Hindawi
Journal of Mathematics
Volume 2022, Article ID 2697155, 12 pages
https://doi.org/10.1155/2022/2697155

Research Article

A Stability Result for a Viscoelastic Wave Equation in the Presence of Finite and Infinite Memories

Adel M. Al-Mahdi,[1,2] **Mohammad M. Al-Gharabli**,[1,2] **Mohammad Kafini**,[2,3] **and Shadi Al-Omari**[1,2]

[1]*The Preparatory Year Program, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia*
[2]*The Interdisciplinary Research Center in Construction and Building Materials, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia*
[3]*Department of Mathematics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia*

Correspondence should be addressed to Shadi Al-Omari; shalomari@kfupm.edu.sa

Received 5 March 2022; Accepted 13 May 2022; Published 17 June 2022

Academic Editor: M. M. Bhatti

In this study, we consider a viscoelastic wave equation in the presence of finite and infinite memories. Under suitable conditions on the variable coefficients and for a wide class of relaxation functions, we establish an explicit and general decay result. Moreover, we use a weaker boundedness condition, on the initial data than the one used in the literature. The proof of our decay result is based on the multiplier method together with some convexity arguments. This study generalizes and improves previous literature outcomes.

1. Introduction

Let us consider an n-dimensional body occupies with a bounded open set $\Omega \subseteq \mathbb{R}^n\,(n \geq 1)$ with smooth boundary $\partial\Omega$. Let $x \longrightarrow u(x,t)$ be the position of the material particle x at time t so that $\forall x \in \Omega$ and $\forall t > 0$; the corresponding motion equation is

$$\begin{aligned} &u_{tt} - \Delta u + \int_0^t g_1(t-s)\operatorname{div}\left(a_1(x)\nabla u(s)\right)\mathrm{d}s \\ &\quad + \int_0^{+\infty} g_2(s)\operatorname{div}\left(a_2(x)\nabla u(t-s)\right)\mathrm{d}s = 0, \\ &u(x,t) = 0, \\ &u(x,-t) = u_0(x,t), \\ &u_t(x,0) = u_1(x). \end{aligned} \tag{1}$$

The functions g_1 and g_2 are called the relaxation (kernels) functions, and they are positive nonincreasing and defined on $\mathbb{R}^+$. The functions a_1 and a_2 are essentially bounded nonnegative defined on Ω. Here, u_0 and u_1 are the history and initial data. This model of materials consist of an elastic part (without memory) and a viscoelastic part, where the dissipation given by the memory is effective.

In this study, we are concerned with the above viscoelastic wave problem (1) and mainly interested in the asymptotic behavior of the solution u when t tends to infinity. In fact, we prove that the solution of the corresponding viscoelastic model decays to zero and no matter how small is the viscoelastic part of the material. Note that the above model is dissipative, and the dissipation is given by the memory term only and the memory is effective only in a part of the body. For materials with memory, the stress depends not only on the present values but also on the entire temporal history of the motion. Therefore, we have to prescribe the history of u before 0. Here, we assume that $u(x,-t) = u_0(x,t), \forall x \in \Omega, \forall t > 0$. Let us mention some other papers related to the problem we address. We start our literature review with the pioneer work of Dafermos [1], in 1970, where the author discussed a certain one-dimensional viscoelastic problem, established some existence results, and then proved that, for smooth monotone decreasing relaxation functions, the solutions go to zero as t goes to infinity. However, no rate of decay has been specified. In [2], a similar result, under a convexity condition on the kernel, has been established. After that, a great deal of attention has been devoted to the study of viscoelastic problems and many existence and long-time behavior results have been established. For example, Hrusa [3] considered a one-dimensional nonlinear viscoelastic equation of the form

$$u_{tt} - cu_{xx} + \int_0^t m(t-s)\left(\psi\left(u_x(x,s)\right)\right)_x \mathrm{d}s = f(x,t), \tag{2}$$

and proved several global existence results for large data. He also proved an exponential decay for strong solutions when $m(s) = e^{-s}$ and ψ satisfies certain conditions. Dassios and Zafiropoulos [4] studied a viscoelastic problem in $\mathbb{R}^3$ and proved a polynomial decay results for exponentially decaying kernels. After that, a very important contribution by Rivera was introduced. In 1994, Rivera [5] considered equations for linear isotropic homogeneous viscoelastic solids of integral type which occupy a bounded domain or the whole space $\mathbb{R}^n$ in the bounded domain case, and for exponentially decaying memory kernel and regular solutions, he showed that the sum of the first and the second energy decays exponentially. For the whole-space case and for exponentially decaying memory kernels, he showed that the rate of decay of energy is of algebraic type and depends on the regularity of the solution. This result was later generalized to a situation, where the kernel is decaying algebraically but not exponentially by Cabanillas and Rivera [6]. In [6], the authors considered the case of bounded domains as well as the case when the material is occupying the entire space and showed that the decay of solutions is also algebraic, at a rate which can be determined by the rate of decay of the relaxation function. This latter result was later improved by Baretto et al. [7], where equations related to linear viscoelastic plates were treated. Precisely, they showed that the solution energy decays at the same rate of the relaxation function.

Decay results for arbitrary growth of the damping term have been considered for the first time in the work of Lasiecka and Tataru [8]. They showed that the energy decays as fast as the solution of an associated differential equation whose coefficients depend on the damping term. Following the method of Lasiecka and Tataru [8], Liu and Zuazua [9] established decay rates for a nonlinear globally distributed damping for the wave equation with no growth assumptions at the origin. The proof of their result is based on constructing a convex function which captures the nonlinearity of the feedback at the origin and combines it, using Jensen's and Young's inequalities, with the use of equivalent energy which satisfies a differential equation, which however is not a priori dissipative. Alabau-Boussouira [10] used weighted integral inequalities and some convexity arguments to establish a semiexplicit formula which leads to decay rates of the energy in terms of the behavior of the nonlinear feedback close to the origin, for which the optimal exponential and polynomial decay rate estimates are only special cases. Alabau-Boussouira [11] presented a general approach based on convexity arguments to establish sharp optimal or quasi-optimal upper energy decay rates for finite and infinite dimensional vibrating damped systems.

Inspired by the experience with frictional damping initiated in the work of Lasiecka and Tataru [8], another step forward was done by Alabau-Boussouira and Cannaras [12] when they considered relaxation functions satisfying

$$g'(t) \leq -H(g(t)). \tag{3}$$

They established an explicit rate of decay under some constraints imposed on the function H.

For partially viscoelastic material, Rivera and Salvatierra [13] showed that the energy decays exponentially and provided the relaxation function decays in a similar fashion and the dissipation is acting on a part of the domain near to the boundary. See also, in this direction, the work of Rivera and Oquendo [14]. The uniform decay of solutions for the viscoelastic wave equation,

$$\begin{aligned} u_{tt} - k_0\Delta u &+ \int_0^t \operatorname{div}[a(x)g(t-\tau)\nabla u(\tau)]\mathrm{d}\tau \\ &+ b(x)h(u_t) + f(u) = 0, \end{aligned} \tag{4}$$

was investigated by Cavalcanti and Oquendo [15] where they considered the condition $a(x) + b(x) \geq \delta > 0$. They established exponential and polynomial stability results based on some conditions on g and the linearity of the function h. After that, Guesmia and Messaoudi [16] extended the work of [15], and they established a general decay result for (1) under the same conditions on a_1 and a_2 used in [15] and for some other conditions for the relaxation functions g_1 and g_2, from which the usual exponential and polynomial decay rates are only special cases. More precisely, they used the conditions $g_1'(t) \leq -\xi(t)g_1(t), \forall t \geq 0$, and

$$\int_0^{+\infty} \frac{g_2(s)}{G^{-1}\left(-g_2'(s)\right)}\mathrm{d}s + \sup_{s\in\mathbb{R}_+}\frac{g_2(s)}{G^{-1}\left(-g_2'(s)\right)} < +\infty, \tag{5}$$

such that

$$\begin{aligned} G(0) = G'(0) &= 0, \\ \lim_{t\longrightarrow+\infty} G'(t) &= +\infty, \end{aligned} \tag{6}$$

where $G\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is an increasing strictly convex function.

In the present work, we consider (1) and under the same conditions on a_1 and a_2 used in [15], and for a large class of the relaxation functions, we prove that problem (1) is stable. Under this class of relaxation functions g_i at infinity, we establish a relation between the decay rate of the solution and the growth of g_i at infinity. In fact, we extend the works of [15, 16]. More precisely, we assume that the relaxations functions g_1 and g_2 satisfy

$$g_i'(t) \leq -\xi_i(t)H_i\left(g_i(t)\right), \quad \forall t \geq 0. \tag{7}$$

In addition, we use a weaker boundedness condition on the history data used in earlier papers such as [17–20].

The paper is organized as follows. In Section 2, we present some material needed for our work. We establish some essential lemmas in Section 3. Section 4 contains the statement and the proof of our main result. We give some examples in Sections 5 to illustrate our energy decay result. Finally, we introduce conclusions and comparison study in Section 6.

2. Preliminaries

In this section, we present some material needed in the proof of our main result. Through this study, we use c to denote a positive generic constant. Now, we start with the following assumptions:

(A1) $g_i\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are differentiable nonincreasing functions such that, for any constant $\beta_0 > 0$ and $i = 1, 2$,

$$\begin{gathered}-\beta_0 g_i(s) \leq g_i'(s),\\ g_i(0) > 0,\\ 1 - \|a_1\|_\infty \int_0^{+\infty} g_1(s)\mathrm{d}s - \|a_2\|_\infty \int_0^{+\infty} g_2(s)\mathrm{d}s = \ell > 0.\end{gathered} \tag{8}$$

and there exist C^1 functions $H_i\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which are linear or strictly increasing and strictly convex C^2 functions on $(0, r]$ for some $r > 0$ with $H_i(0) = H_i'(0) = 0$, $\lim_{s\longrightarrow+\infty} H_i'(s) = +\infty$, $s \mapsto sH_i'(s)$, and $s \mapsto s(H_i')^{-1}(s)$ are convex on $(0, r]$. Moreover, there exist positive nonincreasing differentiable functions $\xi_i\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, such that

$$g_i'(t) \leq -\xi_i(t) H_i(g_i(t)), \quad \forall t \geq 0. \tag{9}$$

(A2) $a_i\colon \overline{\Omega} \longrightarrow \mathbb{R}^+$ are $C^1(\overline{\Omega})$ functions such that, for positive constants δ and a_0 and for $\Gamma_1, \Gamma_2 \subset \partial\Omega$ with meas $(\Gamma_i) > 0, i = 1, 2, \inf_{x\in\overline{\Omega}}(a_1(x) + a_2(x)) \geq \delta$ and

$$\begin{gathered}a_i = 0,\\ \text{or } \inf_{\Gamma_i} a_i(x) \geq 2a_0, \quad i = 1, 2.\end{gathered} \tag{10}$$

(A3) We assume that

$$\int_0^{+\infty} g_2(s) \|\nabla u_0(s)\|^2 \mathrm{d}s < +\infty. \tag{11}$$

Remark 1. The class of relaxation functions satisfying $(A1)$ in the present study is larger than the ones satisfying (5) and (6) used in [16]. In fact, the boundedness of the sup in (5), used in [16], can be interpreted as the inequality in $(A1)$ in the present paper (with $\xi = 1$). Conditions (5) and (6) used in [16] ask also the boundedness of the integral. Moreover, the classical of relaxation functions g that we consider in our example $g(t) = e^{-(t+1)^\beta}, \beta > 0$, satisfy both conditions $(A1)$ in our paper and the one in [16]. So, it is better to consider $(A1)$ used in the present study than the one used in [16].

Remark 2. Hypothesis $(A3)$ is needed for proving the existence and stability results. For the stability, if $(A3)$ holds, then h_0 which is defined in (25) is well defined, so inequality (53) makes sense. Moreover, Hypothesis $(A3)$ is weaker than the one used in [16, 18]; that is, there exists a positive constant M such that

$$\|\nabla u_0(s)\|^2 \leq M, \tag{12}$$

and our decay rate depends on the size of ∇u_0.

Remark 3. If $a_i \neq 0, i = 1, 2$, there exist neighborhoods w_i of $\Gamma_i, i = 1, 2$, such that

$$\inf_{\Omega\cap w_i} a_i(x) \geq a_0 > 0, \quad i = 1, 2. \tag{13}$$

As in [15, 16], let $d = \min\{a_0, \delta\}$ and let $\alpha_i \in C^1(\overline{\Omega}), i = 1, 2$, be such that

$$\begin{cases} 0 \leq \alpha_i(x) \leq a_i(x), \\ \alpha_i(x) = 0, \quad \text{if } a_i(x) \leq \dfrac{d}{4}, \\ \alpha_i(x) = a_i(x), \quad \text{if } a_i(x) \geq \dfrac{d}{2}. \end{cases} \tag{14}$$

Lemma 1. *The functions $\alpha_i, i = 1, 2$, are not identically zero and satisfy*

$$\alpha_1(x) + \alpha_2(x) \geq \frac{d}{2}. \tag{15}$$

Proof

(1) For $x \in \Omega \cap w_i$, we have $a_i(x) \geq a_0 \geq d$, which implies, by (14), that $\alpha_i(x) = a_i(x) \geq d$. Thus, α_i is not identically zero.

(2) If $a_1(x) \geq (d/2)$, then $\alpha_1(x) = a_1(x)$. Consequently, $\alpha_1(x) + \alpha_2(x) \geq a_1(x) \geq (d/2)$. If $a_1(x) < (d/2)$, then $a_2(x) > (d/2)$ which implies, by (14), $\alpha_2(x) = a_2(x) > (d/2)$. Consequently, $\alpha_1(x) + \alpha_2(x) > (d/2)$. This completes the proof.

The existence and uniqueness of the solution of problem (1) is given by the following theorem. □

Theorem 1. *If $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and hypotheses $(A1) - (A3)$ hold, then problem (1) has a unique solution in the class*

$$u \in C^0\left([0, \infty); H_0^1(\Omega)\right) \cap C^1\left([0, \infty); L^2(\Omega)\right). \tag{16}$$

The proof of the above theorem can be obtained, making use, for instance, of the Faedo–Galerkin method and considering standard arguments of density (see [15]).

We define the "modified" energy functional of the weak solution of problem (1) by

$$E(t)=\frac{1}{2}\int_\Omega u_t^2\mathrm{d}x+\frac{1}{2}\int_\Omega\left[1-a_1(x)\int_0^t g_1(s)\mathrm{d}s-a_2(x)\int_0^{+\infty}g_2(s)\mathrm{d}s\right]|\nabla u|^2\mathrm{d}x+\frac{1}{2}(g_1\circ\nabla u)(t)+\frac{1}{2}(g_2\circ\nabla u)(t), \tag{17}$$

where

$$\begin{aligned}(g_1\circ\nabla u)(t)&=\int_\Omega a_1(x)\int_0^t g_1(t-s)|\nabla u(t)-\nabla u(s)|^2\mathrm{d}s\mathrm{d}x,\\ (g_2\circ\nabla u)(t)&=\int_\Omega a_2(x)\int_0^{+\infty}g_2(s)|\nabla u(t)-\nabla u(t-s)|^2\mathrm{d}s\mathrm{d}x.\end{aligned} \tag{18}$$

Lemma 2. *The "modified" energy functional satisfies, along the solution of problem (1), the following:*

$$E'(t)=-\frac{1}{2}g_1(t)\int_\Omega|\nabla u(t)|^2\mathrm{d}x+\frac{1}{2}(g_1'\circ\nabla u)(t)+\frac{1}{2}(g_2'\circ\nabla u)(t)\le 0, \tag{19}$$

where

$$\begin{aligned}(g_1'\circ\nabla u)(t)&=\int_\Omega a_1(x)\int_0^t g_1'(t-s)|\nabla u(t)-\nabla u(s)|^2\mathrm{d}s\mathrm{d}x,\\ (g_2'\circ\nabla u)(t)&=\int_\Omega a_2(x)\int_0^{+\infty}g_2'(s)|\nabla u(t)-\nabla u(t-s)|^2\mathrm{d}s\mathrm{d}x.\end{aligned} \tag{20}$$

Proof. By multiplying the first equation in problem (1) by u_t and integrating over Ω, using integration by parts, hypotheses $(A1)$ and $(A2)$ and some manipulations as in [6,21] and others, we obtain (19) for regular solutions. This inequality remains valid for weak solutions by a simple density argument. □

3. Technical Lemmas

In this section, we present and prove additional lemmas that are pivotal for the principal result.

Lemma 3 (see [22]). *For $i=1,2$, we have*

$$\int_0^1\left(\int_0^t g_i(t-s)(\nabla u(s)-\nabla u(t-s))\mathrm{d}s\right)^2\mathrm{d}x\le C_\alpha(h_i\circ\nabla u)(t), \tag{21}$$

where, for any $0<\alpha<1$,

$$\begin{aligned}C_\alpha&=\int_0^\infty\frac{g_i^2(s)}{\alpha g_i(s)-g_i'(s)}\mathrm{d}s,\\ h_i(t)&=\alpha g_i(t)-g_i'(t).\end{aligned} \tag{22}$$

Furthermore, using the fact that

$$\frac{\alpha g_i^2(s)}{\alpha g_i(s)-g_i'(s)}<g_i(s), \tag{23}$$

and recalling the Lebesgue dominated convergence theorem, one can deduce that

$$\alpha C_\alpha=\int_0^\infty\frac{\alpha g_i^2(s)}{\alpha g_i(s)-g_i'(s)}\mathrm{d}s\longrightarrow 0\text{ as }\alpha\longrightarrow 0. \tag{24}$$

Lemma 4. *Assume that $(A1)-(A3)$ hold. There exists a positive constant M_1 such that*

$$\int_\Omega a_2(x)\int_t^{+\infty}g_2(s)|\nabla u(t)-\nabla u(t-s)|^2\mathrm{d}s\le M_1h_0(t), \tag{25}$$

where $h_0(t)=\int_0^{+\infty}g_2(t+s)(1+\|\nabla u_0(s)\|^2)ds$.

Proof. The proof is identical to the one in [23]. Indeed, we have

$$\begin{aligned}&\int_\Omega a_2(x)\int_t^{+\infty}g_2(s)|\nabla u(t)-\nabla u(t-s)|^2\mathrm{d}s\\ &\le 2\|a_2\|_\infty\|\nabla u(t)\|^2\int_t^{+\infty}g_2(s)\mathrm{d}s+2\|a_2\|_\infty\int_t^{+\infty}g_2(s)\|\nabla u(t-s)\|^2\mathrm{d}s\\ &\le 2\|a_2\|_\infty\sup_{s\ge0}\|\nabla u(s)\|^2\int_0^{+\infty}g_2(t+s)\mathrm{d}s+2\|a_2\|_\infty\int_0^{+\infty}g_2(t+s)\|\nabla u(-s)\|^2\mathrm{d}s\\ &\le\frac{4\|a_2\|_\infty E(s)}{\ell}\int_0^{+\infty}g_2(t+s)\mathrm{d}s+2\|a_2\|_\infty\int_0^{+\infty}g_2(t+s)\|\nabla u_0(s)\|^2\mathrm{d}s\\ &\le\frac{4\|a_2\|_\infty E(0)}{\ell}\int_0^{+\infty}g_2(t+s)\mathrm{d}s+2\|a_2\|_\infty\int_0^{+\infty}g_2(t+s)\|\nabla u_0(s)\|^2\mathrm{d}s\\ &\le M_1\int_0^{+\infty}g_2(t+s)\left(1+\|\nabla u_0(s)\|^2\right)\mathrm{d}s,\end{aligned} \tag{26}$$

where $M_1 = \max\{2\|a_2\|_\infty, (4\|a_2\|_\infty E(0)/\ell)\}$. □

Lemma 5 (see [16]). *Assume that* $(A1)-(A3)$ *hold; then, the functionals,*

$$\begin{aligned}\psi_1(t) &= \int_\Omega u u_t \mathrm{d}x,\\ \psi_2(t) &= -\int_\Omega \alpha_1(x) u_t \int_0^t g_1(t-s)(u(t)-u(s))\mathrm{d}s\mathrm{d}x,\\ \psi_3(t) &= -\int_\Omega \alpha_2(x) u_t \int_0^{+\infty} g_2(s)(u(t)-u(t-s))\mathrm{d}s\mathrm{d}x,\end{aligned} \tag{27}$$

satisfy, along the solution of (1) and for any $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ *and* $\forall t \geq 0$*; the following estimates*

$$\begin{aligned}\psi_1'(t) \leq{}& \int_\Omega u_t^2 \mathrm{d}x - \left[1 - \varepsilon_1 - \|a_1\|_\infty \int_0^{+\infty} g_1(s)\mathrm{d}s - \|a_2\|_\infty \int_0^{+\infty} g_2(s)\mathrm{d}s\right]\int_\Omega |\nabla u|^2 \mathrm{d}x\\ &+ \frac{cC_\alpha}{\varepsilon_1}(h_1 \circ \nabla u)(t) + (h_2 \circ \nabla u)(t),\end{aligned} \tag{28}$$

$$\begin{aligned}\psi_2'(t) \leq{}& -\left[\int_0^t g_1(s)\mathrm{d}s - \varepsilon_2\right]\int_\Omega \alpha_1(x) u_t^2 \mathrm{d}x + \frac{\varepsilon_3}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{c}{\varepsilon_2}(g_1' \circ \nabla u)(t)\\ &+ \frac{cC_\alpha}{\varepsilon_3}(h_1 \circ \nabla u)(t) + (h_2 \circ \nabla u)(t),\end{aligned} \tag{29}$$

and

$$\begin{aligned}\psi_3'(t) \leq{}& -\left[\int_0^{+\infty} g_2(s)\mathrm{d}s - \varepsilon_2\right]\int_\Omega \alpha_2(x) u_t^2 \mathrm{d}x\\ &+ \frac{\varepsilon_3}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{c}{\varepsilon_2}(g_2' \circ \nabla u)(t)\\ &+ \frac{cC_\alpha}{\varepsilon_3}(h_1 \circ \nabla u)(t) + (h_2 \circ \nabla u)(t).\end{aligned} \tag{30}$$

Lemma 6. *Assume that* $(A1)-(A3)$ *hold; then, for a suitable choice of* $N, M, m > 0$ *and for all* $t \geq 0$*, the functional,*

$$\mathscr{L}(t) := NE(t) + M\psi_1(t) + \psi_2(t) + \psi_3(t), \tag{31}$$

satisfies, for any $t > 0$

$$\mathscr{L}'(t) \leq -mE(t) + \frac{1}{4}(g_1 \circ \nabla u + g_2 \circ \nabla u)(t). \tag{32}$$

Proof. Let, for $t_0 > 0$ fixed, $g_0 := \min\{\int_0^{t_0} g_1(s)\mathrm{d}s, \int_0^{+\infty} g_2(s)\mathrm{d}s\}$. Then, direct differentiation of $\mathscr{L}$ and using (19), (28), (29), and (30) leads to

$$\begin{aligned}\mathscr{L}'(t) \leq{}& \left(\frac{N}{2} - \frac{c}{\varepsilon_2}\right)((g_1' \circ \nabla u)(t) + (g_2' \circ \nabla u)(t))\\ &- \int_\Omega [(g_0 - \varepsilon_2)(\alpha_1 + \alpha_2) - M] u_t^2 \mathrm{d}x\\ &+ \left(\frac{cC_\alpha}{\varepsilon_3} + \frac{McC_\alpha}{\varepsilon_1}\right)((h_1 \circ \nabla u)(t) + (h_2 \circ \nabla u)(t))\\ &- [(\ell - \varepsilon_1)M - \varepsilon_3]\int_\Omega |\nabla u|^2 \mathrm{d}x, \quad \forall t \geq t_0.\end{aligned} \tag{33}$$

Recalling that $g_i' = \alpha g_i - h_i$, where $i = 1, 2$. We obtain

$$\begin{aligned}\mathscr{L}'(t) \leq{}& \alpha\left(\frac{N}{2} - \frac{c}{\varepsilon_2}\right)(g_1 \circ \nabla u + g_2 \circ \nabla u)(t)\\ &- \left(\frac{N}{2} - \frac{c}{\varepsilon_2}\right)(h_1 \circ \nabla u + h_2 \circ \nabla u)(t)\\ &- \int_\Omega [(g_0 - \varepsilon_2)(\alpha_1 + \alpha_2) - M] u_t^2 \mathrm{d}x\\ &\times \left(\frac{cC_\alpha}{\varepsilon_3} + \frac{McC_\alpha}{\varepsilon_1}\right)((h_1 \circ \nabla u)(t) + (h_2 \circ \nabla u)(t))\\ &- [(\ell - \varepsilon_1)M - \varepsilon_3]\int_\Omega |\nabla u|^2 \mathrm{d}x, \quad \forall t \geq t_0.\end{aligned} \tag{34}$$

Simplifying the above estimate, we arrive at

$$\begin{aligned}\mathscr{L}'(t) &\leq \alpha\left(\frac{N}{2}-\frac{c}{\varepsilon_2}\right)(g_1\circ\nabla u + g_2\circ\nabla u)(t)\\ &-\left[\frac{N}{2}-\frac{c}{\varepsilon_2}-C_\alpha\left(\frac{c}{\varepsilon_3}+\frac{Mc}{\varepsilon_1}\right)\right](h_1\circ\nabla u + h_2\circ\nabla u)(t)\\ &-\int_\Omega [(g_0-\varepsilon_2)(\alpha_1+\alpha_2)-M]u_t^2\mathrm{d}x\\ &-[(\ell-\varepsilon_1)M-\varepsilon_3]\int_\Omega |\nabla u|^2\mathrm{d}x, \quad t\geq t_0.\end{aligned} \tag{35}$$

By using the fact that $(\alpha_1+\alpha_2)\geq (d/2)$ and choosing

$$\begin{aligned}\varepsilon_1 &= \frac{\ell}{2},\\ \varepsilon_2 &= \frac{g_0}{2},\\ M &= \frac{dg_0}{8},\\ \varepsilon_3 &< \frac{d\ell^2 dg_0}{16},\end{aligned} \tag{36}$$

we obtain

$$(\ell-\varepsilon_1)M-\varepsilon_3 \geq 4(1-\ell). \tag{37}$$

As a consequence of (24), there exists $0<\alpha_0<1$ such that if $\alpha<\alpha_0$, then

$$\alpha C_\alpha < \frac{1}{8[(c/\varepsilon_3)+(Mc/\varepsilon_1)]}. \tag{38}$$

It is clear that (38) gives

$$C_\alpha\left[\frac{N}{2}-\frac{c}{\varepsilon_2}-C_\alpha\left(\frac{c}{\varepsilon_3}+\frac{Mc}{\varepsilon_1}\right)\right] < \frac{N}{2}. \tag{39}$$

Choosing $\alpha=(1/2N)<\alpha_0$, then, for some $m>0$, we have

$$\mathscr{L}'(t)\leq -mE(t)+\frac{1}{4}\left((g_1\circ\nabla u)+(g_2\circ\nabla u)\right)(t). \tag{40}$$

Then, (32) is established. Moreover, one can choose N large enough so that $\mathscr{L}\sim E$. □

Lemma 7. *Assume that* $(A1)-(A3)$ *hold. Then, for all* $t\in\mathbb{R}^+$ *and fixed positive constant* m_0, *we have the following estimates:*

$$\int_0^t E(s)\mathrm{d}s < m_0\left(1+\int_0^t h_0(s)\mathrm{d}s\right), \tag{41}$$

where h_0 *is defined in (25).*

Proof. The proof of this lemma can be established by following the same arguments in [23, 24]. □

Lemma 8. *If* $(A1)-(A3)$ *are satisfied, then we have, for all* $t>0$ *and for* $i=1,2$, *the following estimates:*

$$\begin{aligned}&\int_\Omega a_i(x)\int_0^t g_i(s)|\nabla u(t)-\nabla u(t-s)|^2\mathrm{d}s\mathrm{d}x\\ &\leq \frac{1}{\Lambda_i(t)}H_i^{-1}\left(\frac{\Lambda_i(t)\mu_i(t)}{\xi_i(t)}\right),\end{aligned} \tag{42}$$

where $\Lambda_i(t)=(\lambda_i/1+\int_0^t h_0(s)ds)$, $\lambda_i\in(0,1)$, H_i *and* ξ_i *are introduced in* $(A1)$, *and*

$$\mu_i(t) := -\int_\Omega a_i(x)\int_0^t g_i'(s)|\nabla u(t)-\nabla u(t-s)|^2\mathrm{d}s\mathrm{d}x \leq -cE'(t). \tag{43}$$

Proof. To establish (42), we introduce the following functional:

$$\chi_i(t) := \Lambda_i(t)\int_\Omega a_i(x)\int_0^t |\nabla u(t)-\nabla u(t-s)|^2\mathrm{d}s\mathrm{d}x. \tag{44}$$

Then, using (17) and the fact that E is nonincreasing to get,

$$\begin{aligned}\chi_i(t) &\leq 2\Lambda_i(t)\left(\int_\Omega a_i(x)\int_0^t |\nabla u(t)|^2 + \int_\Omega a_i(x)\int_0^t |\nabla u(t-s)|^2\mathrm{d}s\mathrm{d}x\right)\\ &\leq \frac{4\Lambda_i(t)\|a_i\|_\infty}{\ell}\int_0^t (E(t)+E(t-s))\mathrm{d}s\\ &\leq \frac{8\Lambda_i(t)\|a_i\|_\infty}{\ell}\int_0^t E(s)\mathrm{d}s.\end{aligned} \tag{45}$$

Using (41) and the definition of $\Lambda_i(t)$, (45) becomes

$$\chi_i(t)\leq \frac{8m_0\lambda_i\|a_i\|_\infty}{\ell\left(1+\int_0^t h(s)\mathrm{d}s\right)}\left(1+\int_0^t h(s)\mathrm{d}s\right). \tag{46}$$

Therefore, we can choose $0<\lambda_i<(\ell/8m_0\|a_i\|_\infty)$ such that, for all $t>0$,

$$\chi_i(t)<1. \tag{47}$$

Without loss of the generality, for all $t>0$, we assume that $\chi_i(t)>0$; otherwise, we get an exponential decay from (32). The use of Jensen's inequality and using (43) and (47) give

$$
\begin{aligned}
\mu_1(t) &= \frac{1}{\Lambda_i(t)}\int_0^t \Lambda_i(t)\left(-g_i'(s)\right)\int_\Omega a_i(x)|\nabla u(t)-\nabla u(t-s)|^2\mathrm{d}x\mathrm{d}s\\
&\geq \frac{1}{\Lambda_i(t)}\int_0^t \Lambda_i(t)\xi_i(s)H_i\left(g_i(s)\right)\int_\Omega a_i(x)|\nabla u(t)-\nabla u(t-s)|^2\mathrm{d}x\mathrm{d}s\\
&\geq \frac{\xi_i(t)}{\Lambda_i(t)}\int_0^t H_i\left(\Lambda_i(t)g_i(s)\right)\int_\Omega a_i(x)|\nabla u(t)-\nabla u(t-s)|^2\mathrm{d}x\mathrm{d}s\\
&\geq \frac{\xi_i(t)}{\Lambda_i(t)}H_i\left(\Lambda_i(t)\int_0^t g_i(s)\int_\Omega a_i(x)|\nabla u(t)-\nabla u(t-s)|^2\mathrm{d}x\mathrm{d}s\right)\\
&= \frac{\xi_i(t)}{\Lambda_i}\overline{H_i}\left(\Lambda_i(t)\int_0^t g_i(s)\int_\Omega a_1(x)|\nabla u(t)-\nabla u(t-s)|^2\mathrm{d}x\mathrm{d}s\right).
\end{aligned}
\tag{48}
$$

Hence, (42) is established. □

4. Decay Result

In this section, we state and prove our main result and provide some examples to illustrate our decay results. Let us start introducing some functions and then establishing several lemmas needed for the proof of our main result. Now, for $i = 1, 2$, let us take

$$
\begin{aligned}
\xi &= \min\{\xi_i\},\\
\mu &= \max\{\mu_i\},\\
\lambda &= \max\{\lambda_i\},\\
G &= \left(H_1^{-1}+H_2^{-1}\right)^{-1}.
\end{aligned}
\tag{49}
$$

As in [23], we introduce the following functions:

$$
\begin{aligned}
G_1(t) &:= \int_t^r \frac{1}{sG'(s)}\mathrm{d}s,\\
G_5(t) &= G_1^{-1}\left(c_1\int_0^t \xi(s)\mathrm{d}s\right),
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
G_2(t) &= tG'(t),\\
G_3(t) &= t\left(G'\right)^{-1}(t),\\
G_4(t) &= G_3^*(t),
\end{aligned}
\tag{51}
$$

where G_3^* is the convex conjugate of G_3 in the sense of Young (see [25]). One can easily verify that G_1 is decreasing function over $(0, r]$, and G_2, G_3, and G_4 are convex and increasing functions on $(0, r]$. Furthermore, we introduce the class S of functions $\vartheta\colon \mathbb{R}_+ \longrightarrow (0, r]$ satisfying for fixed $c_1, c_2 > 0$:

$$
\vartheta \in C^1(\mathbb{R}_+), \quad \vartheta \leq 1, \vartheta' \leq 0,
\tag{52}
$$

and

$$
c_2 G_4\left[\frac{c}{d}\Lambda(t)h_0(t)\right] \leq c_1\left(G_2\left(\frac{G_5(s)}{\vartheta(s)}\right) - \frac{G_2\left(G_5(t)\right)}{\vartheta(t)}\right),
\tag{53}
$$

where h_0 is defined in (25) and $d > 0$.

Theorem 2. *Assume that $(A1)-(A3)$ hold; then, there exists a strictly positive constant C such that the solution of (1) satisfies, for all $t \geq 0$,*

$$
E(t) \leq \frac{CG_5(t)}{\vartheta(t)\Lambda(t)},
\tag{54}
$$

where G_5 and ϑ are defined in (50) and (52), respectively.

Proof. Using (25), (32), and (42), then, for some positive constant m and any $t \geq 0$, we obtain

$$
L'(t) \leq -mE(t) + \frac{c}{\Lambda(t)}G^{-1}\left(\frac{\Lambda(t)\mu(t)}{\xi(t)}\right) + ch_0(t).
\tag{55}
$$

Without loss of generality, one can assume that $E(0) > 0$. For $\varepsilon_0 < r$, let the functional $\mathcal{F}$ be defined by

$$
\mathcal{F}(t) := G'\left(\varepsilon_0\frac{E(t)\Lambda(t)}{E(0)}\right)L(t),
\tag{56}
$$

which satisfies $\mathcal{F} \sim E$. By noting that $G'' \geq 0$, $\Lambda' \leq 0$, and $E' \leq 0$, we obtain

$$
\begin{aligned}
\mathcal{F}'(t) &= \varepsilon_0\frac{(qE)'(t)}{E(0)}G''\left(\varepsilon_0\frac{E(t)\Lambda(t)}{E(0)}\right)L(t)\\
&\quad + G'\left(\varepsilon_0\frac{E(t)\Lambda(t)}{E(0)}\right)L'(t)\\
&\leq -mE(t)G'\left(\varepsilon_0\frac{E(t)\Lambda(t)}{E(0)}\right)\\
&\quad + \frac{c}{\Lambda(t)}G'\left(\varepsilon_0\frac{E(t)\Lambda(t)}{E(0)}\right)G^{-1}\left(\frac{\Lambda(t)\mu(t)}{\xi(t)}\right)\\
&\quad + ch_0(t)G'\left(\varepsilon_0\frac{E(t)\Lambda(t)}{E(0)}\right).
\end{aligned}
\tag{57}
$$

Let G^* be the convex conjugate of G in the sense of Young (see [25]); then,

$$
G^*(s) = s\left(G'\right)^{-1}(s) - G\left[\left(G'\right)^{-1}(s)\right], \quad \text{ifs} \in (0,\infty),
\tag{58}
$$

and G^* satisfies the following generalized Young inequality:

$$AB \le G^*(A) + G(B), \quad \text{if } A, B \in (0,\infty). \tag{59}$$

So, with $A = G'(\varepsilon_0 (E(t)\Lambda(t)/E(0)))$ and $B = G^{-1}(\Lambda(t)\mu(t)/\xi(t))$ and using (57)–(59), we arrive at

$$\begin{aligned}\mathcal{F}'(t) &\le -mE(t)G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right) + \frac{c}{\Lambda(t)} G^*\left(G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right)\right) + c\left(\frac{\mu(t)}{\xi(t)}\right) + ch_0(t)G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right)\\ &\le -mE(t)G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right) + c\varepsilon_0 \frac{E(t)}{E(0)} G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right) + c\left(\frac{\mu(t)}{\xi(t)}\right)\\ &\quad + ch_0(t)G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right).\end{aligned} \tag{60}$$

So, multiplying (60) by $\xi(t)$ and using (43) and the fact that $\varepsilon_0 (E(t)\Lambda(t)/E(0)) < r$ give

$$\begin{aligned}\xi(t)\mathcal{F}'(t) &\le -m\xi(t)E(t)G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right) + c\xi(t)\varepsilon_0 \frac{E(t)}{E(0)} G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right)\\ &\quad + c\mu(t)\Lambda(t) + c\xi(t)h_0(t)G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right)\\ &\le -\varepsilon_0\left(\frac{mE(0)}{\varepsilon_0} - c\right)\xi(t)\frac{E(t)}{E(0)} G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right) - cE'(t) + c\xi(t)h_0(t)G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right).\end{aligned} \tag{61}$$

Consequently, recalling the definition of G_2 and choosing ε_0 so that $k = ((mE(0)/\varepsilon_0) - c) > 0$, we obtain, for all $t \in \mathbb{R}_+$,

$$\begin{aligned}\mathcal{F}_1'(t) &\le -k\xi(t)\left(\frac{E(t)}{E(0)}\right)G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right)\\ &\quad + c\xi(t)h_0(t)G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right)\\ &= -k\frac{\xi(t)}{\Lambda(t)}G_2\left(\frac{E(t)\Lambda(t)}{E(0)}\right) + c\xi(t)h_0(t)G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right),\end{aligned} \tag{62}$$

where $\mathcal{F}_1 = \xi\mathcal{F} + cE \sim E$ and satisfies for some $\alpha_1, \alpha_2 > 0$:

$$\alpha_1 \mathcal{F}_1(t) \le E(t) \le \alpha_2 \mathcal{F}_1(t). \tag{63}$$

Since $G_2'(t) = G'(t) + tG''(t)$ and G is strictly increasing and strictly convex on $(0, r]$, we find that $G_2'(t), G_2(t) > 0$ on $(0, r]$. Using the general Young inequality (59) on the last term in (62) with $B = G'(\varepsilon_0 (E(t)\Lambda(t)/E(0)))$ and $A = [(c/d)h_0(t)]$, we have, for $d > 0$,

$$\begin{aligned}ch_0(t)G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right) &= \frac{d}{\Lambda(t)}\left[\frac{c}{d}\Lambda(t)h_0(t)\right]\left(G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right)\right)\\ &\le \frac{d}{\Lambda(t)}G_3\left(G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right)\right) + \frac{d}{\Lambda(t)}G_3^*\left[\frac{c}{d}\Lambda(t)h_0(t)\right]\\ &\le \frac{d}{\Lambda(t)}\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right)\left(G'\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right)\right) + \frac{d}{\Lambda(t)}G_4\left[\frac{c}{d}\Lambda(t)h_0(t)\right]\\ &\le \frac{d}{\Lambda(t)}G_2\left(\varepsilon_0 \frac{E(t)\Lambda(t)}{E(0)}\right) + \frac{d}{\Lambda(t)}G_4\left[\frac{c}{d}\Lambda(t)h_0(t)\right].\end{aligned} \tag{64}$$

Now, combining (62) and (64) and choosing d small enough so that $k_0 = (k - d) > 0$, we arrive at

$$\begin{aligned}\mathscr{F}_1'(t) &\leq -k\frac{\xi(t)}{\Lambda(t)}G_2\left(\varepsilon_0\frac{E(t)\Lambda(t)}{E(0)}\right) + \frac{d\xi(t)}{\Lambda(t)}G_2\left(\varepsilon_0\frac{E(t)\Lambda(t)}{E(0)}\right) + \frac{d\xi(t)}{\Lambda(t)}G_4\left[\frac{c}{d}\Lambda(t)h_0(t)\right]\\ &\leq -k_1\frac{\xi(t)}{\Lambda(t)}G_2\left(\varepsilon_0\frac{E(t)\Lambda(t)}{E(0)}\right) + \frac{d\xi(t)}{\Lambda(t)}G_4\left[\frac{c}{d}\Lambda(t)h_0(t)\right].\end{aligned} \tag{65}$$

Using the equivalent property in (63) and the increasing of G_2, we have

$$G_2\left(\varepsilon_0\frac{E(t)\Lambda(t)}{E(0)}\right) \geq G_2\left(d_0\mathscr{F}_1(t)\Lambda(t)\right). \tag{66}$$

Letting $\mathscr{F}_2(t) \coloneqq d_0\mathscr{F}_1(t)\Lambda(t)$ and recalling $\Lambda' \leq 0$, then we arrive at, for some $c_1, c_2 > 0$,

$$\mathscr{F}_2'(t) \leq -c_1\xi(t)G_2\left(\mathscr{F}_2(t)\right) + c_2\xi(t)G_4\left[\frac{c}{d}\Lambda(t)h_0(t)\right]. \tag{67}$$

Since $d_0\Lambda(t)$ is nonincreasing, using the equivalent property $\mathscr{F}_1 \sim E$ implies that there exists $b_0 > 0$ such that $\mathscr{F}_2(t) \geq b_0E(t)\Lambda(t)$. Let $t \in \mathbb{R}_+$ and $\vartheta(t)$ satisfying (52) and (53). If

$$b_0\Lambda(t)E(t) \leq 2\frac{G_5(t)}{\vartheta(t)}, \tag{68}$$

then, we have

$$E(t) \leq \frac{2}{b_0}\frac{G_5(t)}{\vartheta(t)\Lambda(t)}. \tag{69}$$

If

$$b_0\Lambda(t)E(t) > 2\frac{G_5(t)}{\vartheta(t)}, \tag{70}$$

then, for any $0 \leq s \leq t$, we obtain

$$b_0q(s)E(s) > 2\frac{G_5(t)}{\vartheta(t)}, \tag{71}$$

since $\Lambda(t)E(t)$ is a nonincreasing function. Therefore, for any $0 \leq s \leq t$, we have

$$\mathscr{F}_2(s) > 2\frac{G_5(t)}{\vartheta(t)}. \tag{72}$$

Using (72), recalling the definition of G_2, the fact that G_2 is convex, and $0 < \vartheta \leq 1$, we have, for any $0 \leq s \leq t$ and $0 < \epsilon_1 \leq 1$,

$$\begin{aligned}G_2\left(\varepsilon_1\vartheta(s)\mathscr{F}_2(s) - \varepsilon_1G_5(s)\right) &\leq \varepsilon_1\vartheta(s)G_2\left(\mathscr{F}_2(s) - \frac{G_5(s)}{\vartheta(s)}\right)\\ &\leq \varepsilon_1\vartheta(s)\mathscr{F}_2(s)G'\left(\mathscr{F}_2(s) - \frac{G_5(s)}{\vartheta(s)}\right) - \varepsilon_1\vartheta(s)\frac{G_5(s)}{\vartheta(s)}G'\left(\mathscr{F}_2(s) - \frac{G_5(s)}{\vartheta(s)}\right)\\ &\leq \varepsilon_1\vartheta(s)\mathscr{F}_2(s)G'\left(\mathscr{F}_2(s)\right) - \varepsilon_1\vartheta(s)\frac{G_5(s)}{\vartheta(s)}G'\left(\frac{G_5(s)}{\vartheta(s)}\right).\end{aligned} \tag{73}$$

Now, we let

$$\mathscr{F}_3(s) = \varepsilon_1\vartheta(s)\mathscr{F}_2(s) - \varepsilon_1G_5(s), \tag{74}$$

where ε_1 small enough so that $\mathscr{F}_3(0) \leq 1$. Then, (73) becomes, for any $0 \leq s \leq t$,

$$G_2\left(\mathscr{F}_3(s)\right) \leq \varepsilon_1\vartheta(s)G_2\left(\mathscr{F}_2(s)\right) - \varepsilon_1\vartheta(s)G_2\left(\frac{G_5(s)}{\vartheta(s)}\right). \tag{75}$$

Furthermore, we have

$$\mathscr{F}_3'(t) = \varepsilon_1\vartheta'(t)\mathscr{F}_2(t) + \varepsilon_1\vartheta(s)\mathscr{F}_2'(t) - \varepsilon_1G_5'(t). \tag{76}$$

Since $\vartheta' \leq 0$ and using (67), then, for any $0 \leq s \leq t$, $0 < \varepsilon_1 \leq 1$, we obtain

$$\begin{aligned}\mathscr{F}_3'(t) &\leq \varepsilon_1\vartheta(t)\mathscr{F}_2'(t) - \varepsilon_1G_5'(t)\\ &\leq -c_1\varepsilon_1\xi(t)\vartheta(t)G_2\left(\mathscr{F}_2(t)\right)\\ &\quad + c_2\varepsilon_1\xi(t)\vartheta(s)G_4\left[\frac{c}{d}\Lambda(t)h_0(t)\right] - \varepsilon_1G_5'(t).\end{aligned} \tag{77}$$

Then, using (53) and (75), we obtain

$$\begin{aligned}\mathscr{F}_3'(t) \leq& -c_1\xi(t)G_2\left(\mathscr{F}_3(t)\right) + c_2\varepsilon_1\xi(t)\vartheta(t)G_4\left[\frac{c}{d}\Lambda(t)h_0(t)\right]\\ &- c_1\varepsilon_1\xi(t)\vartheta(t)G_2\left(\frac{G_5(t)}{\vartheta(t)}\right) - \varepsilon_1G_5'(t).\end{aligned} \tag{78}$$

From the definition of G_1 and G_5, we have

$$G_1(G_5(s)) = c_1\int_0^s \xi(\tau)\mathrm{d}\tau. \quad (79)$$

Hence,

$$G_5'(s) = -c_1\xi(s)G_2(G_5(s)). \quad (80)$$

Now, we have

$$\begin{aligned}
&c_2\varepsilon_1\xi(t)\vartheta(t)G_4\left[\frac{c}{d}\Lambda(t)h_0(t)\right] - c_1\varepsilon_1\xi(t)\vartheta(t)G_2\left(\frac{G_5(t)}{\vartheta(t)}\right) - \varepsilon_1 G_5'(t)\\
&= c_2\varepsilon_1\xi(t)\vartheta(t)G_4\left[\frac{c}{d}\Lambda(t)h_0(t)\right] - \varepsilon_1\xi(t)\vartheta(t)G_2\left(\frac{G_5(t)}{\vartheta(t)}\right) + c\varepsilon_1\xi(t)G_2(G_5(t))\\
&= \varepsilon_1\xi(t)\vartheta(t)\left(c_2G_4\left[\frac{c}{d}\Lambda(t)h_0(t)\right] - c_1G_2\left(\frac{G_5(t)}{\vartheta(t)}\right) + c_1\frac{G_2(G_5(t))}{\vartheta(t)}\right).
\end{aligned} \quad (81)$$

Then, according to (53), we obtain

$$\varepsilon_1\xi(t)\vartheta(t)\left(c_2G_4\left[\frac{c}{d}\Lambda(t)h_0(t)\right] - c_1G_2\left(\frac{G_5(t)}{\vartheta(t)}\right) - c_1\frac{G_2(G_5(t))}{\vartheta(t)}\right) \le 0. \quad (82)$$

Then, (78) gives

$$\mathcal{F}_3'(t) \le -c_1\xi(t)G_2(\mathcal{F}_3(t)). \quad (83)$$

Thus, from (83) and the definition of G_1 and G_2 in (50) and (51), we obtain

$$(G_1(\mathcal{F}_3(t)))' \ge c_1\xi(t). \quad (84)$$

Integrating (84) over $[0,t]$, we obtain

$$G_1(\mathcal{F}_3(t)) \ge c_1\int_0^t \xi(s)\mathrm{d}s + G_1(\mathcal{F}_3(0)). \quad (85)$$

Since G_1 is decreasing, $\mathcal{F}_3(0) \le 1$ and $G_1(1) = 0$, then

$$\mathcal{F}_3(t) \le G_1^{-1}\left(c_1\int_0^t \xi(s)\mathrm{d}s\right) = G_5(t). \quad (86)$$

Recalling that $\mathcal{F}_3(t) = \varepsilon_1\vartheta(t)\mathcal{F}_2(t) - \varepsilon_1G_5(t)$, we have

$$\mathcal{F}_2(t) \le \frac{(1+\varepsilon_1)}{\varepsilon_1}\frac{G_5(t)}{\vartheta(t)}. \quad (87)$$

Similarly, recall that $\mathcal{F}_2(t) \coloneqq d_0\mathcal{F}_1(t)\Lambda(t)$; then,

$$\mathcal{F}_1(t) \le \frac{(1+\varepsilon_1)}{d_0\varepsilon_1}\frac{G_5(t)}{\vartheta(t)\Lambda(t)}. \quad (88)$$

Since $\mathcal{F}_1 \sim E$, then, for some $b>0$, we have $E(t) \le b\mathcal{F}_1$, which gives

$$E(t) \le \frac{b(1+\varepsilon_1)}{d_0\varepsilon_1}\frac{G_5(t)}{\vartheta(t)\Lambda(t)}. \quad (89)$$

From (69) and (89), we obtain the following estimate:

$$E(t) \le c_3\left(\frac{G_5(t)}{\vartheta(t)\Lambda(t)}\right), \quad (90)$$

where $c_3 = \max\{(2/b_0), (b(1+\varepsilon_1)/d_0\varepsilon_1)\}$. □

5. Examples

Let $g(t) = (a/(1+t)^\beta)$, where $\beta > 1$ and $0 < a < \beta - 1$ so that $(A1)$ is satisfied. In this case $\xi(t) = \beta a^{(-1/\beta)}$ and $G(t) = t^{(\beta+1/\beta)}$. Then, there exist positive constants $a_i (i = 0, \ldots, 3)$ depending only on a, β such that

$$\begin{aligned}
G_4(t) &= a_0 t^{\beta+1/\beta},\\
G_2(t) &= a_1 t^{\beta+1/\beta},\\
G_1(t) &= a_2\left(t^{(-1/\beta)} - 1\right),\\
G_5(t) &= (a_3 t + 1)^{-\beta}.
\end{aligned} \quad (91)$$

We will discuss two cases.

Case 1. If

$$m_0(1+t)^r \le 1 + \|\nabla u_0\|^2 \le m_1(1+t)^r, \quad (92)$$

where $0 < r < \beta - 1$ and $m_0, m_1 > 0$, then we have, for some positive constants $a_i (i = 4, \ldots, 7)$, depending only on a, β, m_0, m_1, r, the following:

$$a_4(1+t)^{-\beta+1+r} \le h_0(t) \le a_5(1+t)^{-\beta+1+r}, \quad (93)$$

$$\frac{\lambda}{\Lambda(t)} \ge a_6\begin{cases} 1+\ln(1+t), & \beta - r = 2,\\ 2, & \beta - r > 2,\\ (1+t)^{-\beta+r+2}, & 1 < \beta - r < 2, \end{cases} \quad (94)$$

$$\frac{\lambda}{\Lambda(t)} \le a_7\begin{cases} 1+\ln(1+t), & \beta - r = 2,\\ 2, & \beta - r > 2,\\ (1+t)^{-\beta+r+2}, & 1 < \beta - r < 2. \end{cases} \quad (95)$$

We notice that condition (53) is satisfied if

$$(t+1)^\beta\Lambda(t)h_0(t)\vartheta(t) \le a_8\left(1 - (\vartheta)^{1/\beta}\right)^{(\beta/\beta+1)}, \quad (96)$$

where $a_8 > 0$ depending on a, β, c_1, and c_2. Choosing $\vartheta(t)$ as the following,

$$\vartheta(t) = \lambda \begin{cases} (1+t)^{-p}, p = r+1 & \beta - r \geq 2, \\ (1+t)^{-p}, p = \beta - 1, & 1 < \beta - r < 2, \end{cases} \tag{97}$$

so that (52) is valid. Moreover, using (93) and (94), we see that (96) is satisfied if $0 < \lambda \leq 1$ is small enough, and then, (53) is satisfied. Hence, (54) and (95) imply that, for any $t \in \mathbb{R}_+$,

$$E(t) \leq a_9 \begin{cases} (1 + \ln(1+t))(1+t)^{-(\beta - r - 1)}, & \beta - r = 2, \\ (1+t)^{-(\beta - r - 1)}, & \beta - r > 2 \text{ or } 1 < \beta - r < 2. \end{cases} \tag{98}$$

Thus, estimate (98) gives $\lim_{t \longrightarrow +\infty} E(t) = 0$.

Case 2. If $m_0 \leq 1 + \|\nabla u_0\|^2 \leq m_1$. That is, $r = 0$ in (92), as it was assumed in many papers in the literature. So, it is clear that estimate (98) gives $\lim_{t \longrightarrow +\infty} E(t) = 0$ when $r = 0$.

6. Conclusion

Comparing our result with some earlier results in literature, we find that the result in this study improves and extends some earlier decay results in the literature as follows:

(i) Our decay result is obtained with more general relaxation function that is used in the literature such as the one in [15,16].

(ii) If $m_0 (1+t)^r \leq 1 + \|\nabla u_0\|^2 \leq m_1 (1+t)^r$, where $0 < r < \beta - 1$ and $m_0, m_1 > 0$, our estimate (98) extends and improves the decay rate $(1+t)^{\widetilde{\beta}}$ obtained in [17] where $0 < \widetilde{\beta}$ is small enough and $\beta > 2$.

(iii) If $m_0 \leq 1 + \|\nabla u_0\|^2 \leq m_1$, our estimate (98) gives a better decay rate than the ones $(1+t)^{\widetilde{\beta}}$, where $0 < \widetilde{\beta} < ((q-1)/2)$ is obtained in [18], and also better than the decay rate $(1+t)^{\widetilde{\beta}}$, where $0 < \widetilde{\beta}$ is small enough and obtained in [19]. Moreover, our estimate (98) provides a better decay than $(1+t)^{(-(q^2 - q - 1)/q)}$ obtained in [20].

Data Availability

No data were used to support this study.

Disclosure

A version of this work has been presented as Arxive (https://Arxiv.org) [26].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors thank King Fahd University of Petroleum and Minerals (KFUPM) for its continuous supports. This work was supported by KFUPM, under project no SB201026.

References

[1] C. M. Dafermos, "Asymptotic stability in viscoelasticity," *Archive for Rational Mechanics and Analysis*, vol. 37, no. 4, pp. 297–308, 1970.
[2] C. M. Dafermos, "An abstract volterra equation with applications to linear viscoelasticity," *Journal of Differential Equations*, vol. 7, no. 3, pp. 554–569, 1970.
[3] W. J. Hrusa, "Global existence and asymptotic stability for a semilinear hyperbolic volterra equation with large initial data," *SIAM Journal on Mathematical Analysis*, vol. 16, no. 1, pp. 110–134, 1985.
[4] G. Dassios and F. Zafiropoulos, "Equipartition of energy in linearized 3-d viscoelasticity," *Quarterly of Applied Mathematics*, vol. 48, no. 4, pp. 715–730, 1990.
[5] J. E. Muñoz Rivera and H. D. Fernández Sare, "Stability of timoshenko systems with past history," *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 482–502, 2008.
[6] J. E. Muñoz Rivera and E. C. Lapa, "Decay rates of solutions of an anisotropic inhomogeneousn-dimensional viscoelastic equation with polynomially decaying kernels," *Communications in Mathematical Physics*, vol. 177, no. 3, pp. 583–602, 1996.
[7] J. E. M. Rivera, E. C. Lapa, and R. Barreto, "Decay rates for viscoelastic plates with memory," *Journal of Elasticity*, vol. 44, no. 1, pp. 61–87, 1996.
[8] I. Lasiecka and D. Tataru, "Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping," *Differential Integral Equations*, vol. 6, no. 3, pp. 507–533, 1993.
[9] W.-J. Liu and E. Zuazua, "Decay rates for dissipative wave equations," *Ricerche di Matematica*, vol. 48, pp. 61–75, 1999.
[10] F. Alabau-Boussouira, "Convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems," *Applied Mathematics and Optimization*, vol. 51, no. 1, pp. 61–105, 2005.
[11] F. Alabau-Boussouira, "A unified approach via convexity for optimal energy decay rates of finite and infinite dimensional vibrating damped systems with applications to semi-discretized vibrating damped systems," *Journal of Differential Equations*, vol. 248, no. 6, pp. 1473–1517, 2010.
[12] F. Alabau-Boussouira and P. Cannarsa, "A general method for proving sharp energy decay rates for memory-dissipative evolution equations," *Comptes Rendus Mathématique*, vol. 347, no. 15-16, pp. 867–872, 2009.
[13] J. E. Muñoz Rivera and A. Peres Salvatierra, "Asymptotic behaviour of the energy in partially viscoelastic materials," *Quarterly of Applied Mathematics*, vol. 59, no. 3, pp. 557–578, 2001.
[14] J. E. M. Rivera and H. P. Oquendo, "Exponential stability to a contact problem of partially viscoelastic materials," *Journal of Elasticity*, vol. 63, no. 2, pp. 87–111, 2001.
[15] M. M. Cavalcanti and H. P. Oquendo, "Frictional versus viscoelastic damping in a semilinear wave equation," *SIAM Journal on Control and Optimization*, vol. 42, no. 4, pp. 1310–1324, 2003.
[16] A. Guesmia and S. A. Messaoudi, "A general decay result for a viscoelastic equation in the presence of past and finite history memories," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 1, pp. 476–485, 2012.
[17] A. Guesmia and N. E. Tatar, "Some well-posedness and stability results for abstract hyperbolic equations with infinite

memory and distributed time delay," *Communications on Pure & Applied Analysis*, vol. 14, no. 2, pp. 457–491, 2015.

[18] A. Guesmia, "Asymptotic stability of abstract dissipative systems with infinite memory," *Journal of Mathematical Analysis and Applications*, vol. 382, no. 2, pp. 748–760, 2011.

[19] A. Guesmia and S. Messaoudi, "A new approach to the stability of an abstract system in the presence of infinite history," *Journal of Mathematical Analysis and Applications*, vol. 416, no. 1, pp. 212–228, 2014.

[20] A. M. Al-Mahdi and M. M. Al-Gharabli, "New general decay results in an infinite memory viscoelastic problem with nonlinear damping," *Boundary Value Problems*, vol. 2019, no. 1, 15 pages, 2019.

[21] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and J. A. Soriano, "Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping," *Electronic Journal of Differential Equations (EJDE*, vol. 2002, no. 44, pp. 1–14, 2002.

[22] M. I. Mustafa, "Optimal decay rates for the viscoelastic wave equation," *Mathematical Methods in the Applied Sciences*, vol. 41, no. 1, pp. 192–204, 2018.

[23] A. Guesmia, "New general decay rates of solutions for two viscoelastic wave equations with infinite memory," *Mathematical Modelling and Analysis*, vol. 25, no. 3, pp. 351–373, 2020.

[24] A. M. Al-Mahdi, "General stability result for a viscoelastic plate equation with past history and general kernel," *Journal of Mathematical Analysis and Applications*, vol. 490, no. 124216, 19 pages, 2020.

[25] V. I. Arnol'd, "Mathematical methods of classical mechanics," *Springer Science & Business Media*, vol. 60, 2013.

[26] A. M. Al-Mahdi, M. M. Al-Gharabli, M. Kafini, and S. Al-Omari, "Stability result for a viscoelastic wave equation in the presence of finite and infinite memories," 2021, https://arxiv.org/abs/2112.13256.