us to

$$\frac{w(z)-1}{w(z)+1} = \frac{-\sum_{n=k+1}^{\infty}\left(1+\mathbb{E}_{k+1}/\varepsilon(k+1)\right) n b_n z^{n-1}}{2+2\sum_{n=2}^{k} n b_n z^{n-1} + \sum_{n=k+1}^{\infty}\left(1-\mathbb{E}_{k+1}/\varepsilon(k+1)\right) n b_n z^{n-1}}, \tag{92}$$

which reduces to

$$\left|\frac{w(z)-1}{w(z)+1}\right| \leq \frac{\left(1+\mathbb{E}_{k+1}/\varepsilon(k+1)\right)\sum_{n=k+1}^{\infty} n|b_n|}{2-2\sum_{n=2}^{k} n|b_n| - \left(1-\mathbb{E}_{k+1}/\varepsilon(k+1)\right)\sum_{n=k+1}^{\infty} n|b_n|}. \tag{93}$$

Now,

$$\left|\frac{w(z)-1}{w(z)+1}\right| \leq 1. \tag{94}$$

if

$$\sum_{n=2}^{k} n|b_n| + \sum_{n=k+1}^{\infty} n|b_n| \leq 1. \tag{95}$$

It is sufficient to show that the left hand side of (28) is bounded above by $\sum_{n=2}^{\infty} \mathbb{E}_n/\varepsilon|b_n|$ if

$$\sum_{n=2}^{k} n|b_n| + \sum_{n=k+1}^{\infty} n|b_n| \leq \sum_{n=2}^{\infty} \frac{\mathbb{E}_n}{\varepsilon}|b_n|, \tag{96}$$

which leads to the following expression:

$$\sum_{n=2}^{\infty}\left(\frac{\mathbb{E}_n}{\varepsilon} - n\right)|b_n| \geq 0. \tag{97}$$

The result (89) is sharp with respect to the function given by (62). □ □

In the next theorem, we will find the radii of starlikeness for the class $\beta - \mathrm{UCV}_q^{\lambda}[A,B]$.

Theorem 7. *Let $y(z) \in \beta - UCV_q^{\lambda}[A,B]$. Then, $y(z)$ is a convex function of order $\alpha \in [0,1)$ in $|z| < r = r_1(\alpha)$, where*

$$r_1(\alpha) = \left(\frac{\mathbb{E}_n(1-\alpha)}{\varepsilon\left(q[n-1]_q + (1-\alpha)\right)}\right)^{1/n-1}, \quad n = 2, 3, \ldots, \tag{98}$$

where $\mathbb{E}_n$ is defined by (29) and $\varepsilon = (1+q)|B-A|$.

Proof. Let $y(z) \in \beta - \mathrm{UCV}_q^{\lambda}[A,B]$. Then, by the theorem,

$$\sum_{n=2}^{\infty} \frac{\mathbb{E}_n}{\varepsilon} |b_n| < 1, \tag{99}$$

where $\mathbb{E}_n$ is defined by (29) and $\varepsilon = (1+q)|B-A|$. For $\alpha \in [0,1)$, we need to show that

$$\left| \frac{\partial_q \left(z \partial_q R_q^\lambda y(z) \right)}{\partial_q R_q^\lambda y(z)} \right| < 1 - \alpha, \tag{100}$$

that is,

$$\begin{aligned} \left| \frac{\partial_q \left(z \partial_q R_q^\lambda y(z) \right) - \partial_q R_q^\lambda y(z)}{\partial_q R_q^\lambda y(z)} \right| &= \left| -\frac{\sum_{n=2}^{\infty} q[n-1]_q [n]_q \psi_{n-1} b_n z^{n-1}}{1 - \sum_{n=2}^{\infty} [n]_q \psi_{n-1} b_n z^{n-1}} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} q[n-1]_q [n]_q \psi_{n-1} |b_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} [n]_q \psi_{n-1} |b_n| |z|^{n-1}} \\ &< 1 - \alpha. \end{aligned} \tag{101}$$

Thus, $|\partial_q (z \partial_q R_q^\lambda y(z)) - \partial_q R_q^\lambda y(z) / \partial_q R_q^\lambda y(z)| \leq 1 - \alpha$ if

$$\left(\frac{q[n-1]_q}{1-\alpha} + 1 \right) [n]_q \psi_{n-1} |b_n| |z|^{n-1} \leq 1. \tag{102}$$

According to theorem (99), inequality (102) will be true if

$$\left(\frac{q[n-1]_q}{1-\alpha} + 1 \right) |z|^{n-1} \leq \frac{\mathbb{E}_n}{\varepsilon}. \tag{103}$$

Solving (103) for $|z|$, we obtain

$$|z|^{n-1} \leq \frac{\mathbb{E}_n (1-\alpha)}{\varepsilon \left(q[n-1]_q + (1-\alpha) \right)}. \tag{104}$$

Setting $|z| = r(\alpha)$ in (104), we have

$$r(\alpha) = \left(\frac{\mathbb{E}_n (1-\alpha)}{\varepsilon \left(q[n-1]_q + (1-\alpha) \right)} \right)^{1/n-1}, \tag{105}$$

which is the required result. □ □

3. Conclusion

In this article, we have applied the q-Ruscheweyh differential operator to define and study a new class $\beta - \mathrm{UCV}_q^\lambda [A, B]$ of q-convex functions associated with the conic domain. This class generalizes the classes $\beta - \mathrm{UCV}[A, B]$, $C[A, B]$, $K(\beta, \alpha)$, $C(\alpha)$, and C which have been defined and studied earlier. This fact has been illustrated above with details and proper referencing. The results presented include sufficiency criteria related to Taylor series coefficients, the coefficient bounds, and the ratios of partial sums to their infinite sum for functions of the class $\beta - \mathrm{UCV}_q^\lambda [A, B]$.

Data Availability

No data were used in this article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors' Contributions

All authors contributed equally and approved the final manuscript.

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