Hindawi
Journal of Mathematics
Volume 2022, Article ID 2684474, 8 pages
https://doi.org/10.1155/2022/2684474

Research Article

Epimorphisms, Dominions, and Various Classes of Saturated Semigroups

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Received 17 June 2022; Accepted 25 July 2022; Published 31 August 2022

Academic Editor: Hassan Raza

In this paper, we discussed some saturated classes of $\mathscr{H}$-commutative semigroups, left (right) regular semigroups, medial semigroups, and paramedial semigroups. The results of this paper significantly extend the long standing result about normal bands that normal bands were saturated and, thus, significantly broaden the class of saturated semigroups.

1. Introduction

Epimorphic and dominion-related properties for the various classes of semigroups are internally linked to zigzags, amalgams, and semigroup embeddings. This trend of studying the properties of a semigroup class has been one of the hot pursuits areas in semigroup theory and other universal algebras worldwide, and a lot of original work has been conducted and is still being explored. In general, the theory of semigroups has a significant influence on the theory of computers, languages, and automata where the equation of zigzags can be represented in terms of zigzag languages. Now we come to the basic notion and fundamental concept of our study which are as follows: Suppose that a semigroup S has a subsemigroup named U. Using the definition from Isbell [17], U dominates an element d of S if for all semigroups T and homomorphisms $\beta, \gamma\colon S \longrightarrow T$, $u\beta = u\gamma$ for every $u \in U$, implies $d\beta = d\gamma$. The dominion of U in S is denoted by $\mathrm{Dom}(U, S)$ and includes all elements of S dominated by U. It is simple to prove that $\mathrm{Dom}(U, S)$ is a subsemigroup of S that contains U. It is said that if $\mathrm{Dom}(U, S) = U$, then U is *close d* in S and U is absolutely closed if $\mathrm{Dom}(U, S) = U$ for all generally comprising semigroups S. If $\mathrm{Dom}(U, S) \neq S$ for all generally comprising semigroup S, then the semigroup U is said to be saturated. Saturation of a semigroup class occurs when all of its members are saturated.

A morphism from $\alpha\colon A \longrightarrow B$ is referred to as a epimorphism (in short epi) in a category $\mathscr{C}$ if for all $C \in \mathscr{C}$ and for all morphisms $\beta, \gamma\colon B \longrightarrow C$, $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$. It is very clear that a morphism $\alpha\colon S \longrightarrow T$ is epi if and only if the inclusion $i\colon S\alpha \longrightarrow T$ is epi. It is also simple to check that if $\mathrm{Dom}(U, S) = S$, then the inclusion map $i\colon U \longrightarrow S$ is epi for any subsemigroup U of a semigroup S. This scenario also implies that U is epimorphically embedded or dense in S. This also means that every onto morphism is an epimorphism.

Surjective morphisms are epimorphisms in the case of category theory, while the opposite is not always true in general. For example, in the categories of Sets, Abelian Groups, and Groups, epis are onto, but generally it is not true in the categories of semigroups and rings.

Dominions have a great connection with epimorphisms as it follows from the above definitions that all epis from a semigroup U are onto states that all morphic images of U are saturated.

The structure of the paper is as follows: in Section 3, we have extended Khan's result from commutativity to

$\mathscr{H}$-commutativity and shown that any $\mathscr{H}$-commutative semigroup satisfying a nontrivial identity of which at least one side has no repeated variable is saturated. In Section 4, it has been shown that all right [left] regular medial semigroups are saturated, while in Section 5, we have proved that all medial semigroups satisfying the identities $x^r = x$ and $xy = xy^2$ are, respectively, saturated; and all paramedial semigroups satisfying the identity $axy = axay$ are saturated.

2. Preliminaries

Isbell's Zigzag theorem gives a highly useful tool for describing semigroup dominions.

Result 1 (see [17, Theorem 2.3] or [15, theorem VII.2.13]). Let U be a subsemigroup of a semigroup S and let $d \in S$, then $d \in \text{Dom}(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:

$$\begin{aligned} d &= a_0 y_1 = x_1 a_1 y_1 = x_1 a_2 y_2 \\ &= x_2 a_3 y_2 = \cdots = x_m a_{2m-1} y_m = x_m a_{2m}, \end{aligned} \tag{1}$$

where $m \geq 1, a_i \in U, (i = 0, 1, \ldots, 2m), \quad x_i, y_i \in S (i = 1, 2, \ldots, m)$, and

$$\begin{aligned} a_0 &= x_1 a_1, \\ a_{2m-1} y_m &= a_{2m}, \\ a_{2i-1} y_i &= a_{2i} y_{i+1}, \quad (1 \leq i \leq m-1). \\ x_i a_{2i} &= x_{i+1} a_{2i+1}, \end{aligned} \tag{2}$$

When factoring S over U, this is known as a zigzag with the value d, length m and spine $a_0, a_1, \ldots, a_{2m}$.

Results 1's equations are referred to throughout the remainder of the paper as "the zigzag equations."

Result 2. ([19, Result 3]). Let U be any subsemigroup of a semigroup S and let $d \in \text{Dom}(U, S) \backslash U$. If (1) is a zigzag of minimum length m over U with value d, then $x_i, y_i \in S \backslash U$ for $i = 1, 2, \ldots, m$.

In the following results, let U and S be any semigroups with U dense in S.

Result 3. ([19, Result 4]). For any $d \in S \backslash U$ and k any positive integer, if (1) is a zigzag of minimum length m over U with value d, then there exists $b_1, b_2, \ldots, b_k \in U$ and $d_k \in SU$ such that $d = b_1 b_2, \ldots, b_k d_k$.

If there are any symbols or terms that are not explained, we refer the readers to Clifford, Preston, and Howie [8, 15]. Furthermore, bracketed assertions or conceptions are dual to the other claims or notions in what follows.

In semigroup theory, ring theory, and elsewhere, there have been several efforts to find the classes of algebras which are saturated [7]. It was proved by Gardner [9, Theorem 2.10] that, in the class of all rings, any regular ring is saturated but Higgins [12, Corollary 4] established that not every regular semigroup is saturated. Any class of generalized inverse semigroups, on the other hand, is saturated and had been shown by Higgins in [8]. There must be at least one side of any identity defining a semigroup variety that has no repeating variables in order for it to be saturated [13, Theorem 6]. Commutative and heterotypical varieties of semigroups have already been dealt with, however, it remains an unanswered question how to identify all saturated semigroups (see [13, 14, 18]). The following semigroups are not saturated: commutative cancellative semigroups, subsemigroups of finite inverse semigroups [17], commutative periodic semigroups [14], and bands, since Trotter [24] has produced a band with a correctly epimorphically embedded subband. In this direction, a very recent significant and remarkable work have been made by Ahanger and Shah on partially ordered semigroups (posemigroups), and commutative posemigroups (see [1–3], [23]).

Now, we begin with the class of $\mathscr{H}$-commutative semigroups whose concept was first developed by Tully [25]. In [19], Nagy presented a new concept of $\mathscr{H}$-commutativity, i.e., for all $a, b \in S$, there exists $x \in S^1$ such that $ab = bxa$. He also found that the two characterizations coincide (Theorem 5.1, 18). Recently, the structure of semigroups of this class has been explored by Alam, Higgins, and Khan [5].

A semigroup S is known as left (right) quasi commutative if, for all $a, b \in S$, there exists a positive integer r such that $ab = b^r a \, (ab = ba^r)$. A semigroup S is called quasi commutative semigroup if it is both left [right] quasi commutative semigroup. It can be easily seen that all quasi commutative semigroups are $\mathscr{H}$-commutative, but this is not always be true for the converse case [see Ch.8, 19].

An element x of a semigroup S is called left [right] regular if $x = yx^2 \, [x = x^2 y]$ for some $y \in S$, or in other words, $x \mathscr{L} x^2 \, [x \mathscr{R} x^2]$. If all elements of S is left [right] regular, then S is called a left [right] regular semigroup, see [20]. A medial semigroup is a semigroup which fulfills the identity $abcd = acbd$, as shown in [Ch. 9, 19].

Protic [21] introduced the concept of paramedial semigroups as a generalization of externally commutative semigroups. A semigroup is called paramedial semigroup if it satisfies the paramedial law: $wxyz = zxyw$ for all $w, x, y, z \in S$. For further information and related results, see [10, 11, 16, 19].

3. Saturated Class of $\mathscr{H}$-Commutative Semigroups

The general question of identifying all saturated varieties of semigroups has been open for long, though lot of efforts had been made over the last four decades. For example, there was an answer to the question for commutative varieties (Higgins [14], Khan [18]) and heterotypical varieties (Higgins [13]). Readers may refer [10, 11] for the related notions, results, and materials on the topic. Our first finding extends Khan's result [18, Theorem 3.4] providing an essential condition for the saturation of commutative varieties. Authors had previously extended this result to quasi commutative semigroups [6, Theorem 2.5]. Since the class of $\mathscr{H}$-commutative semigroups contains the class of quasi commutative semigroups, it is worth to explore whether this result may further be extended to the class of $\mathscr{H}$-commutative semigroups. Here, we have answered the above question and generalized authors's result for the class of

$\mathscr{H}$-commutative semigroups. This class, infact, also contains the class of commutative semigroups.

If the semigroup is commutative and at least one side has no repeating variables, Khan [18] established that it was saturated. This conclusion was extended in [6] for quasi commutative semigroups. Saturation of a $\mathscr{H}$-commutative semigroup fulfilling a nontrivial identity of which at least one side has no repeating variable is further shown in the current article.

Theorem 1. *If a $\mathscr{H}$-commutative semigroup U satisfies a nontrivial identity I of which at least one side has no repeated variable, then U is saturated.*

Proof. Since one side of the identity I has no repeated variable, the identity I has the following form:

$$x_1x_2\cdots x_n = w(x_1, x_2, \ldots x_n, x_{n+1}, \ldots, x_{n+m}), \quad (m \geq 0). \tag{3}$$

□

Proposition 1. *In any $\mathscr{H}$-commutative semigroup S, for any $a, b, c \in S$ and positive integer $r \geq 2$, there exists $w \in S^1$ such that*

$$(abc)^r = a^r b^r c^r w. \tag{4}$$

Proof. We shall use induction on r to prove the proposition. For $r = 2$, by repeated application of $\mathscr{H}$-commutativity of S and for some $w_1, w_2, w_3, w_4 \in S^1$, we have

$$\begin{aligned}(abc)^2 &= (abc)(abc)\\ &= a(bc)a(bc)\\ &= a^2w_1(bc)(bc)\\ &= a^2(bc)(bc)w_2w_1\\ &= a^2b^2w_3c^2w_2w_1\\ &= a^2b^2c^2w_4w_3w_2w_1.\end{aligned} \tag{5}$$

So the result holds for $r = 2$.

Similarly, for $r = 3$, by repeated application of $\mathscr{H}$-commutativity of S, the case for $r = 2$ and for some $w_1, w_2, \ldots, w_8 \in S^1$, we have

$$\begin{aligned}(abc)^3 &= (abc)^2(abc),\\ &= a^2b^2c^2w_4w_3w_2w_1(abc)\\ &= a^2(abc)w_5b^2c^2w_4w_3w_2w_1\\ &= a^3bb^2w_6ccw_5c^2w_4w_3w_2w_1\\ &= a^3b^3cw_7w_6cw_5c^2w_4w_3w_2w_1\\ &= a^3b^3cc^2w_8w_7w_6cw_5w_4w_3w_2w_1\\ &= a^3b^3c^3(w_8w_7w_6cw_5w_4w_3w_2w_1).\end{aligned} \tag{6}$$

Thus the result is true for $r = 3$.

For all positive integers less than or equal to $r - 1$, suppose inductively that the conclusion is true. Thus, we have

$$(abc)^{r-1} = a^{r-1}b^{r-1}c^{r-1}(w_{4r-8}w_{4r-9}, \ldots, w_3w_2w_1), \tag{7}$$

for some $w_{4r-8}, w_{4r-9}, \ldots, w_3, w_2, w_1 \in S^1$.

Now for positive integer r, we repeat the application of $\mathscr{H}$-commutativity of S for some $w_{4r-7}, w_{4r-6}, w_{4r-5}, w_{4r-4} \in S^1$. Therefore, we have

$$\begin{aligned}(abc)^r &= (abc)^{r-1}(abc)\\ &= a^{r-1}b^{r-1}c^{r-1}(w_{4r-8}w_{4r-9}\cdots w_3w_2w_1)(abc)\\ &= a^{r-1}(abc)w_{4r-7}b^{r-1}c^{r-1}(w_{4r-8}w_{4r-9}\cdots w_3w_2w_1)\\ &= a^rbcw_{4r-7}b^{r-1}c^{r-1}(w_{4r-8}w_{4r-9}\cdots w_3w_2w_1)\\ &= a^r(bb^{r-1})w_{4r-6}cw_{4r-7}c^{r-1}(w_{4r-8}w_{4r-9}\cdots w_3w_2w_1)\\ &= a^rb^rcw_{4r-5}w_{4r-6}w_{4r-7}c^{r-1}(w_{4r-8}w_{4r-9}\cdots w_3w_2w_1)\\ &= a^rb^rcc^{r-1}w_{4r-4}w_{4r-5}w_{4r-6}w_{4r-7}(w_{4r-8}w_{4r-9}\cdots w_3w_2w_1)\\ &= a^rb^rc^r(w_{4r-4}w_{4r-5}w_{4r-6}\cdots w_3w_2w_1),\\ &= a^rb^rc^rw\end{aligned} \tag{8}$$

where $w = w_{4r-4}w_{4r-5}w_{4r-6}\cdots w_3w_2w_1 \in S^1$. □

Returning to the proof of the theorem, we assume to the contrary that U is not saturated, so $\mathrm{Dom}(U, S) = S$ for some semigroup S containing U properly. □

Lemma 1. *$xa = xa^2t\,[ay = a^2ty]$ for some $t \in U^1$ and, all $a \in U$, $x \in SU\,[y \in SU]$.*

Proof. By Result 3, as $x \in SU$, we have

$$x = x'u_1u_2\cdots u_n, \tag{9}$$

for some $u_1, u_2, \ldots, u_n \in U$ and $x' \in SU$.

Since U satisfies (3), we have the two cases: Case (1): identity (3) is heterotypical and Case (2): identity (3) is homotypical. □

Case 1. Suppose identity (3) is heterotypical. Then $|x_{n+k}|_w \geq 1$ for some $1 \leq k \leq m$ ($|x|_w$, for any word w, denotes how many times the variable x appears in the word w). Now, we have

$$\begin{aligned}xa &= x'u_1u_2\cdots u_na\\ &= x'w\big(u_1, u_2, \ldots, u_n, u_{n+1}, \ldots, u_{n+k-1}, a^2,\\ &\quad u_{n+k+1}, \ldots, u_{n+m-1}, u_{n+m}\big)a,\end{aligned} \tag{10}$$

for any $u_{n+1}, \ldots, u_{n+k-1}, u_{n+k+1}, \ldots, u_{n+m-1}, u_{n+m} \in U$.

Since U is $\mathscr{H}$-commutative and $w(u_1, u_2, \ldots, u_n, u_{n+1}, \ldots, u_{n+k-1}, u_{n+k+1}, \ldots, u_{n+m-1}, u_{n+m})$ contains the element a^{2r} ($r = |y_{n+k}|w$), by using equality (9) and for some $w, w\prime \in U^1$, we have

$$\begin{aligned}
xa &= x' u_1 u_2 \cdots u_n a \\
&= x' w\big(u_1, u_2, \ldots, u_n, u_{n+1}, \ldots, u_{n+k-1}, a^2, \\
&\quad u_{n+k+1}, \ldots, u_{n+m-1}, u_{n+m}\big)a \\
&= x' w\left(u_1, u_2, \ldots, u_n, u_{n+1}, \ldots, u_{n+k-1},\right. \\
&\quad \left. a, u_{n+k+1}, \ldots, u_{n+m-1}, u_{n+m}\right) w a^{r+1} \\
&= x' u_1 u_2 \cdots u_n w a^{r+1} \\
&= x w a^{r+1} \\
&= x a^2 w\prime w a^{r-1} \\
&= x a^2 t,
\end{aligned} \tag{11}$$

where $t = wwa^{r-1} \in U^1$.

Case 2. Suppose identity (3) is homotypical. According to the nontriviality of the identity (3), we may consider that (3) is in the following form:

$$x_1 x_2 \ldots x_n = w(x_1, x_2, \ldots, x_n), \tag{12}$$

where $|x_k|_w \geq 1$ for all $1 \leq k \leq n$ and $|x_j|_w = r \geq 2$ for some $1 \leq j \leq n$. Since U is $\mathscr{H}$-commutative and by using equality (9) and (12) for some $p, w, w\prime \in U^1$, we have

$$\begin{aligned}
xa &= x' d_1 d_2, \ldots, d_{j-1} d_j d_{j+1}, \ldots, d_n a, \\
&= \big(x' d_1 d_2, \ldots, d_{j-1} d_j\big)\big(d_{j+1}, \ldots, d_n\big)a \\
&= \big(x' d_1 d_2, \ldots, d_j\big)\big((ap)\big(d_{j+1}, \ldots, d_n\big)\big) \\
&= x' w\big(d_1, d_2, \ldots, d_{j-1}, d_j a, p d_{j+1}, \ldots, d_n\big) \\
&= x' w(d_1, d_2, \ldots, d_n) w p^{|x_{j+1,w}} a^r \\
&= x' d_1 d_2, \ldots, d_n w p^{|x_{j+1,w}} a^r \\
&= x w p^{|x_{j+1}|_w} a^r \\
&= x a^2 w\prime w p^{|x_{j+1}|_w} a^{r-2} \\
&= x a^2 t,
\end{aligned} \tag{13}$$

as required, where $t = w\prime w p^{|x_{j+1}|_w} a^{r-2} \in U^1$. □

Returning back to the proof of the theorem, let $d \in S \backslash U$ be any element. As $\mathrm{Dom}(U, S) = S$, let (1) be a zigzag for d in S over U of minimal length m. Now, by using equalities (1), Lemma 1 for some $t_1, t_3 \cdots t_{2m-1} \in U^1$, and $\mathscr{H}$-commutativity of U for some $w_1, w_2, w_3, w_4, w', w^*, w^{**}, w^{***} \in U^1$

$$\begin{aligned}
d &= x_1 a_1 y_1 \\
&= x_1 a_1^2 t_1 y_1 \\
&= x_1 a_1 (a_1 t_1) y_1 \\
&= x_1 a_1 (t_1 w_1 a_1) y_1 \\
&= x_1 a_1 t_1 w_1 a_1 y_1 \\
&= x_1 a_1 t_1 w_1 a_2 y_2 \\
&= x_1 (a_1 t_1 w_1) a_2 y_2 \\
&= x_1 (a_2)(w_2)(a_1 t_1 w_1) y_2 \\
&= x_2 a_3 (w_2 a_1 t_1 w_1) y_2 \\
&= x_2 a_3 (w_2 a_1 t_1 w_1) y_2 \\
&= x_2 a_3^2 t_3 (w_2 a_1 t_1 w_1) y_2 \\
&= x_2 a_3 (a_3)(t_3 w_2 a_1 t_1 w_1) y_2 \\
&= x_2 a_3 (t_3 w_2 a_1 t_1 w_1)(w_3)(a_3) y_2 \\
&= x_2 a_3 (t_3 w_2)(a_1)(t_1 w_1 w_3) a_3 y_2 \\
&= x_2 a_3 (t_3 w_2 t_1 w_1 w_3)(w_4)(a_1) a_3 y_2 \\
&= x_2 a_3 (w\prime) a_1 a_3 y_2 \\
&\ \ \vdots \\
&= x_m a_{2m-1} (w^*) a_{2m-3} a_{2m-5} \ldots a_5 a_3 a_1 (a_{2m-1} y_m) \\
&= x_{m-1} a_{2m-2} (w^*) a_{2m-3} a_{2m-5} \ldots a_5 a_3 a_1 a_{2m} \\
&= x_{m-1} (a_{2m-2} (w^*)) a_{2m-3} a_{2m-5} \ldots a_5 a_3 a_1 a_{2m} \\
&= x_{m-1} a_{2m-3} a_{2m-3} \\
&\quad (a_{2m-2} (w^{**} w^*)) a_{2m-5} \ldots a_3 a_1 a_{2m} \\
&= x_{m-2} a_{2m-4} a_{2m-3} a_{2m-2} \\
&\quad (w^{**} w^*) a_{2m-5} \ldots a_5 a_3 a_1 a_{2m} \\
&= x_{m-2} a_{2m-5} a_{2m-5} a_{2m-4} a_{2m-3} a_{2m-2} \\
&\quad (w^{***} w^{**} w^*) \ldots a_5 a_3 a_1 a_{2m} \\
&\ \ \vdots \\
&= x_1 a_1 a_1 a_2 a_3 \ldots a_{2m-6} a_{2m-5} a_{2m-4} a_{2m-3} a_{2m-2} (v) a_{2m}) \\
&= a_0 a_1 a_2 a_3 \ldots a_{2m-6} a_{2m-5} a_{2m-4} a_{2m-3} a_{2m-2} (w *) a_{2m} \in U \\
&\Rightarrow d \in U.
\end{aligned} \tag{14}$$

This is a contradiction which shows that $\mathrm{Dom}(U, S) \neq S$, where U is saturated. □

As a corollary, we have the following interesting result:

Corollary 1. *Classes of all quasi commutative semigroups satisfies a nontrivial identity of which at least one side has no repeated variable are saturated.*

Since the class of weakly commutative semigroups is a wider class than the class of H-commutative semigroups, so one may arise the open problem as follows:

Open Problem 1. Whether the class of all weakly commutative semigroups satisfying a nontrivial identity of which at least one side has no repeated variable is saturated. If not, then under what condition this class may be saturated?

4. Saturated Classes of Medial Semigroups

Normal bands have long been known to be closed [22], and the class of generalized inverse semigroups (regular semigroups whose set of idempotents form normal bands) has been saturated (see [11]). So, it is natural to ask whether this result can be extended to the class of left [right] regular semigroups.

In the present section, we prove that the class of left [right] regular medial semigroups are saturated. We also show that medial semigroups satisfying the identities $x^r = x, r \geq 2$, and $xy = xy^2$ are saturated and consequently deduce the known fact that normal bands are saturated.

Lemma 2. *Let S be any semigroup with a medial subsemigroup U and such that $Dom(U, S) = S$. Then $xaby = xbay$ for all $a, b \in U$ and $x, y \in S$.*

Proof. When $x, y \in U$, then there is nothing to prove. So assume that $x \in S \backslash U$. As $\mathrm{Dom}(U, S) = S$, by Result 3 and based on property U, we obtain

$$x = x_1 u, \quad \text{for some } u \in U \text{ and } x_1 \in S \backslash U. \tag{15}$$

Now, if $y \in U$, then

$$\begin{aligned} xaby &= x_1 uaby, \\ &= x_1 ubay \\ &= xbay, \end{aligned} \tag{16}$$

as required.

In the other case, i.e., when $y \in S \backslash U$, then

$$y = vy_1, \quad \text{for some } v \in U \text{ and } y_1 \in S \backslash U. \tag{17}$$

So

$$\begin{aligned} xaby &= x_1 uabvy_1, \\ &= x_1 ubavy_1 \\ &= xbay. \end{aligned} \tag{18}$$

This completes the proof of the lemma. □

Theorem 2. *All right regular medial semigroups are saturated.*

Proof. Suppose all right regular medial semigroups are not saturated. So there is a right regular medial semigroup U and a semigroup S containing U as a proper subsemigroup and such that $\mathrm{Dom}(U, S) = S$. For any $d \in \mathrm{Dom}(U, S) \backslash U$, if (1) is a zigzag for d in S over U of minimal length m, then by using equalities (1), Lemma 2 and based on property U for some $z_1, z_3, \cdots z_{2m-1} \in U$, we have

$$\begin{aligned}
d &= a_0 y_1, \\
&= x_1 a_1 y_1 \\
&= x_1 a_1^2 z_1 y_1 \\
&= x_1 a_1 (a_1 z_1) y_1 \\
&= x_1 (a_1 z_1) a_1 y_1 \\
&= x_1 (a_1 z_1) a_1 y_1 \\
&= x_1 (a_1 z_1) a_2 y_2 \\
&= x_1 a_2 (a_1 z_1) y_2 \\
&= x_2 a_3 (a_1 z_1) y_2 \\
&= x_2 a_3^2 z_3 (a_1 z_1) y_2 \\
&= x_2 (a_3 z_3)(a_1 z_1) a_3 y_2 \\
&\qquad \vdots \\
&= x_{m-1} (a_{2m-3} z_{2m-3})(a_{2m-5} z_{2m-5}) \ldots (a_3 z_3)(a_1 z_1) a_{2m-3} y_{m-1} \\
&= x_{m-1} (a_{2m-3} z_{2m-3})(a_{2m-5} z_{2m-5}) \ldots (a_3 z_3)(a_1 z_1) a_{2m-2} y_m \\
&= x_{m-1} a_{2m-2} (a_{2m-3} z_{2m-3})(a_{2m-5} z_{2m-5}) \cdot (a_3 z_3)(a_1 z_1) y_m \\
&= x_m a_{2m-1} (a_{2m-3} z_{2m-3})(a_{2m-5} z_{2m-5}) \ldots (a_3 z_3)(a_1 z_1) y_m \\
&= x_m a_{2m-1}^2 z_{2m-1} (a_{2m-3} z_{2m-3})(a_{2m-5} z_{2m-5}) \ldots (a_3 z_3)(a_1 z_1) y_m \\
&= x_m a_{2m-1} z_{2m-1} (a_{2m-3} z_{2m-3}) \\
&\quad (a_{2m-5} z_{2m-7}) \ldots (a_3 z_3)(a_1 z_1) a_{2m-1} y_m \\
&= x_{m-1} a_{2m-2} z_{2m-1} (a_{2m-3} z_{2m-3})(a_{2m-5} z_{2m-5}) \ldots (a_3 z_3)(a_1 z_1) a_{2m} \\
&= x_{m-1} a_{2m-3} a_{2m-2} z_{2m-1} z_{2m-3} (a_{2m-5} z_{2m-5}) \ldots (a_3 z_3)(a_1 z_1) a_{2m} \\
&= x_{m-2} a_{2m-4} a_{2m-2} z_{2m-1} z_{2m-3} (a_{2m-5} z_{2m-5}) \ldots (a_3 z_3)(a_1 z_1) a_{2m} \\
&= x_{m-2} a_{2m-5} a_{2m-4} a_{2m-2} z_{2m-1} z_{2m-3} z_{2m-5} \ldots (a_3 z_3)(a_1 z_1) a_{2m} \\
&= x_{m-3} a_{2m-6} a_{2m-4} a_{2m-2} z_{2m-1} z_{2m-3} z_{2m-5} \ldots (a_3 z_3)(a_1 z_1) a_{2m} \\
&\qquad \vdots \\
&= x_2 a_4 a_6 a_8 z_{2m-1} z_{2m-3} z_{2m-5} \ldots z_7 z_5 (a_3 z_3)(a_1 z_1) a_{2m} \\
&= x_2 a_3 a_4 a_6 a_8 z_{2m-1} z_{2m-3} z_{2m-5} \cdots z_7 z_5 z_3 (a_1 z_1) a_{2m} \\
&= x_1 a_2 a_4 a_6 a_8 z_{2m-1} z_{2m-3} z_{2m-5} \ldots z_7 z_5 z_3 (a_1 z_1) a_{2m} \\
&= x_1 a_1 a_2 a_4 a_6 a_8 z_{2m-1} z_{2m-3} z_{2m-5} \ldots z_7 z_5 z_3 z_1 a_{2m} \\
&= a_0 a_2 a_4 a_6 a_8 \ldots a_{2m-6} a_{2m-4} a_{2m-2} z_{2m-1} z_{2m-3} z_{2m-5} \ldots z_7 z_5 z_3 z_1 a_{2m} \\
&= a_0 z_1 a_2 z_3 a_4 z_5 a_6 z_7 a_8 \ldots a_{2m-6} z_{2m-5} a_{2m-4} z_{2m-3} a_{2m-2} z_{2m-1} \ldots a_{2m} \\
&\Rightarrow d \in U.
\end{aligned} \tag{19}$$

This is a contradiction and, so, $\mathrm{Dom}(U, S) \neq S$. Thus U is saturated. □

Here, we generalize the result that the variety of normal bands was saturated [11] to medial semigroups fulfilling the identity $x^r = x$ $(r \geq 2)$. □

Theorem 3. *Any medial semigroup satisfying the identity $x^r = x$ $(r \geq 2)$ is saturated.*

Proof. Assume that U is a nonsaturated medial semigroup fulfilling the identity $x^r = x$ $(r \geq 2)$. So there is a semigroup S containing U as a proper subsemigroup and such that $\mathrm{Dom}(U, S) = S$. For any $d \in \mathrm{Dom}(U, S) \backslash U$, if (1) is a zigzag for d in S over U of least length m, then by using equalities (1), Lemma 2 and based on property U, it implies that

$$\begin{aligned}
d &= a_0 y_1, \\
&= x_1 a_1 y_1 \\
&= x_1 a_1^r y_1 \\
&= x_1 a_1^{r-1} a_1 y_1 \\
&= x_1 a_1^{r-1} a_2 y_2 \\
&= x_1 a_2 a_1^{r-1} y_2 \\
&= x_2 a_3 a_1^{r-1} y_2 \\
&= x_2 a_3^r a_1^{r-1} y_2 \\
&\vdots \\
&= x_{m-1} a_{2m-3}^r \left(a_{2m-5}^{r-1} \ldots a_5^{r-1} a_3^{r-1} a_1^{r-1}\right) y_{m-1} \\
&= x_{m-1} a_{2m-3} a_{2m-3}^{r-1} \left(a_{2m-5}^{r-1} \ldots a_5^{r-1} a_3^{r-1} a_1^{r-1}\right) y_{m-1} \\
&= x_{m-1} a_{2m-3}^{r-1} \left(a_{2m-5}^{r-1} \ldots a_5^{r-1} a_3^{r-1} a_1^{r-1}\right) a_{2m-3} y_{m-1} \\
&= x_{m-1} \left(a_{2m-3}^{r-1} a_{2m-5}^{r-1} \ldots a_5^{r-1} a_3^{r-1} a_1^{r-1}\right) a_{2m-2} y_m \\
&= x_{m-1} a_{2m-2} \left(a_{2m-3}^{r-1} a_{2m-5}^{r-1} \ldots a_5^{r-1} a_3^{r-1} a_1^{r-1}\right) y_m \\
&= x_m a_{2m-1} \left(a_{2m-3}^{r-1} a_{2m-5}^{r-1} \ldots a_5^{r-1} a_3^{r-1} a_1^{r-1}\right) y_m \\
&= x_m a_{2m-1} a_{2m-1}^{r-1} \left(a_{2m-3}^{r-1} a_{2m-5}^{r-1} \ldots a_5^{r-1} a_3^{r-1} a_1^{r-1}\right) y_m \\
&= x_m a_{2m-1}^{r-1} \left(a_{2m-3}^{r-1} a_{2m-5}^{r-1} \ldots a_5^{r-1} a_3^{r-1} a_1^{r-1}\right) \left(a_{2m-1} y_m\right) \\
&= x_m a_{2m-1} a_{2m-1}^{r-2} \left(a_{2m-3}^{r-1} a_{2m-5}^{r-1} \ldots a_5^{r-1} a_3^{r-1} a_1^{r-1}\right) a_{2m} \\
&= x_{m-1} a_{2m-2} a_{2m-1}^{r-2} \left(a_{2m-3}^{r-1} a_{2m-5}^{r-1} \ldots a_5^{r-1} a_3^{r-1} a_1^{r-1}\right) a_{2m} \\
&= x_{m-1} a_{2m-3}^{r-1} \left(a_{2m-5}^{r-1} \ldots a_5^{r-1} a_3^{r-1} a_1^{r-1}\right) a_{2m-2} a_{2m-1}^{r-2} \left(a_{2m}\right) \\
&= x_{m-1} a_{2m-3} a_{2m-3}^{r-2} \left(a_{2m-5}^{r-1} \ldots a_5^{r-1} a_3^{r-1} a_1^{r-1}\right) a_{2m-2} a_{2m-1}^{r-2} a_{2m} \\
&= x_{m-2} a_{2m-4} a_{2m-3}^{r-2} \left(a_{2m-5}^{r-1} \ldots a_5^{r-1} a_3^{r-1} a_1^{r-1}\right) a_{2m-2} a_{2m-1}^{r-2} a_{2m} \\
&= x_{m-2} a_{2m-5}^{r-1} \left(a_{2m-7}^{r-1} \ldots a_5^{r-1} a_3^{r-1} a_1^{r-1}\right) a_{2m-4} a_{2m-3}^{r-2} a_{2m-2} a_{2m-1}^{r-2} a_{2m} \\
&\vdots \\
&= x_2 a_3^{r-1} a_1^{r-1} a_4 a_5^{r-2} a_6 a_7^{r-2} \ldots a_{2m-4} a_{2m-3}^{r-2} a_{2m-2} a_{2m-1}^{r-2} a_{2m} \\
&= x_2 a_3 a_3^{r-2} a_1^{r-1} a_4 a_5^{r-2} a_6 a_7^{r-2} \ldots a_{2m-4} a_{2m-3}^{r-2} a_{2m-2} a_{2m-1}^{r-2} a_{2m} \\
&= x_1 a_2 a_3^{r-2} a_1^{r-1} a_4 a_5^{r-2} a_6 a_7^{r-2} \ldots a_{2m-4} a_{2m-3}^{r-2} a_{2m-2} a_{2m-1}^{r-2} a_{2m} \\
&= x_1 a_1^{r-1} a_2 a_3^{r-2} a_4 a_5^{r-2} a_6 a_7^{r-2} \ldots a_{2m-4} a_{2m-3}^{r-2} a_{2m-2} a_{2m-1}^{r-2} a_{2m} \\
&= x_1 a_1 a_1^{r-2} a_2 a_3^{r-2} a_4 a_5^{r-2} a_6 a_7^{r-2} \ldots a_{2m-4} a_{2m-3}^{r-2} a_{2m-2} a_{2m-1}^{r-2} a_{2m} \\
&= a_0 a_1^{r-2} a_2 a_3^{r-2} a_4 a_5^{r-2} a_6 a_7^{r-2} \ldots a_{2m-4} a_{2m-3}^{r-2} a_{2m-2} a_{2m-1}^{r-2} a_{2m} \\
&\Rightarrow d \in U.
\end{aligned} \tag{20}$$

This is a contradiction implying that $\mathrm{Dom}(U, S) \neq S$. Therefore U is saturated. □

Classes of left [right] commutative semigroups and externally commutative semigroups satisfying the identity $xy = xy^2$ were saturated was shown in [4]. Here, we further extend this result for a class of medial semigroups.

Lemma 3. *Let U be a semigroup fulfilling the identity $xy = xy^2$ and let S be a semigroup containing U as a proper subsemigroup and such that $\mathrm{Dom}(U, S) = S$. Then $xay = xa^2y$ for all $a \in U$ and $x, y \in S$.*

Proof. If $x \in S$, then the proof follows trivially. In the other case, i.e., when $x \in S \backslash U$, then, by Result 3, $x = x_1 u$ for some $u \in U$ and $x_1 \in S \backslash U$. Now, based on property U, we obtain $xay = x_1 uay = x_1 ua^2 y = xa^2 y$. □

Theorem 4. *Medial semigroups satisfying the identity $xy = xy^2$ are saturated.*

Proof. Take any semigroup U fulfilling the identities $xy^2 = xy$ and $xyzw = xzyw$. If, by contradictory U is non-saturated, then there is a semigroup S containing U properly and such that $\mathrm{Dom}(U, S) = S$. For any $d \in \mathrm{Dom}(U, S) \backslash U$, if (1) is a zigzag for d in S over U of minimal length m, then by using equalities (1), Lemma 2, Lemma 3, and by the property of U, we have

$$\begin{aligned}
d &= a_0 y_1, \\
&= x_1 a_1 y_1 \\
&= x_1 a_1^2 y_1 \\
&= x_1 a_1 a_1 y_1 \\
&= x_1 a_1 a_2 y_2 \\
&= x_1 a_2 a_1 y_2 \\
&= x_2 a_3 a_1 y_2 \\
&= x_2 a_3^2 a_1 y_2 \\
&= x_2 a_3 a_1 a_3 y_2 \\
&\vdots \\
&= x_{m-1} a_{2m-3} a_1 a_3 \ldots a_{2m-7} a_{2m-5} y_{m-1} \\
&= x_{m-1} a_1 a_3 \ldots a_{2m-7} a_{2m-5} a_{2m-3}^2 y_{m-1} \\
&= x_{m-1} a_1 a_3 \ldots a_{2m-7} a_{2m-5} a_{2m-3} a_{2m-2} y_m \\
&= x_{m-1} a_{2m-2} a_1 a_3 \ldots a_{2m-7} a_{2m-5} a_{2m-3} y_m \\
&= x_m a_{2m-1} a_1 a_3 \ldots a_{2m-7} a_{2m-5} a_{2m-3} y_m \\
&= x_m a_{2m-1}^2 a_1 a_3 \ldots a_{2m-7} a_{2m-5} a_{2m-3} y_m \\
&= x_m a_{2m-1} a_1 a_3 \ldots a_{2m-7} a_{2m-5} a_{2m-3} a_{2m-1} y_m \\
&= x_m a_{2m-1} a_1 a_3 \ldots a_{2m-7} a_{2m-5} a_{2m-3} a_{2m} \\
&= x_{m-1} a_{2m-2} a_1 a_3 \ldots a_{2m-7} a_{2m-5} a_{2m-3} a_{2m} \\
&= x_{m-1} a_{2m-3} a_1 a_3 \ldots a_{2m-7} a_{2m-5} a_{2m-2} a_{2m} \\
&= x_{m-2} a_{2m-4} a_1 a_3 \ldots a_{2m-7} a_{2m-5} a_{2m-2} a_{2m} \\
&= x_{m-2} a_{2m-5} a_1 a_3 \ldots a_{2m-7} a_{2m-4} a_{2m-2} a_{2m} \\
&\vdots \\
&= x_2 a_3 a_1 a_4 a_6 \ldots a_{2m-6} a_{2m-4} a_{2m-2} a_{2m} \\
&= x_1 a_2 a_1 a_4 a_6 \ldots a_{2m-6} a_{2m-4} a_{2m-2} a_{2m} \\
&= x_1 a_1 a_2 a_4 a_6 \ldots a_{2m-6} a_{2m-4} a_{2m-2} a_{2m} \\
&= a_0 a_2 a_4 a_6 \ldots a_{2m-6} a_{2m-4} a_{2m-2} a_{2m} \in U.
\end{aligned} \tag{21}$$

A contradiction and, so, $\mathrm{Dom}(U,S) \neq S$. Hence U is saturated. □

Corollary 2. *The variety* $[xy = xy^2, xyz = xzy]$ $([xy = xy^2, xyz = yxz])$ *of semigroups is saturated.*

Corollary 3. *Varieties of left [right] normal bands and normal bands are saturated.*

5. Saturated Class of Paramedial Semigroups

It was shown, in [4], that a medial semigroup fulfilling the identity $axy = axay$ was saturated. In the following, this result is extended to paramedial semigroups by showing that a paramedial semigroup with the identity $axy = axay$ is saturated.

Proposition 2. *For any proper subsemigroup U of a semigroup S, if $\mathrm{Dom}(U,S) = S$ and the identity $axy = axay$ is satisfied by U, then $xay = xazay$ and $xaby = xabay$ for all $a, b \in U$ and $x, y \in S\backslash U$, where $x = x'z$ for some $z \in U$ and $x' \in S\backslash U$.*

Proof. Now, as $y \in S\backslash U$, let $y = wy'$ for some $w \in U$ and $y' \in S\backslash U$. So, based on property U, we obtain

$$\begin{aligned} xay &= x'zawy \\ &= x'zazwy' \\ &= x'z(azw)y' \\ &= x'z(azaw)y' \\ &= (x'z)aza(wy') \\ &= xazay \\ &\text{and} \\ xaby &= xabwy' \\ &= xabawy' \\ &= xabay, \end{aligned} \tag{22}$$

as required. □

Theorem 5. *Any paramedial semigroup satisfying the identity $axy = axay$ is saturated.*

Proof. If to the contrary, a paramedial semigroup U satisfying the identity $axy = axay$ is nonsaturated, then there is a semigroup S with proper containment of U such that $\mathrm{Dom}(U,S) = S$. Now, for any $d \in \mathrm{Dom}(U,S)\backslash U$, if (1) is a zigzag for d in S over U of minimal length m, then by using equalities (1), Proposition 2, and by the property of U, we have

$$\begin{aligned} d &= a_0 y_1 \\ &= x_1 a_1 y_1 \\ &= x_1 a_1 z_1 a_1 y_1 \\ &= x_1 a_1 z_1 a_2 y_2 \\ &= x_1 a_1 z_1 a_1 a_2 y_2) \\ &= x_1 a_2 z_1 a_1 a_1 y_2 \\ &= x_2 a_3 (z_1 a_1^2) y_2 \\ &= x_2 a_3 (z_1 a_1^2) a_3 y_2 \\ &\vdots \\ &= x_{m-1} a_{2m-3} (z_1 a_1^2) a_3 a_5 \dots a_{2m-7} a_{2m-5} a_{2m-3} y_{m-1} \\ &= x_{m-1} a_{2m-3} (w) (a_3 a_5 \dots a_{2m-7} a_{2m-5}) a_{2m-2} y_m, \quad (\text{where } w = z_1 a_1^2) \\ &= x_{m-1} a_{2m-2} (w) (a_3 a_5 \dots a_{2m-7} a_{2m-5}) a_{2m-3} y_m \\ &= x_m a_{2m-1} (w a_3 a_5 \dots a_{2m-7} a_{2m-5} a_{2m-3}) y_m \\ &= x_m a_{2m-1} (w a_3 a_5 \dots a_{2m-7} a_{2m-5} a_{2m-3}) a_{2m-1} y_m \\ &= x_{m-1} (a_{2m-2}) (w) (a_3 a_5 \dots a_{2m-7} a_{2m-5}) (a_{2m-3}) a_{2m} \\ &= x_{m-1} ((a_{2m-2}) (w) (a_3 a_5 \dots a_{2m-7} a_{2m-5}) (a_{2m-3})) a_{2m} \\ &= x_{m-1} ((a_{2m-3}) (w) (a_3 a_5 \dots a_{2m-7} a_{2m-5}) (a_{2m-2})) a_{2m} \\ &= x_{m-2} a_{2m-4} (w) (a_3 a_5 \dots a_{2m-7}) (a_{2m-5}) a_{2m-2} a_{2m} \\ &= x_{m-2} a_{2m-5} (w) (a_3 a_5 \dots a_{2m-7}) a_{2m-4} a_{2m-2} a_{2m} \\ &\vdots \\ &= x_2 a_3 (w) a_4 a_6 \dots a_{2m-4} a_{2m-2} a_{2m} \\ &= x_2 a_3 (z_1 a_1^2) a_4 a_6 \dots a_{2m-4} a_{2m-2} a_{2m}, \quad (\text{as } w = z_1 a_1^2 \in U) \\ &= x_1 (a_2) (z_1) (a_1) (a_1) a_4 a_6 \dots a_{2m-4} a_{2m-2} a_{2m} \\ &= x_1 a_1 (z_1) (a_1) a_2 a_4 a_6 \dots a_{2m-4} a_{2m-2} a_{2m} \\ &= a_0 (z_1 a_1) a_2 a_4 a_6 \dots a_{2m-4} a_{2m-2} a_{2m}. \end{aligned} \tag{23}$$

Thus $d \in U$, which is a contradiction. Consequently, $\mathrm{Dom}(U,S) \neq S$ and, thus, U is saturated and the proof of the theorem is completed. □

6. Conclusion

In the present paper, authors have successfully proved that any H-commutative semigroup satisfying a nontrivial identity of which at least one side has no repeated variable is saturated. Then it has been shown that all right [left] regular medial semigroups are saturated. Finally, we have proved that all medial semigroups satisfying the identities $x^r = x$ $(r \geq 2)$ and $xy = xy^2$ are, respectively, saturated; and all paramedial semigroups satisfying the identity $axy = axay$ are saturated.

The results obtained in the paper have their immense utility as they imply that all epis from these classes are onto and open avenues and hope to explore further classes of semigroups for which epis are onto; for example, we list a few open problems in this direction to look into by researchers:

(i) Is it possible to extend the results proved in the paper for semigroups satisfying identities other than used in the paper?

(ii) The determination of all saturated classes of bands has been unanswered for long and an effort may be made in this direction.

(iii) To explore whether the extension of Theorem 1 for the class of weakly commutative semigroups which is wider and larger class of H-commutative semigroups is possible or not?

Data Availability

No data were used to support the study

Conflicts of Interest

The authors declare that they have no conflict of interests.

Acknowledgments

This research has been funded by Research Deanship at the University of Ha'il, Saudi Arabia, through project number RG-20 189.

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