Research Article

Orderings of Extreme Claim Amounts from Heterogeneous and Dependent Weibull-G Insurance Portfolios

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1. Introduction

In the context of insurance, annual premium is the amount paid by the policyholder to purchase the insurance policy in order to transfer risk. In determining the proper annual premium, the stochastic behavior of the extreme claim amounts is critical. In need, the largest claim amount is contained in the LCR (largest claim reinsurance) and the ECOMOR (excedent du cout moyen relatif) reinsurance treaties, which are widely used in pricing insurance portfolios of the policyholder (cf. Ammeter [1] and Hansjörg Albrecher and Teugels [2]). Consider an insurance portfolio consisting of \( n \) business. Let \( X_1, X_2, \ldots, X_n \) be a set of nonnegative random variables with \( X_i \) denoting the \( i \)th claim size in given insurance portfolio, for \( i = 1, 2, \ldots, n \) and let \( I_1, I_2, \ldots, I_n \) be a group of Bernoulli random variables, independent of the \( X_i \)'s, with \( I_i \) indicating whether the claim \( X_i \) occurs or not and such that \( E[I_i] = p_i \) for \( i = 1, 2, \ldots, n \). Thus, the risk vector of the insurance portfolio can be expressed as \( (I_1X_1, I_2X_2, \ldots, I_nX_n) \). The past decades have witnessed the rich developments in the investigation of stochastic properties of different quantities for the individual risk model. For example, Zhang et al. [3, 4] investigated how the heterogeneity among occurrence probabilities and claim severities affects the aggregate claim numbers and aggregate claim amount for an insurance portfolio. Zhang et al. [5] established sufficient conditions for comparing extreme claim amounts arising from two sets of heterogeneous Weibull-G insurance portfolios with respect to the usual stochastic order under Archimedean copula dependence. Second, we establish some sufficient conditions to compare the smallest claim amounts in the sense of the usual stochastic and hazard rate orders when the claim severities are independent or dependent heterogeneous Weibull-G insurance portfolios. Finally, to illustrate the key theoretical insights, some numerical examples are offered.

Many scholars have been studying insurance and reinsurance in recent decades, particularly the largest and smallest claim amounts. For example, Barmalzan and Payandeh Najafabadi [6] investigated the ordering properties of the smallest claim amounts from a set of Weibull heterogeneous portfolios in the sense of the convex transform order and the right spread order. Barmalzan et al. [7] discussed the likelihood ratio order and dispersive order between the smallest claim amounts from two sets of independent heterogeneous Weibull claims. Barmalzan et al. [8] investigated the smallest claim amounts in terms of the usual stochastic and hazard rate orders, as well as established the usual stochastic order between the largest claim amounts for a general scale model. Balakrishnan et al. [9] gave the stochastic orderings between the largest claim amounts and ranges from two sets of independent heterogeneous portfolio risks. Zhang et al. [5] established sufficient conditions
to stochastically compare the largest and smallest claim amounts arising from two sets of heterogeneous insurance portfolios in the sense of the usual stochastic, hazard rate, likelihood ratio, right-spread, and convex transform orders. Nadeb et al. [13] obtained some stochastic comparisons results between two heterogeneous portfolio risks with chain majorized parameters on the aggregate claim amount when both claim occurrence probabilities and claim severities are dependent. Torrado and Navarro [19] investigated the extreme claim amounts in dependent individual risk models. Inspired by these outstanding works, we will investigate the stochastic behaviors of the largest and smallest claim amounts with dependent insurance portfolios, and provide some quite significantly perceptions and insights into determining the annual premium for an insurant, we try to establish the stochastic comparisons of the largest and smallest claim amounts from two different sets of the dependent and heterogeneous Weibull-G random variables under Archimedean copula.

The remainder of this paper is proceed as follows. In Section 2, we first introduce some key definitions and notation pertinent to stochastic orders, multivariate majorization, vector majorization, related orders and copulas. Section 3 present the usually stochastic order of the independent largest claim amounts from two sets of independent heterogeneous portfolio risks with chain majorized parameters. Section 4, we develop some stochastic comparisons of the smallest claim amounts in two cases: (i) when claim severities are independent, and then (ii) when claim severities are Archimedean copula dependent. Conclusions and discussions are given in Section 6.

2. Preliminaries

In this section, we recall some basic definitions and lemmas, which are useful to obtain the main result in subsequent developments. Throughout this paper, the terms increasing and decreasing are used for monotone non-decreasing and monotone nonincreasing, respectively, and “" means that both sides of the equality have the same sign. We denote \( \mathbb{R} = ( -\infty, +\infty ), \mathbb{R}^+ = ( 0, +\infty ), \) and \( h^{-1} \) the inverse function of strictly monotone function \( h, f, f' \) and \( f^{(k)} \) the first, second and \( k \) th derivatives of a differentiable function \( f \), respectively. All random variables mentioned are assumed to be absolutely continuous.

2.1. Stochastic Orders. For a nonnegative random variable \( X \), denote the distribution function, survival function, density function, hazard rate function and reversed hazard rate function of \( X \) by \( F_X(x), F_{1X}(x), f_X(x), h_X(x) \) and \( \overline{F}_X(x), \) respectively. Then, some fundamental stochastic orders are follows.

Definition 1 (Definition 1.A.1 on Page 3, 1.B.1 on Page 16, and 1.B.6 on Page 36 of Shaked and Shanthikumar [20]). \( X \) is said to be larger than \( Y \) in the

(i) Usual stochastic order (denoted by \( X \geq_s Y \)) if \( F_X(x) \geq F_Y(x) \) for all \( x \in \mathbb{R}^+ \)

(ii) Hazard rate order (denoted by \( X \geq_{hr} Y \)) if and only if \( f_X(x)/f_Y(x) \) is increasing in \( x \in \mathbb{R}^+ \), or equivalently, \( h_X(x) \leq h_Y(x) \) for all \( x \in \mathbb{R}^+ \)

The hazard rate order is generally known to imply the usual stochastic order, while the opposite is not true.
Interested readers may refer to Müller and Stoyan [21] and Shaked and Shanthikumar [20] for comprehensive discussions on various stochastic orders and their applications in different contexts. Besides, the hazard function also known as Mills’s ratio (cf. Škanata [22]) in the context of actuarial risk science, and the hazard function are seen as the fundamental feature that may be used to describe probabilistic component of risk and may be used to describe time evolution of the probabilistic component of risk. For more discussion of the hazard function in actuarial risk, one can refer to the work of Škanata [22].

2.2. Majorizations. The concept of the majorization order is a key tool in establishing various inequalities arising from actuarial science, reliability theory and other applied probability fields. Let $x_{1:n} \leq x_{2:n} \leq \ldots \leq x_{n:n}$ be the increasing arrangement of the components of the vector $x = (x_1, x_2, \ldots, x_n)$, then we have the following definition.

**Definition 2.** (Definition A.1 on Page 8 and A.2 on Page 42 of Marshall et al. [23]). A vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is said to

(i) Majorize another vector $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, denoted by $x \succ y$, if $\sum_{i=1}^j x_i \leq \sum_{i=1}^j y_i$ for $j = 1, 2, \ldots, n - 1$, and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$.

(ii) Weakly submajorize another vector $y \in \mathbb{R}^n$, denoted by $x \succeq y$, if $\sum_{i=j}^n x_i \geq \sum_{i=j}^n y_i$ for every $j = 1, 2, \ldots, n$.

It is clear that $x \succ y$ implies $x \succeq y$. For an elaborate discussion on the theory of the majorization order and their applications, one may refer to Marshall et al. [23] and Kundu et al. [24].

**Definition 3** (Definition 3.1 on Page 80 of Marshall et al. [23]). We say that a real-valued function $\phi$, defined on a set $\mathcal{A} \subseteq \mathbb{R}^n$, is Schur-convex (Schur-concave) on $\mathcal{A}$, if and only if $x \succ y$ implies $\phi(x) \geq [\leq] \phi(y)$, for any $x, y \in \mathcal{A}$.

**Lemma 1** (Theorem A.4 on Page 84 of Marshall et al. [23]). Consider the real-valued continuously differentiable function $\phi$ on $\mathcal{J}^n$, where $\mathcal{J} \subseteq \mathbb{R}$ is an open interval. Then, $\phi$ is Schur-convex (Schur-concave) on $\mathcal{J}^n$ if and only if $\phi$ is symmetric on $\mathcal{J}^n$, and for all $i \neq j$ and all $z \in \mathcal{J}^n$,

$$z_i - z_j \left( \frac{\partial \phi(z)}{\partial z_i} - \frac{\partial \phi(z)}{\partial z_j} \right) \geq [\leq] 0,$$

where $\partial \phi(x)/\partial z_i$ denotes the partial derivative of $\phi$ with respect to its $i$th argument.

**Lemma 2** (Definition A.4 on Page 15 Marshall et al. [23]). Consider a real-valued function $\phi$, defined on a set $\mathcal{A} \subseteq \mathbb{R}^n$. Then, $x \succ y$ implies $\phi(x) \geq [\leq] \phi(y)$ if and only if $\phi$ is increasing (decreasing) and Schur-convex (Schur-concave) on $\mathcal{A}$.

**Lemma 3** (Definition B.2 on Page 90 Marshall et al. [23]). Consider the composition function $\psi(x) = \phi(g(x_1), g(x_2), \ldots, g(x_n))$, where $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$ and $g: \mathbb{R}^n \rightarrow \mathbb{R}$ are two real-valued functions. If $\phi$ is decreasing and Schur-convex in $x$, and $g$ is concave, then $\psi(x)$ is Schur-convex in $x$.

Now, we describe the notion of the majorization on matrices. A square matrix $I_n$, of order $n$, is said to be a permutation matrix if each row and column has a single entry as 1, and all other entries are zero. It is obvious that a permutation matrix can be constructed by interchanging rows (or columns) of the $n \times n$ identity matrix $I_n$. The $T$-transform matrix has the form

$$T_w = wI_n + (1 - w)I_n, \quad 0 \leq w \leq 1,$$

where $I_n$ is the identity matrix and $I_n$ is a permutation matrix that only interchanges two co-ordinates. It should be mentioned here that all permutation matrices used in the $T$-transform matrix are matrices that interchange only two columns of the identity matrix $I_n$. For two $T$-transform matrices $T_w = wI_n + (1 - w)I_n$ and $T_{w'} = w'2I_n + (1 - w')I_n$, they are said to be have the same structure if $I_1 = I_2$. It is well-known that a finite product of $T$-transform matrices with the same structure is also a $T$-transform matrix, while it is not necessarily a $T$-transform matrix for different structures (see Balakrishnan et al. [25]). Below, we introduce the concept of the multivariate majorization.

**Definition 4** (Definition 6 on Page 4 of Marshall et al. [23]). Consider two $m \times n$ matrices $U = \{u_{ij}\}$ and $S = \{s_{ij}\}$, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Then, a matrix $U$ is larger than matrix $S$ in the chain majorization, denoted by $U \gg S$, if there exists a finite set of $n \times n$ $T$-transform matrices $T_{u_1}, T_{u_2}, \ldots, T_{u_n}$ such that $S = UT_{u_1}T_{u_2} \cdots T_{u_n}$. Interested readers may refer to Marshall et al. [23] for an elaborate discussion on the theory of vector and matrix majorizations and their applications in various contexts. For simplicity, let us denote

$$\mathcal{M}_n = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x_i, y_i > 0 \text{ and } (x_i - x_j)(y_i - y_j) \geq 0, i, j = 1, 2, \ldots, n\},$$

$$\mathcal{D}_+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq x_2 \geq \ldots \geq x_n\},$$

$$\mathcal{J}_+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 \leq x_2 \leq \ldots \leq x_n\}. $$
Next, we present some results that are required for established on the multivariate chain majorizations. First, we present to lemmas that are useful for comparison results when heterogeneity in two parameters.

**Lemma 4** (Theorem 2 on Page 4 of Balakrishnan et al. [25]). Consider the differentiable function \( \phi: \mathbb{R}_+^n \to \mathbb{R}_+ \). Then,
\[
\phi(U) \leq \phi(V) \quad \text{for all } U, V, \text{ such that } U \in \mathcal{M}_2 \text{ and } U \succ V,
\]
if and only if
\[
(i) \phi(U) = \phi(U\Pi) \quad \text{for all permutation matrices } \Pi
\]
\[
(ii) \sum_{i=1}^2 (u_{ik} - u_{ij})[\phi_{ik}(U) - \phi_{ij}(U)] \leq 0 \quad \text{for all } j, k = 1, 2, \text{ where } \phi_{ij}(U) = \partial \phi(U)/\partial u_{ij}
\]

**Lemma 5** (Theorem 3 on Page 4 of Balakrishnan et al. [25]). Consider the differentiable function \( \phi: \mathbb{R}_+^n \to \mathbb{R}_+ \), and let the function \( \phi_n: \mathbb{R}_+^{n^2} \to \mathbb{R}_+ \), be defined as \( \phi_n = \prod_{i=1}^n \phi(u_{ii}, u_{ij}) \). If \( \phi_2 \) satisfies (3), then \( \phi_n(U) \geq \phi_n(V) \), where \( U \in S_n \) and \( V = UT \).

### 2.3. Archimedean Copula

Now, let us review the concept of the copula. Let \( X = (X_1, X_2, \ldots, X_n) \) be a random vector. The joint distribution and the survival functions of \( X \) are denoted by \( H \) and \( H^c \), respectively. Further, we assume that \( F_{X_1}, F_{X_2}, \ldots, F_{X_n} \) are the marginal distribution functions and the survival functions of \( X_1, X_2, \ldots, X_n \). When the random variables \( X_1, X_2, \ldots, X_n \) are dependent, the joint cumulative distribution function can be represented in terms of the copula \( C \) as
\[
H(x_1, x_2, \ldots, x_n) = C(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)).
\]

Further, the joint survival function can also be represented in terms of the survival copula \( C^c \), which is given by
\[
H^c(x_1, x_2, \ldots, x_n) = C^c(F_{X_1}^c(x_1), F_{X_2}^c(x_2), \ldots, F_{X_n}^c(x_n)).
\]

It is noted that the copula is a multivariate distribution function of a random vector having uniform marginal distributions in the interval \([0,1]\).

### 3.1. Insurance Portfolios of Independent Risks

Throughout the paper, we use \( Y_{n,n} \) and \( Y_{n,n}^* \) to denote the largest claim amounts from the insurance portfolios \( (I_{1,1}, I_{1,2}, \ldots, I_{n,n}) \) and \( (I_{1,1}^*, I_{1,2}^*, \ldots, I_{n,n}^*) \), respectively.

The following Theorem 1 provides some sufficient conditions for comparing the largest claim amounts in terms of the usual stochastic order.

**Definition 5** (Expression 4.6.1 on Page 151 of Nelsen [26]). For a decreasing and continuous function \( \psi: \mathbb{R}^n \to [0,1] \) such that \( \psi(0) = 1, \psi(+\infty) = 0 \). Denote \( \phi = \psi^{-1} \) the pseudo-inverse of \( \psi(x) \), and \( \psi(k) \) the kth derivative of the \( \psi(x) \) \((k = 1, 2, \ldots, n \in \mathbb{N}) \). If \((-1)^k \psi(k)(x) \geq 0 \) for all \( 0 \leq k \leq n - 2 \), and \((-1)^{n-2} \psi^{(n-2)}(x) \) is decreasing and convex, then
\[
C_\psi(u_1, u_2, \ldots, u_n) = \left( \sum_{i=1}^n \phi(u_i) \right), \quad u_i \in [0,1],
\]

is called an Archimedean copula with the generator \( \psi \).

Archimedean copulas are widely used to model the dependence among variables due to their mathematical tractability as well as their ability to model a wide range of dependence structures.

**Definition 6** (Li and Fang [27]). A function \( f \) is said to be super-additive if \( f(x + y) \geq f(x) + f(y) \) for all \( x \) and \( y \) in the domain of \( f \).

Based on Lemma A.1 in Li and Fang [27]; it is known that, for two \( n \)-dimensional Archimedean copulas \( C_{\psi_1}(u) \) and \( C_{\psi_2}(u) \) with respective generators \( \psi_1 \) and \( \psi_2 \) and pseudo-inverses \( \phi_1 \) and \( \phi_2 \), if \( \psi_1 \) is super-additive, then \( C_{\psi_1}(u) \leq C_{\psi_2}(u) \) for all \( u \in [0,1]^n \). For more detailed discussions on copulas and their properties, one may refer to Nelsen [26] and McNeil and Nešlehová [28].

### 3. On Usual Stochastic Ordering of Largest Claim Amounts

Consider two sets of heterogeneous and independent non-negative random observations from Weibull-G claim severities. Suppose that \( \omega(y_x) = F(y)/F(y_c) \). In this section, we establish the comparison results between the largest claim amounts arising from these sets in the sense of the usual stochastic order. Assume \( X_1, X_2, \ldots, X_n \) are non-negative random variables and \( I_1, I_2, \ldots, I_n \) are Bernoulli random variables. Let \( Y_i = I_iX_i, i = 1, 2, \ldots, n \), then the survival function of \( Y_i \) can be expressed as
\[
\bar{F}_{Y_i}(x) = \mathbb{P}(Y_i > x) = \mathbb{P}(X_iI_i > x | I_i = 1)p_i + \mathbb{P}(X_iI_i > x | I_i = 0)(1 - p_i)
\]
\[
= p_i\bar{F}_{X_i}(x), x \in \mathbb{R}^+, \quad i = 1, 2, \ldots, n.
\]

**Theorem 1.** Suppose \( X_1, X_2, X_3, X_4 \) are independent non-negative random variables with \( X_i \sim W(x, \beta, \gamma) \) \((i = 1, 2, 3, 4) \). Let \( I_{p_1}, I_{p_2}, I_{p_3}, I_{p_4} \) are independent Bernoulli random variables and independent of \( X_3, X_4 \). Let \( \mathbb{E}(I_{p_i}) = p_i |\mathbb{E}(I_{p_i}) = p_i|, \quad i = 1, 2, 3, 4 \). Assume that \( h: [0,1] \to (\infty, 0) \) is a differentiable and strictly decreasing convex function. Then, for \( (h(p), \alpha) \in \mathcal{M}_2 \), it holds that
\[
\begin{pmatrix}
  h(p_1) & h(p_2) \\
  a_1 & a_2
\end{pmatrix} \gg \begin{pmatrix}
  h(p'_1) & h(p'_2) \\
  a'_1 & a'_2
\end{pmatrix} \Rightarrow Y_{2,2} \geq Y'_{2,2}.
\]

(12)

**Proof.** Without loss of generality, let us assume that 
\( u_1 = h(p_1) \geq h(p_2) = u_2 > 0 \), \( u'_1 = h(p'_1) \geq h(p'_2) = u'_2 > 0 \),
then we have \( a_1 \geq a_2 > 0 \) and \( a'_1 \geq a'_2 > 0 \). Denotes the inverse function of \( h \) by \( h^{-1} \), and set \( p_i = h^{-1}(u_i) \) \((i = 1, 2)\), the distribution function of the largest claim amounts \( Y_{2,2} \) and \( Y'_{2,2} \) can be expressed as

\[
\varphi(u, \alpha) = (u_1 - u_2) \left[ \frac{\partial F_{Y_{2,2}}(x)}{\partial u_1} - \frac{\partial F_{Y'_{2,2}}(x)}{\partial u_2} \right] + (\alpha_1 - \alpha_2) \left[ \frac{\partial F_{Y_{2,2}}(x)}{\partial \alpha_1} - \frac{\partial F_{Y'_{2,2}}(x)}{\partial \alpha_2} \right],
\]

(15)

For any \( u_1 \geq u_2, \alpha_1 \geq \alpha_2, \) equation (15) can be rewritten as

\[
\varphi(u, \alpha) = (u_1 - u_2) F_{Y_{2,2}}(x)[m(u_2, \alpha_2) - m(u_1, \alpha_1)] + (\alpha_1 - \alpha_2) F_{Y'_{2,2}}(x)[n(u_1, \alpha_1) - n(u_2, \alpha_2)],
\]

(18)

where

\[
m(u, \alpha) = \frac{\partial h^{-1}(u)}{\partial u} \frac{e^{-\alpha u\theta(y)}}{1 - h^{-1}(u)e^{-\alpha u\theta(y)}},
\]

\[
n(u, \alpha) = \frac{h^{-1}(u)e^{-\alpha u\theta(y)}w^\theta(y)}{1 - h^{-1}(u)e^{-\alpha u\theta(y)}}.
\]

(19)

Owing to \( h^{-1}(u) \) is also strictly decreasing convex, we obtain

\[
\frac{\partial m(u, \alpha)}{\partial u} \equiv \left[ 1 - h^{-1}(u)e^{-\alpha u\theta(y)} \right] \frac{\partial^2 h^{-1}(u)e^{-\alpha u\theta(y)}}{\partial u^2} + \left[ \frac{\partial h^{-1}(u)}{\partial u} \right] e^{-2\alpha u\theta(y)} \geq 0,
\]

(20)

\[
\frac{\partial m(u, \alpha)}{\partial \alpha} = -w^\theta(y)h^{-1}(u)e^{-\alpha u\theta(y)} \left[ 1 - h^{-1}(u)e^{-\alpha u\theta(y)} \right] + e^{-2\alpha u\theta(y)} \frac{\partial h^{-1}(u)}{\partial u} \geq 0.
\]

(21)

It follows that

\[
m(u_1, \alpha_1) \geq m(u_2, \alpha_1) \geq m(u_2, \alpha_2),
\]

(22)

which implies
and thus $\varphi(u, \alpha) \leq 0$, which confirms the condition (ii) of Lemma 4. The desired result is proved.

**Remark 1.** Theorem 1 shows that the more the heterogeneity among the distributions of claim severities and occurrence probabilities, the larger the largest claim amounts, in the sense of the usual stochastic order.

Taking $T$-transform matrix as $T = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, in Theorem 1, we immediately obtain a lower bound on the survival function of the largest claim amounts.

**Corollary 1.** Under the setup of Theorem 1, $1 - [1 - h^{-1}(\bar{h}(p))e^{-\bar{\varphi}(\alpha, \beta, y)}]^{2}$ is a low bound of $T_{Y_{2}, 2}(x)$, that is, for any $x \in \mathbb{R}^{*}$, we have

$$
T_{Y_{2}, 2}(x) \geq 1 - \left[1 - h^{-1}(\bar{h}(p))e^{-\bar{\varphi}(\alpha, \beta, y)}\right]^{2}.
$$

Next, we give a numerical example to illustrate the effectiveness of Theorem 1 and Corollary 1.

**Example 1.** Suppose the baseline random variable follows Weibull $(0.02, 2)$ distribution. Suppose $X_{1}, X_{2}, [X_{1}^{*}, X_{2}^{*}]$ are independent random variables with $X_{i} \sim W - G(0.02, 2, \alpha_{i}, 3)$, $X_{i}^{*} \sim W - G(0.02, 2, \alpha_{i}^{*}, 3)$, $i = 1, 2$. Let $\{I_{1}, I_{p_{1}}, I_{p_{1}}, I_{p_{1}}^{*} \}$, are independent Bernoulli random variables, independent of $X_{i}$, with $E[I_{p_{1}}] = p_{1}$, $i = 1, 2$. To plot the whole survival curves of $Y_{2, 2}$ and $Y_{2, 2}^{*}$ in $[0, \infty)$, we perform the transformation $e^{-x^{2}}$ to $[0, \infty) \rightarrow (0, 1)$. Then, it is obvious that $Y_{2, 2} \geq_{st} Y_{2, 2}^{*}$ is equivalent with $e^{-Y_{2, 2}^{*}} \leq_{st} e^{-Y_{2, 2}}$.

(i) Suppose $(\alpha_{1}, \alpha_{2}) = (3, 1)$, $(\alpha_{1}^{*}, \alpha_{2}^{*}) = (1.72, 2.28)$, $(p_{1}, p_{2}) = (e^{-2}, e^{1})$ and $(p_{1}^{*}, p_{2}^{*}) = (e^{-1.36}, e^{-1.64})$. Taking $h(p) = -\ln p$ and considering the following T-transform matrix,

$$
T_{0.36} \equiv 0.36 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.64 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

Proof. Setting $\varphi_{n}(p, \alpha) = F_{Y_{n}}(x)$ and $\varphi(p, \alpha) = 1 - Pe^{-\varphi_{n}(\alpha, x)}$, for any $x \geq 0$, we then have $\varphi_{n}(p, \alpha) = \prod_{i=1}^{n} \varphi(p_{i}, \alpha_{i})$. According to Theorem 1, $\varphi_{2}$ satisfies in (7) of Lemma 4. Now, the required result follows immediately from Lemma 5.

**Remark 2.** Note that the distribution in Theorem 2 and Theorem 1 are the special case of the proportional hazard rate model, hence, our theoretical findings verify the case of the proportional hazard rate model. Besides, Theorem 2 result indicates that more heterogeneity among some specified transformed occurrence probabilities and scale parameter $\alpha$ in the sense of the chain majorization order

we then have

$$
\begin{pmatrix}
(h(p_{1}^{*})) & (h(p_{2}^{*})) & \cdots & (h(p_{n}^{*})) \\
\alpha_{1}^{*} & \alpha_{2}^{*} & \cdots & \alpha_{n}^{*}
\end{pmatrix}
\begin{pmatrix}
(h(p_{1})) & (h(p_{2})) & \cdots & (h(p_{n})) \\
\alpha_{1} & \alpha_{2} & \cdots & \alpha_{n}
\end{pmatrix}

T_{0.36}
$$

leads to the greater survival function of the largest claim amount.

It is easy to check that the conditions of Theorem 1 are all satisfied. As is seen in Figure 1(a), the graph of $T_{Y_{2}, 2}(x)$ is always over the zero plane, that is, the difference functions $T_{Y_{2}, 2}(x) - T_{Y_{2}, 2}^{*}(x)$ is always non-negative for all $x = -\ln u$ and $u \in (0, 1]$, which coincides with the result of Theorem 1.

(i) Consider the following T-transform matrix, with other conditions unchanged,

$$
T_{0.5} = 0.5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.5 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

**Theorem 2.** Suppose $X_{1}, X_{2}, \ldots, X_{n}[X_{1}^{*}, X_{2}^{*}, \ldots, X_{n}^{*}]$ are independent nonnegative random variables with $X_{i} \sim W - G(\alpha_{i}, \beta, y)[X_{i}^{*} \sim W - G(\alpha_{i}^{*}, \beta, y)]$, $i = 1, 2, \ldots, n$. Let $\{I_{p_{1}}, I_{p_{2}}, \ldots, I_{p_{n}}, [I_{p_{1}}, I_{p_{1}}, I_{p_{1}}^{*}] \}$ are independent Bernoulli random variables, independent of $X_{i}[X_{i}^{*}]$, with $E[I_{p_{1}}] = p_{1}$, $E[I_{p_{1}}^{*} \mid E[I_{p_{1}}] = p_{1}^{*}]$, $i = 1, 2, \ldots, n$. Then, under the same settings as in Theorem 1, for $(h(p), \alpha) \in \mathcal{M}_{2}$, it holds that

$$
T_{Y_{n}} \geq_{st} Y_{n}^{*}.
$$

leads to the greater survival function of the largest claim amount.

The following Theorem 3 gives a sufficient condition under which $Y_{2, 2} \geq_{st} Y_{2, 2}^{*}$ holds when the baseline distribution follows exponential. We only provide the result of $n = 2$ case.

**Theorem 3.** Suppose $X_{1}, X_{2}[X_{1}^{*}, X_{2}^{*}]$ are two independent non-negative random variables with $X_{i} \sim W - Exp(\alpha, \beta, \gamma)[X_{i}^{*} \sim W - Exp(\alpha, \beta, \gamma^{*})]$, $i = 1, 2$. Let $\{I_{p_{1}}, I_{p_{2}}, [I_{p_{1}}, I_{p_{1}}, I_{p_{1}}^{*}] \}$ are independent Bernoulli random variables, independent of $X_{i}[X_{i}^{*}]$, with $E[I_{p_{1}}] = p_{1}$, $E[I_{p_{1}}^{*} \mid E[I_{p_{1}}] = p_{1}^{*}]$, $i = 1, 2$. Assume that
Note that the baseline distribution $F(x) = 1 - e^{-\gamma x}$, $x > 0$, which implies $u(F(x)) = e^{\gamma x} - 1$. Set $p_i = h^{-1}(u_i)$, for $i = 1, 2$. Then, the distribution function of the largest claim amounts $Y_{2:2}$ is given by

$$F_{Y_{2:2}}(x) = \prod_{i=1}^{2} \left[ 1 - h^{-1} (u_i) e^{-\alpha \gamma x - \beta \gamma^{\prime}} \right].$$

(31)

Clearly, for fixed $x \geq 0$, $F_{Y_{2:2}}(x)$ is the permutation invariant on $M_2$, which confirms the condition (i) of Lemma 4. On the other hand, we need prove the condition (ii) of Lemma 4, for any $x \geq 0$ and $1 = i \neq j = 2$, set

\begin{align}
\varphi(u, \gamma) &= (u_1 - u_2) \left[ \frac{\partial F_{Y_{2:2}}(x)}{\partial u_1} - \frac{\partial F_{Y_{2:2}}(x)}{\partial u_2} \right] + (\gamma_1 - \gamma_2) \left[ \frac{\partial F_{Y_{2:2}}(x)}{\partial \gamma_1} - \frac{\partial F_{Y_{2:2}}(x)}{\partial \gamma_2} \right],
\end{align}

(32)

where $u = (u_1, u_2)$ and $\gamma = (\gamma_1, \gamma_2)$. Then, we have

\begin{align}
\frac{\partial F_{Y_{2:2}}(x)}{\partial u_i} &= -F_{Y_{2:2}}(x) \frac{d h^{-1}(u_i)}{d u_i} \frac{e^{-\gamma \gamma^{\prime} e^{\gamma x}}}{1 - h^{-1}(u_i) e^{-\gamma \gamma^{\prime} e^{\gamma x}}} \alpha \beta x F_{Y_{2:2}}(x) \frac{h^{-1}(u_i) e^{\gamma x} (e^{\gamma x} - 1)^{\beta - 1}}{e^{\gamma \gamma^{\prime} e^{\gamma x} - 1}}.
\end{align}

(33)

Similarly to the proof of Theorem 1, equation (32) can be represented as

\begin{align}
\varphi(u, \gamma) &= (u_1 - u_2) F_{Y_{2:2}}(x) [k(u_2, \gamma_2) - k(u_1, \gamma_1)] + \alpha \beta x (\gamma_1 - \gamma_2) F_{Y_{2:2}}(x) [I(u_1, \gamma_1) - I(u_2, \gamma_2)],
\end{align}

(34)

where

**Figure 1:** (a) Plot of the difference functions $F_{Y_{2:2}}(x) - F_{Y_{2:2}}(x)$, for all $x = -\ln u$ and $u \in (0, 1)$.
where \[ \phi \geq 0 \] and \( \phi \equiv \phi (z) \geq 0 \). It follows from \( \frac{d\varphi_3(z)}{dz} = z \varphi_3 + z \varphi_3, z \geq 0 \) that \( \varphi_3(0) = 0 \). Combining \( \varphi_2(1) = 0 \) with \( \varphi_1(1) = 0 \), we conclude that \( \varphi_1(t) > 0 \), which implies \( \partial l(u, y)/\partial \gamma \leq 0 \). Basing on \( u_1 \geq u_2 \) and \( \gamma_1 \geq \gamma_2 \), it follows that

\[
\begin{align*}
l(u_1, y_1) &
\leq l(u_2, y_1) \leq l(u_2, y_2),
\end{align*}
\]

which confirms the second side of (32) is also non-positive. Upon combining (38) with (44), we have \( \varphi(u, y) \leq 0 \); hence, condition (ii) of Lemma 4 is verified. The proof of desired result is completed. \( \square \)

Without loss of generality, assume \( u_1 \geq u_2, \gamma_1 \geq \gamma_2 \). Owing to \( h^{-1} \) is a strictly decreasing convex function, we have

\[
\begin{align*}
k(u_1, y_1) &\geq k(u_2, y_1) \geq k(u_2, y_2),
\end{align*}
\]

Further, we have

\[
\begin{align*}
\varphi_1(t) &= \left( a\beta(t-1)^\beta - \beta \right) t + 1 \right) e^{\alpha(t-1)^\beta} + \beta t - 1, \quad t \geq 1.
\end{align*}
\]

where

\[
\varphi_1(t) = \left( a\beta(t-1)^\beta - \beta \right) t + 1 \right) e^{\alpha(t-1)^\beta} + \beta t - 1, \quad t \geq 1.
\]

For any \( t > 1 \), it follows that

\[
\begin{align*}
\frac{d\varphi_2(t)}{dt} &= 1 - e^{\alpha(t-1)^\beta} + e^{\alpha(t-1)^\beta} \alpha(t-1)^\beta \left( 1 + a\beta(t-1)^\beta \right) t + \beta t(t-1)^\beta + \frac{\beta}{t-1} \\
&\geq 1 - e^{\alpha(t-1)^\beta} + \alpha(t-1)^\beta e^{\alpha(t-1)^\beta} = \varphi_3(\alpha(t-1)^\beta),
\end{align*}
\]

which confirms the second side of (32) is also non-positive. Upon combining (38) with (44), we have \( \varphi(u, y) \leq 0 \); hence, condition (ii) of Lemma 4 is verified. The proof of desired result is completed. \( \square \)
Remark 3. According to Theorem 3, more heterogeneity among some specified transformed occurrence probabilities and scale parameter $\gamma$ in terms of the chain majorization order causes the greater survival function of the largest claim amount with two independent insurance portfolios. The proof of the following Theorem 4 is similar to that of Theorem 2 and thus omitted here.

\[
\left( h(p_{1}^*) \ h(p_{2}^*) \ \cdots \ h(p_{n}^*) \right)_{\gamma_1 \ \gamma_2 \ \cdots \ \gamma_n} = \left( \frac{h(p_1)}{\gamma_1} \ \frac{h(p_2)}{\gamma_2} \ \cdots \ \frac{h(p_n)}{\gamma_n} \right)
\]

Remark 4. Note that the distribution in Theorem 3 and Theorem 4 are the special case of the scaled model in Barmalan et al. [8]; hence, our established results check the setting of the scaled model. Meanwhile, according to Theorem 4, more heterogeneity among the scale parameter $\gamma$ and the specified transformed occurrence probabilities with respect to the chain majorization order obtains the greater survival function of the largest claim amount from multiple independent insurance portfolios.

Naturally, one can question Theorems 3 and 4 whether holds for other baseline function, unfortunately, the following numerical example gives a negative answer.

Example 2. Suppose the baseline random variable follows Frechet $(0, \gamma, 1)$ distribution. Suppose $X_1, X_2, X'_1, X'_2$ are independent random variables with $X_i \sim W - G(0, \gamma_i, 4, 0.6, 2)$, $X'_i \sim W - G(0, \gamma_i, 4, 0.6, 2)$, $i = 1, 2$, $\beta = 2$. Let $I_{p_1}$, $I_{p_2}$, $[I_{p_1}', I_{p_2}']$, are independent Bernoulli random variables, independent of $X_i$s, with $E(I_{p_i}) = p_i, i = 1, 2$. To plot the whole survival curves of $X_{22}$ and $X'_{22}$ in $[0, \infty)$. Suppose $(\gamma_1, \gamma_2) = (0.2, 0.5)$, $(\gamma_1', \gamma_2') = (0.425, 0.275)$, $(p_1, p_2) = (e^{-3}, e^{-1})$ and $(p'_1, p'_2) = (e^{-1.25}, e^{-1.75})$. Taking $h(p) = -\ln p$ and considering the following T-transform matrix,

\[
\begin{bmatrix}
T_{0.25} = 0.25 \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} + 0.64 \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\end{bmatrix}
\]

We have then
\[
\left( \frac{h(p_{1}^*)}{\gamma_1'} \ \frac{h(p_{2}^*)}{\gamma_2'} \right) = \left( \frac{h(p_1)}{\gamma_1} \ \frac{h(p_2)}{\gamma_2} \right)T_{0.36} ;
\]

thus, a matrix \( h(p_{1}^*) \ h(p_{2}^*) \) is larger than matrix \( h(p_1) \ h(p_2) \) in the chain majorization, that is, it is easy to check that the conditions of Theorem 3 are all satisfied. As is seen in Figure 2, the survival functions $F_{Y_{nn}}(x)$ and $F_{Y'_{nn}}(x)$ are each crossing, which states Theorem 3 is not necessarily hold for other baseline function.

3.2. Insurance Portfolios of Dependent Risks. In this section, we use the usual stochastic order to compare the largest claim amounts $Y_{nn}$ and $Y'_{nn}$ arising from two sets of dependent heterogeneous insurance portfolios when occurrence probabilities are dependent and claim severities are also dependent.

To proceed, let us define $0 = (0, 0, \ldots, 0)$, $1 = (1, 1, \ldots, 1)$, $p(\lambda) = P(I = \lambda)$ and

\[
A_k = \{ \lambda | \lambda_i = 0 \ or \ 1, \ i = 1, 2, \ldots, n, \lambda_1 + \lambda_2 + \cdots + \lambda_n = k \}, \ \ k = 0, 1, \ldots, n.
\]

For all $1 \leq i \neq j \leq n$ and $k = 1, 2, \ldots, n - 1$, denote
\[
A_{ij}^k(0, 0) = \{ \lambda \in A_k \lambda_i = 0, \lambda_j = 0 \}.
\]

Obviously, $A_{ij}^k(1, 1) = A_{n+1}^k(0, 0) = \emptyset$ and
\[
A_k = A_{ij}^k(1, 0) \cup A_{ij}^k(0, 1) \cup A_{ik}^j(1, 1) \cup A_{ik}^j(0, 0).
\]

With the help of the above notations, Cai and Wei [29] presented characterizations of stochastic arrangement increasing(SAI) Bernoulli random vector, which is very useful to model the positive dependence among occurrence probabilities.

**Theorem 4.** Suppose $X_1, X_2, \ldots, X_n, X'_1, X'_2, \ldots, X'_n$ are independent nonnegative random variables with $X_i \sim W - G(\alpha, \beta, \gamma)$, $X'_i \sim W - G(\alpha', \beta, \gamma)$, $i = 1, 2, \ldots, n$. Let $I_{p_1}$, $I_{p_2}, \ldots, I_{p_n}$, $[I_{p_1}', I_{p_2}']$, are independent Bernoulli random variables, independent of $X_i$s, with $E(I_{p_i}) = p_i, [E(I_{p_i}') = p_i']$, $i = 1, 2, \ldots, n$. Then, under the same settings as in Theorem 1, for $(h(p), \gamma) \in \mathcal{M}_2$, it holds that

\[
\left( h(p_1) \ h(p_2) \ \cdots \ h(p_n) \right) \Rightarrow Y_{nn} \geq_s Y'_{nn}.
\]

\[
T_{0.36} = 0.25 \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} + 0.64 \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

**Lemma 6.** Cai and Wei [29]). A multivariate Bernoulli random vector $I$ is SAI if and only if $P(\tau_j(\lambda)) \leq P(\lambda)$, for any $\lambda \in A_i^j(0, 1), 1 \leq j \leq n$ and $k = 1, 2, \ldots, n - 1$.

**Theorem 5.** Suppose $I$ is SAI, $X_i \sim W - G(\alpha_i, \beta, \gamma)$ and the associated Archimedean copula is with $\psi_1(\phi_1 = \psi_1)$ and $X'_i \sim W - G(\alpha'_i, \beta, \gamma)$, and the associated Archimedean copula with generator $\psi_2(\phi_2 = \psi_2)$, for all $i = 1, 2, \ldots, n$ and $\alpha_i, \alpha'_i, \gamma_i \in \mathbb{R}^+$, $\gamma_i \in \mathcal{S}$. Assume the following conditions hold:

(i) $(\alpha_1, \alpha_2, \ldots, \alpha_n) < (\alpha_1, \alpha_2, \ldots, \alpha_n)$ for $\alpha, \alpha \in \mathcal{S}$

(ii) $\phi_2 \circ \psi_1$ is super additive
Then, we have \( Y_{n:n} \geq Y_{n:n}^* \).

**Proof.** We need to show that \( F_{Y_{n:n}}(x) \leq F_{Y_{n:n}^*}(x) \), for all \( x \in \mathbb{R}^+ \). By the nature of majorization order, it is sufficient to prove the desired result when \( (\alpha^*_i, \alpha^*_j) \succeq (\alpha_i, \alpha_j) \), for some pair \( 1 \leq i \leq j \leq n \), and \( \alpha_s = \alpha^*_s \) for \( s \neq i, j \). Let

\[
F^{ij}_{1} (x) = \left( F_{Y_{1}} (X), \ldots, F_{Y_{i-1}} (X), F_{Y_{i+1}} (X), \ldots, F_{Y_{j-1}} (X), F_{Y_{j+1}} (X), \ldots, F_{Y_{n}} (X) \right).
\]

(50)

The distribution function of \( Y_{n:n} \) is given by

\[
F_{Y_{n:n}} (x) = \mathbb{P} \left( \max \left\{ I_{\lambda_i} X_1, I_{\lambda_2} X_2, \ldots, I_{\lambda_n} X_n \right\} \leq x \right)
= \sum_{k=0}^{n} \sum_{\lambda \in \mathcal{A}_k} \mathbb{P} \left( I_{\lambda_1} X_1 \leq x, \ldots, I_{\lambda_n} X_n \leq x \mid \lambda = \lambda \right) \mathbb{P} (\lambda)
= p(0) + p(1) \left( \max \left\{ X_1, \ldots, X_n \right\} \leq x \right) + \sum_{k=1}^{n-1} \sum_{\lambda \in \mathcal{A}_k} \mathbb{P} \left( \max \left\{ I_{\lambda_1} X_1, \ldots, I_{\lambda_n} X_n \right\} \leq x \right)
= p(0) + p(1) \psi_1 \left[ \sum_{i=1}^{n} \phi_i \left( 1 - e^{-a_i \left( u(y, x) \right)^{1/2}} \right) \right]
+ \sum_{k=1}^{n-1} \sum_{\lambda \in \mathcal{A}_k^{(1,0)}} \mathbb{P} \left( \lambda_i X_i \leq x, \lambda_j X_j \leq x, \lambda_s X_s \leq x, s \neq i, j \right)
+ \sum_{k=1}^{n-1} \sum_{\lambda \in \mathcal{A}_k^{(0,1)}} \mathbb{P} \left( \lambda_i X_i \leq x, \lambda_j X_j \leq x, \lambda_s X_s \leq x, s \neq i, j \right)
+ \sum_{k=1}^{n-1} \sum_{\lambda \in \mathcal{A}_k^{(0,0)}} \mathbb{P} \left( \lambda_i X_i \leq x, \lambda_j X_j \leq x, \lambda_s X_s \leq x, s \neq i, j \right)
\]
\[ + \sum_{k=1}^{n-1} \sum_{\lambda \in A_k^{i,j}} p(\lambda) \mathbb{P}(\lambda, X_i \leq x, \lambda, X_j \leq x, \lambda, X_s \leq x, s \neq i, j) = p(0) + p(1) \psi_1 \left[ \sum_{i=1}^{n} \phi_1 \left( 1 - e^{-a_i(w(y,x))^\rho} \right) \right] \]

\[ + \sum_{k=1}^{n-1} \sum_{\lambda \in A_k^{i,j}} p(\lambda) \psi_1 \left[ \phi_1(1) + \phi_1 \left( 1 - e^{-a_i w_i^\rho (y,x)} \right) + \sum_{s \neq i,j} \phi_1 \left( 1 - p_s e^{-a_i w_i^\rho (y,x)} \right) \right] \]

\[ + \sum_{k=1}^{n-1} \sum_{\lambda \in A_k^{i,j}} p(\lambda) \psi_1 \left[ \phi_1(1) + \phi_1 \left( 1 - e^{-a_i w_i^\rho (y,x)} \right) + \sum_{s \neq i,j} \phi_1 \left( 1 - p_s e^{-a_i w_i^\rho (y,x)} \right) \right] \]

\[ + \sum_{k=1}^{n-1} \sum_{\lambda \in A_k^{i,j}} p(\lambda) \psi_1 \left[ \phi_1(1) + \phi_1 \left( 1 - e^{-a_i w_i^\rho (y,x)} \right) + \sum_{s \neq i,j} \phi_1 \left( 1 - p_s e^{-a_i w_i^\rho (y,x)} \right) \right] \]

\[ (51) \]

Similarly,

\[ F_{\gamma, x}(x) = p(0) + p(1) \psi_2 \left[ \sum_{i=1}^{n} \phi_2 \left( 1 - e^{-a_i^\gamma (w(y,x))^\rho} \right) \right] \]

\[ + \sum_{k=1}^{n-1} \sum_{\lambda \in A_k^{i,j}} p(\lambda) \psi_2 \left[ \phi_2(1) + \phi_2 \left( 1 - e^{-a_i^\gamma w_i^\rho (y,x)} \right) + \sum_{s \neq i,j} \phi_2 \left( 1 - p_s e^{-a_i^\gamma w_i^\rho (y,x)} \right) \right] \]

\[ + \sum_{k=1}^{n-1} \sum_{\lambda \in A_k^{i,j}} p(\lambda) \psi_2 \left[ \phi_2(1) + \phi_2 \left( 1 - e^{-a_i^\gamma w_i^\rho (y,x)} \right) + \sum_{s \neq i,j} \phi_2 \left( 1 - p_s e^{-a_i^\gamma w_i^\rho (y,x)} \right) \right] \]

\[ + \sum_{k=1}^{n-1} \sum_{\lambda \in A_k^{i,j}} p(\lambda) \psi_2 \left[ \phi_2(1) + \phi_2 \left( 1 - e^{-a_i^\gamma w_i^\rho (y,x)} \right) + \sum_{s \neq i,j} \phi_2 \left( 1 - p_s e^{-a_i^\gamma w_i^\rho (y,x)} \right) \right] \]

\[ + \sum_{s \neq i,j} \phi_2 \left( 1 - p_s e^{-a_i^\gamma w_i^\rho (y,x)} \right) \]

\[ (52) \]

From the convexity and Schur-convexity of \(-\sum_{i=1}^{n} \phi_1(u_i)\) and the concavity of \(F(x; \alpha) = 1 - e^{-aw^\rho (yx)}\) in \(\alpha\), we know that \(-\sum_{i=1}^{n} \phi_1(u_i)\) is Schur-convex in \(\alpha\) by applying Lemma 3. Hence, according to the super additivity of \(\phi_2 \psi_1\) and \(\alpha^\ast \preceq m\alpha\), we have

\[ \psi_2 \left[ \sum_{i=1}^{n} \phi_2 \left( 1 - e^{-a_i^\gamma (w(y,x))^\rho} \right) \right] \geq \psi_1 \left[ \sum_{i=1}^{n} \phi_1 \left( 1 - e^{-a_i (w(y,x))^\rho} \right) \right] \geq \psi_1 \left[ \sum_{i=1}^{n} \phi_1 \left( 1 - e^{-a (w(y,x))^\rho} \right) \right], \quad x \in \mathbb{R}^n. \]

\[ (53) \]

According to \(\phi_2 \psi_1\) is superadditive and \((\alpha^\ast, \alpha^\ast) \preceq m(\alpha, \alpha)\), for any \(x \in \mathbb{R}^n\), we have
\[
\psi_2 \left[ \phi_2 (1) + \phi_2 (1) + \sum_{s \neq i,j} \phi_2 \left( 1 - p_s e^{-a_i w^d (y,x)} \right) \right] \geq \psi_1 \left[ \phi_1 (1) + \phi_1 (1) + \sum_{s \neq i,j} \phi_1 \left( 1 - p_s e^{-a_i w^d (y,x)} \right) \right],
\]

(54)

\[
\psi_2 \left[ \phi_2 \left( 1 - e^{-a_i w^d (y,x)} \right) + \phi_2 \left( 1 - e^{-a_i w^d (y,x)} \right) + \sum_{s \neq i,j} \phi_2 \left( 1 - p_s e^{-a_i w^d (y,x)} \right) \right] \\
\geq \psi_1 \left[ \phi_1 \left( 1 - e^{-a_i w^d (y,x)} \right) + \phi_1 \left( 1 - e^{-a_i w^d (y,x)} \right) + \sum_{s \neq i,j} \phi_1 \left( 1 - p_s e^{-a_i w^d (y,x)} \right) \right],
\]

(55)

On the other hand, the concavities of both \( \psi_1 \sum_{i=1}^n \phi_i (u_i) \) and \( F(x; \alpha) \) imply that \( \psi \sum_{i=1}^n \phi (F(x; \alpha)) \) is also concave in \( \alpha \), which leads to the fact that

\[
\psi_1 \left[ \phi_1 (1) + \phi_1 \left( 1 - e^{-a_i w^d (y,x)} \right) + \sum_{s \neq i,j} \phi_1 \left( 1 - p_s e^{-a_i w^d (y,x)} \right) \right] \\
+ \psi_1 \left[ \phi_1 \left( 1 - e^{-a_i w^d (y,x)} \right) + \phi_1 (1) + \sum_{s \neq i,j} \phi_1 \left( 1 - p_s e^{-a_i w^d (y,x)} \right) \right]
\]

(56)

is Schur-concave in \( \alpha \) by proposition I.3.C.1 on Page 80 of Marshall et al. [23]; for any \( x \in \mathbb{R}^n \). Hence, it follows from the superadditivity of \( \phi_2 \psi_1 \) and \( (\alpha^*, \alpha^*) \preceq m (\alpha_i, \alpha_j) \), for all \( x \in \mathbb{R}^n \).

\[
\psi_2 \left[ \phi_2 (1) + \phi_2 \left( 1 - e^{-a_i w^d (y,x)} \right) + \sum_{s \neq i,j} \phi_2 \left( 1 - p_s e^{-a_i w^d (y,x)} \right) \right] \\
+ \psi_2 \left[ \phi_2 (1) + \phi_2 \left( 1 - e^{-a_i w^d (y,x)} \right) + \sum_{s \neq i,j} \phi_2 \left( 1 - p_s e^{-a_i w^d (y,x)} \right) \right] \\
\geq \psi_1 \left[ \phi_1 (1) + \phi_1 \left( 1 - e^{-a_i w^d (y,x)} \right) + \sum_{s \neq i,j} \phi_1 \left( 1 - p_s e^{-a_i w^d (y,x)} \right) \right] \\
+ \psi_1 \left[ \phi_1 (1) + \phi_1 \left( 1 - e^{-a_i w^d (y,x)} \right) + \sum_{s \neq i,j} \phi_1 \left( 1 - p_s e^{-a_i w^d (y,x)} \right) \right]
\]

(57)

Observe that
\[
F_{\tau_{ij}}(x) - F_{\tau_{ij}}(x) = p(1) \left\{ \sum_{s=1}^{n} \phi_{s} \left[ \left( 1 - e^{-\alpha \omega(x)} \right) \right] - \sum_{s=1}^{n} \phi_{s} \left[ \left( 1 - e^{-\alpha \omega(x)} \right) \right] \right\} \\
\quad + \sum_{k=1}^{n} \sum_{k \neq j} \left( \phi_{k} (F(x; a_j)) + \sum_{s=1}^{n} \phi_{s} \left[ 1 - p e^{-\alpha \omega(x)} \right] \right) \\
\quad + \sum_{k=1}^{n} \sum_{k \neq j} \left( \phi_{k} (F(x; a_j)) + \sum_{s=1}^{n} \phi_{s} \left[ 1 - p e^{-\alpha \omega(x)} \right] \right) \\
\quad + \sum_{k=1}^{n} \sum_{k \neq j} \left( \phi_{k} (F(x; a_j)) + \sum_{s=1}^{n} \phi_{s} \left[ 1 - p e^{-\alpha \omega(x)} \right] \right) \\
\quad + \sum_{k=1}^{n} \sum_{k \neq j} \left( \phi_{k} (F(x; a_j)) + \sum_{s=1}^{n} \phi_{s} \left[ 1 - p e^{-\alpha \omega(x)} \right] \right) \\
\quad \geq 0.
\]
where the first inequality is from (53)–(55). The second inequality is due to Lemma 6 and the last inequality is based on (57). Hence, the desired result follows.

**Remark 5.** In the light of Theorem 5, we know that more heterogeneity among the distribution functions of claim severities leads to a larger value of the largest claim amount in terms of the usual stochastic ordering when occurrences probabilities are positively dependent via SAI and claim severities are also dependent via a general copula that is concave and Schur-concave.

Next, we give a numerical example to illustrate the result of Theorem 5.

\[
\mathcal{F}_{Y_{3.3}} (x) = 1 - \left[ p(0, 0, 0) + p(1, 1, 1)\psi_1 \left[ \phi_1 \left( 1 - e^{-a_{\alpha 1} (e^{\beta_1 x} - 1)^\beta} \right) + \phi_1 \left( 1 - e^{-a_{\beta 2} (e^{\gamma_2 x} - 1)^\gamma} \right) + \phi_1 \left( 1 - e^{-a_{\gamma 3} (e^{\delta_3 x} - 1)^\delta} \right) \right] \right.
\]

\[
+ p(1, 0, 0)\psi_1 \left[ \phi_1 \left( 1 - e^{-a_{\alpha 1} (e^{\beta_1 x} - 1)^\beta} \right) + \phi_1 (1) \right] \]

\[
+ p(0, 1, 0)\psi_1 \left[ \phi_1 (1) + \phi_1 (1) \right] \]

\[
+ p(0, 1, 0)\psi_1 \left[ \phi_1 (1) + \phi_1 (1) \right] \]

\[
+ p(1, 1, 0)\psi_1 \left[ \phi_1 (1) \right] + \phi_1 (1) \]

\[
\left. \right] + p(1, 1, 1)\psi_1 \left[ \phi_1 (1) \right] + \phi_1 (1) \right] \]

\[
+ p(0, 1, 1)\psi_1 \left[ \phi_1 (1) \right] + \phi_1 (1) \right] \]

\[
= \left( \psi_1 (1) \right) \cdot (59) \]

The survival functions of \( Y_{3.3} \) and \( Y_{3.3}^- \), denoted as \( \mathcal{F}_{Y_{3.3}} (x) \) and \( \mathcal{F}_{Y_{3.3}^-} (x) \), are plotted in Figure 3, respectively, for all \( x = -luu \) and \( uu \in (0, 1] \), from which we see that survival function \( \mathcal{F}_{Y_{3.3}^-} (x) \) is always larger than that of \( \mathcal{F}_{Y_{3.3}} (x) \). Thus, the effectiveness of Theorem 5 is validated.

The proof of the following Theorem 6 is similar to that of Theorem 5, and thus omitted here.

**Theorem 6.** Suppose \( I \) is SAI, \( X_i \sim W - G(\alpha_i, \beta, \gamma_i) \) and the associated Archimedean copula is with \( \psi_1 (\phi_1 = \psi_1^{-1}) \) and \( X_i^* \sim W - G(\alpha_i, \beta, \gamma_i^*) \), and the associated Archimedean copula with generator \( \psi_2 (\phi_2 = \psi_2^{-1}) \) for all \( i = 1, 2, \ldots, n \) and \( \gamma_i, \gamma_i^*, \alpha_i \in \mathbb{R}^+ \), \( \alpha_i \in \mathbb{S} \). Assume the following conditions hold:

(i) \((\gamma_1, \gamma_2, \ldots, \gamma_n) \leq (\gamma_1, \gamma_2, \ldots, \gamma_n)^* \), for \( \gamma, \gamma^* \in \mathbb{S} \).

(ii) \( \phi_2 \circ \psi_1 \) is super additive.

Then, we have \( Y_{n: n} \geq \alpha Y_{n: n}^* \).

**Remark 6.** Theorem 6 means more heterogeneity among the distribution functions of claim severities leads to a larger value of the largest claim amount among scale parameter \( \gamma \) in terms of the usual stochastic ordering when occurrences probabilities are positively dependent via SAI and claim severities are also dependent via a Archimedean copula.

**Example 3.** Suppose \( X_{11}, X_{21}, X_{31} [X_{11}^*, X_{21}^*, X_{31}^*] \) are three non-negative random variables, let \( F(x) = 1 - e^{-(\gamma x)} \), and generators \( \psi_1 (x) = (\eta_1 x + 1)^{-1/\eta_1} \), \( \eta_1 > 0 \), \( \psi_2 (x) = e^{(1/\eta_2)(1-e^x)} \), \( 0 < \eta_2 \leq 1 \). Set \( \alpha^* = (0.6, 0.5, 0.2) \), \( \beta = (0.8, 1.6) \), \( \eta_1 = 1.6 \), \( \eta_2 = 0.8 \), \( \gamma = (0.12, 0.06, 0.04) \), \( p(0, 0, 0) = 0.16 \), \( p(1, 0, 0) = 0.05 \), \( p(0, 0, 1) = 0.17 \), \( p(0, 1, 0) = 0.2 \), \( p(1, 1, 0) = 0.02 \), \( p(1, 0, 1) = 0.1 \), \( p(0, 1, 1) = 0.18 \), and \( p(1, 1, 1) = 0.15 \). It is easy to verify that the conditions of Theorem 5 are all satisfied. The survival function of \( Y_{3.3} \) can be expressed as

\[
4. On the Orderings of Smallest Claim Amounts

In this section, we study the stochastic comparisons of the smallest claim amounts with respect to the usual stochastic and hazard rate orders when the smallest claim amounts are independent and dependent.

4.1. Insurance Portfolios of Independent Risks. In this part, we discuss the stochastic comparisons between the smallest claim amounts from two insurance portfolios with W-G claims in the sense of the usual stochastic and hazard rate orders. Suppose \( X_{11}, X_{21}, \ldots, X_{n1} \) are independent, the survival function of the smallest claim amounts \( Y_{1: n} \) among \( n \) claims can be expressed as

\[
\mathcal{F}_{Y_{1: n}} = \prod_{i=1}^{n} \left[ p_i e^{-a_i \phi (\gamma_i x)} \right], \quad x \in \mathbb{R}^+, \alpha_i, \beta, \gamma_i \in \mathbb{R}^+. \quad (60)
\]

**Theorem 7.** Suppose \( X_{11}, X_{21}, \ldots, X_{n1} [X_{11}^*, X_{21}^*, \ldots, X_{n1}^*] \) are independent nonnegative random variables with \( X_i \sim W - G(\alpha_i, \beta, \gamma_i) [X_i^* \sim W - G(\alpha_i^*, \beta, \gamma_i^*)] \). Let \( I_{p_1}, I_{p_2}, \ldots, I_{p_n}, [I_{p_1}^*, I_{p_2}^*, \ldots, I_{p_n}^*] \) be a set of independent Bernoulli random variables, independent of \( X_i, X_i^* \), with \( E(I_{p_i}) = p_i, E(I_{p_i}^*) = p_i^* \), \( i = 1, 2, \ldots, n \). If
(i) $\beta \geq 1$ and $w(y; x)$ is an increasing convex function for $x \in \mathbb{R}_+$

(ii) $\prod_{i=1}^n p_i \leq \prod_{i=1}^n p'_i$

(iii) $(a_1, a_2, \ldots, a_n)^T \geq (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then, for

$$Y_{1:n}^* \geq wY_{1:n}$$

Proof. Note that

$$\frac{F_{Y_{1:n}^*}(x)}{F_{Y_{1:n}}(x)} = \left(\frac{\prod_{i=1}^n p'_i}{\prod_{i=1}^n p_i}\right)e^{-\sum_{i=1}^n \alpha_i (w(y; x))^{\beta_i}} = \frac{F_{X_{1:n}^*}(x)}{F_{X_{1:n}}(x)}$$

According to (3.1), the hazard rate function of $X_{1:n}$ can be presented as

$$r_{X_{i:n}}(x) = \beta \sum_{i=1}^n \alpha_i (w(y; x))^{\beta_i - 1}w'(y; x).$$

Let

$$\psi(\alpha) = \sum_{i=1}^n \alpha_i (w(y; x))^{\beta_i - 1}w'(y; x).$$

Then,

$$\psi(\alpha_1, \alpha_2, \ldots, \alpha_n) = \psi(a_1, a_2, \ldots, a_n).$$

Note that $w(x)$ is increasing and convex in $x$, then $x \geq 0$, $\beta \geq 1$ and $i \leq j$ imply $y_i(w(y; x))^{\beta_i} \leq y_j(w(y; x))^{\beta_j}$ and $w_i(y; x) \leq w_j(y; x)$. It follows that (65) is nonnegative, and hence, $\psi(\alpha)$ is Schur-convex in $\alpha$. By Lemma 1, we have

$$\psi(a_1, a_2, \ldots, a_n) \geq \psi(a_1^*, a_2^*, \ldots, a_n^*).$$

that is $r_{X_{i:n}}(x) \geq r_{X_{i:n}^*}(x)$, which is equivalent to $F_{X_{i:n}^*}(x)/F_{X_{i:n}}(x)$ is increasing in $x$. Note that, when $\prod_{i=1}^n p'_i / \prod_{i=1}^n p_i \geq 1$ implies

$$\lim_{x \to 0} \frac{F_{Y_{i:n}^*}(x)}{F_{Y_{i:n}}(x)} = \lim_{x \to 0} \frac{\prod_{i=1}^n p'_i F_{X_{i:n}^*}(x)}{\prod_{i=1}^n p_i F_{X_{i:n}}(x)} \geq \frac{F_{X_{i:n}^*}(0)}{F_{X_{i:n}}(0)},$$

that is, $F_{Y_{i:n}^*}(x)/F_{Y_{i:n}}(x)$ is increasing in $x \in \mathbb{R}_+$. The desired result is proved.

$\square$
Remark 7. According to Theorem 7, more heterogeneity among the scale parameter $\alpha$ obtains the greater the hazard rate function of the smallest claim amounts.

The following Example 4 illustrates the result of Theorem 7.

Example 4. Assume $X_1, X_2, X_3, X_4, X_5, X_6$ are independent non-negative random variables with $X_i - W - G(\alpha_i, \beta, \gamma_i)[X_i^* \sim W - G(\alpha_i^*, \beta, \gamma_i)]$, $i = 1, 2, 3, 4, 5, 6$. Taking the baseline distribution as $F(x) = 1 - e^{-\gamma x}$, and let $\alpha = (0.32, 0.26, 0.33) < \beta = (0.19, 0.23, 0.4) = \alpha, \beta \in (1.1, 3.1)$. For convenience, let $p_i = (0.26, 0.49, 0.59)$, $p_{i1} = (0.45, 0.64, 0.88)$.

On the other hand, set $\beta = 0.65$ and $\gamma = (0.11, 0.12, 0.13)$ with the remaining conditions, from 4(b) see that the difference of the hazard rate functions $h_{Y_1}(x) - h_{Y_2}(x) \geq 0$, for all $x = -\ln u$, $u \in (0, 1]$, hence, the condition $\beta > 1$ of Theorem 7 is sufficient and not necessarily necessary for the need of mathematical proof.

Theorem 8. Suppose $X_1, X_2, \ldots, X_n[X_1^*, X_2^*, \ldots, X_n^*]$ are independent non-negative random variables with $X_i - W - G(\alpha_i, \beta, \gamma_i)[X_i^* \sim W - G(\alpha_i^*, \beta, \gamma_i)]$. Let $I_{p_1}, I_{p_2}, \ldots, I_{p_n}$ be a set of independent Bernoulli random variables, independent of $X_i$s with $E(I_{p_i}) = p_i, E(I_{p_i}) = p_i^*$, $i = 1, 2, \ldots, n$. If $\beta \geq 2$, and

(i) $w(\cdot)$ is increasing convex and twice differentiable, and $w''(x)$ is increasing.

(ii) $\prod_{i=1}^n p_i \leq \prod_{i=1}^n p_i^*$.

(iii) $(\gamma_1^*, \gamma_2^*, \ldots, \gamma_n^*) < \gamma_1, \gamma_2, \ldots, \gamma_n$.

Then, for $\alpha, \gamma, \gamma^* \in \mathcal{D}_s$, we have

$$Y_{1: n} \leq hr^* Y_{1: n}.$$  \hfill (68)

Proof. In view point of Theorem 7, the hazard rate function of $X_{1:n}$ is

$$r_{X_{1:n}}(x) = \beta \sum_{i=1}^n \alpha_i w_i^{\beta-1}(y)(y)x = \sum_{i=1}^n \alpha_i w_i^{\beta-1}(y)(y)x.$$  \hfill (69)

For convenience, let

$$\psi(y) = \sum_{i=1}^n \alpha_i w_i^{\beta-1}(y)(y)x,$$  \hfill (70)

we have, for $i = 1, 2, \ldots, n$

$$\frac{\partial \psi(y)}{\partial \gamma_i} = a_i w_i^{\beta-2} \left( y, x \right) \left[ w(y, x) w_i^\prime(y, x) + (\beta - 1) y_i x \left( w(y, x)^2 + y_i w(y, x) w_i^\prime(y, x) \right) \right].$$  \hfill (71)

where

$$\psi_1(y) = \left[ w(y, x) w_i^\prime(y, x) + (\beta - 1) y_i x \left( w(y, x)^2 + y_i w(y, x) w_i^\prime(y, x) \right) \right] \left( \alpha_i w_i^{\beta-2} (y, x) - \alpha_i \left( w(y, x) \right)^{\beta-2} \right),$$

$$\psi_2(y) = \left[ w(y, x) w_i^\prime(y, x) - w(y, x) w_i^\prime(y, x) + (\beta - 1) y_i x \left( w(y, x)^2 \right) - (\beta - 1) y_i x \left( w(y, x)^2 \right) + y_i w(y, x) w_i^\prime(y, x) - y_i w(y, x) w_i^\prime(y, x) \right] \left( y, x \right)^{\beta-2}(y, x).$$

From the increasing convexity of $w(\cdot)$ and the increasing property of $w^\prime(y, x)$, it holds that $\psi_1(y) \geq 0$ and $\psi_2(y) \geq 0$. Thus, for any $i, j$, we have

$$\left( y_i - y_j \right) \left( \frac{\partial \psi(y)}{\partial \gamma_i} - \frac{\partial \psi(y)}{\partial \gamma_j} \right) \geq 0,$$  \hfill (75)

which shows that $\psi(y) \leq \text{schur-convex in } y$. According to condition (iii), we further have

$$\psi(y_1, y_2, \ldots, y_n) \geq \psi(y_1^*, y_2^*, \ldots, y_n^*).$$  \hfill (76)

Similar to the proof Theorem 7 we can prove that $Y_{1: n} \leq hr^* Y_{1: n}$. The proof is finished. \hfill $\square$
Remark 8. As per Theorem 8, more heterogeneity among the scale parameter \( \gamma \) claim severities obtains the greater the hazard rate function of the smallest claim amounts. As one anonymous reviewer suggested, Theorem 3.10(iii) in Torrado and Navarro [19] holds for dependent insurance portfolios, they give the necessary and sufficient conditions for the hazard rate order between smallest claim amounts from portfolios with arbitrary distribution functions. Our Theorem 7 and Theorem 8 are independent case and for a specific distribution function (Weibull-G Distributions), it is evident that Theorem 3.10(iii) of Torrado and Navarro [19] is broader than our Theorem 7 and 8.

Next, Example 5 is provided to illustrate the result of Theorem 8.

Example 5. Let \( F(x) = 1 - (1 + x/y)^{-\mu} \), \( \mu > 2 \), \( y > 0 \), then \( w(x) = (1 + x/y)^{\mu} \). Set \( y = (4.65, 4.55, 4.5) \), \( \mu = (2.1, 4.1) \), \( \alpha = (0.12, 0.06, 0.01) \), \( (p_1, p_2, p_3) = (0.3, 0.4, 0.5) \), \( (p_1^*, p_2^*, p_3^*) = (0.55, 0.038, 0.66) \). One can verify that the conditions of Theorem 8 are all satisfied. Figure 5(a) plots the difference \( h_{Y_{1:3}}(x) - h_{Y_{1:3}}^*(x) \) of two hazard rate function, for all \( x = -\ln u, u \in (0,1) \), which check the result of Theorem 8.

On the other hand, set \( \beta = 0.001, \mu = 2.5 \), \( \alpha = (0.13, 0.12, 0.11) \), \( (p_1, p_2, p_3) = (0.2, 0.3, 0.4) \), \( (p_1^*, p_2^*, p_3^*) = (0.75, 0.68, 0.86) \) with the remaining conditions, from 5(b) see that the difference of the hazard rate functions \( h_{Y_{1:3}}(x) - h_{Y_{1:3}}^*(x) \) is crossing at line \( y = 0 \), \( x \in (0,10) \), hence, the condition \( \beta \geq 2 \) of Theorem 8 is an necessarily condition and not is removed.

4.2. Insurance Portfolios of Dependent Risks. As described by Barmalzan et al. [11]; the independent assumption among portfolio claim amounts is not usually suitable in practice and we shall need to explore portfolio risks with statistical dependent claims. In this subsection, we establish usual stochastic order of the smallest claim amounts with dependent heterogeneous Weibull-G insurance portfolios under the Archimedean survival copula. To establish the stochastic comparisons of the smallest claim amounts, we first give two Lemmas.

Lemma 7. Suppose \( X_i \sim W - G(\alpha_i, \beta_i, \gamma_i) \), \( X_i^* \sim W - G(\alpha_i^*, \beta_i, \gamma_i) \) \( (i = 1, 2, \ldots, n) \), and they have the associated Archimedean survival copula with the generator \( \psi_1 \) or \( \psi_2 \) denoting the pseudo-inverse of \( \psi_1 \) by \( \phi_i \) \( (i = 1, 2) \). If \( \phi_i \circ \psi_1 \) is super additive, and \( \psi_1 \) or \( \psi_2 \) is log-convex. Then for any \( (\alpha_1, \alpha_2, \ldots, \alpha_n) \succ m (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*) \) implies

\[
X_{1:n} \preceq_{st} X_{1:n}^*.
\] (77)

Proof. The distribution function of Weibull-G distribution is increasing in \( \alpha \) and the survival function of Weibull-G is log-concave in \( \alpha \), which satisfies conditions in Proposition 3.16 of Torrado [30]. Hence, according to Proposition 3.16 of Torrado [30]; we can easily obtain Lemma 7.

The following result can be proven by applying a similar method to that of Lemma 7, and thus the detailed proof is omitted for brevity.

Lemma 8. Suppose \( X_i \sim W - G(\alpha_i, \beta_i, \gamma_i) \), \( X_i^* \sim W - G(\alpha_i, \beta_i, \gamma_i^*) \) \( (i = 1, 2, \ldots, n) \), and they have the associated Archimedean survival copula with the generator \( \psi_1 \) or \( \psi_2 \). Denote the pseudo-inverse of \( \psi_i \) by \( \phi_i \) \( (i = 1, 2) \). If \( \phi_1 \circ \psi_1 \) is super additive, and \( \psi_1 \) or \( \psi_2 \) is log-convex. Then for any \( \alpha, \gamma, \gamma^* \in \mathbb{R}_+ \), \( \beta \geq 1 \), \( (\gamma_1^*, \gamma_2^*, \ldots, \gamma_n^*) \prec m (\gamma_1, \gamma_2, \ldots, \gamma_n) \) implies

\[
X_{1:n} \preceq_{st} X_{1:n}^*.
\] (78)

Proof. The distribution function of Weibull-G is increasing in \( \gamma \) and the survival function of Weibull-G is log-concave in \( \gamma \) for \( \beta \geq 1 \), which checks conditions in Proposition 3.16 of Torrado [30]. Thus, according to Proposition 3.16 of Torrado [30]; we can easily establish Lemma 8.
Remark 9. Observe that Lemmas 7 and 8 can be easily proved from Theorem 3.10 (iii) of Torrado and Navarro [19]. Thus, Lemma 7 and 8 are particular cases of Theorem 3.10 (iii) in Torrado and Navarro [19].

Next, we give some sufficient conditions for the usual stochastic order between the smallest claim amounts from dependent heterogeneous portfolios risks.

Theorem 9. Suppose \(-X_i \sim W - G(\alpha_i, \beta_i, \gamma_i)\) \([X_i^* \sim W - G(\alpha_i, \beta_i, \gamma_i^*)]\) \((i = 1, \ldots, n)\), and they have the associated Archimedean survival copula with the generator \(\psi_{1:3}\). Let \(I_{p_1}, I_{p_2}, \ldots, I_{p_n} [I_{p_1^*}, I_{p_2^*}, \ldots, I_{p_n^*}]\) be a set of independent Bernoulli random variables, independent of \(X_i^* \mid X_i\)'s, with \(E(I_{p_i}) = p_i[E(I_{p_i}^*) = p_i^*], \) \((i = 1, 2, \ldots, n)\). Further, suppose \(\phi_2^o \psi_{1:3}\) is super-additive, \(\psi_1\) or \(\psi_2\) is log-convex. Then, for any \(\prod_{i=1}^n p_i \leq \prod_{i=1}^n p_i^*\), and \((\alpha_1^*, \alpha_2^*, \ldots, \alpha_n) < m (\alpha_1, \alpha_2, \ldots, \alpha_n)\) implies
\[Y_{1:n} \preceq Y^*_{1:n}\] (79)

Proof. From (60), we have
\[F_{Y_{1:n}}(x) = e^{-\sum_{i=1}^n \alpha_i(w(y_i, x))\prod_{i=1}^n p_i = F_{X_{1:n}}((x))\prod_{i=1}^n p_i}\] (80)

And thus, the desired result follows immediately from assumption \(\prod_{i=1}^n p_i \leq \prod_{i=1}^n p_i^*\) and Lemma 7.

Remark 10. As per Theorem 9, we know that more heterogeneity among the distribution functions of claim severities leads to a smaller value of the smallest claim amount in the stochastic order under Archimedean copula dependence.

The next numerical Example 6 illustrates the result of Theorem 9.

Example 6. Under the setting of Theorem 9. Let \(F(x) = e^{-\gamma x^\mu}, \gamma > 0, \mu > 1,\) and generators \(\psi_1(x) = e^{(1/\eta_1^*) x^{(1-\gamma)}}, 0 < \eta_1^* \leq 1, \) \(\psi_2(x) = (\eta_2 x + 1)^{(-1/\eta_2)}\), \(\eta_2 > 0\). One can check that \(\psi_1(x)\) and \(\psi_2(x)\) is log-concave and log-convex, respectively. It can be seen that \(\phi_2^o \psi_1(x) = e^{e^{\eta_2^* (\alpha_i + 1/\eta_1^*)}} (\eta_2^* (\eta_1 + \eta_2 e^\gamma) \geq 0, \) hence, \(\phi_2^o \psi_1(x)\) is super-additive. Set \(\mu = 1.9, \) \(\beta \in (1, 1.5), \) \(\gamma_i = 0.85, \) \(\eta_i = 1.6, \) \(\alpha \in (2.6, 2, 1.3, 0.9) \) \(3.0, \) \(2.3, 1.4, 0.1) = \alpha^*, \gamma \in (0.16, 0.15, 0.13, 0.11), \(p_1, p_2, p_3) = (0.78, 0.38, 0.29, 0.37), (p_1^*, p_2^*, p_3^*) = (0.82, 0.42, 0.31, 0.40),\) as is seen in Figure 6. \(F_{Y_{1:n}}(x) - F_{Y^*_{1:n}}(x) \geq 0, \) for all \(x = \ln u, u \in (0, 1)\).

One natural question arises, whether the requirements \(\phi_2^o \psi_1\) is super additive in Theorem 9 could be removed or not? The following example provides a negative answer.

Example 7. Under the setup of Example 6, set \(\mu \in (1, 1.4, 1), \beta = 1.28,\) let two generators \(\psi_1(x) = (\eta_i x + 1)^{-1/\eta_i}, \) \(\psi_2(x) = (\eta_i x + 1)^{1-1/\eta_i}, \) \(\eta_i > 0, i = 1, 2, \) Note that \(\log(\psi_1(x)) = \log((\eta_i x + 1)^{-1}) \geq 0, \) for \(x \geq 0,\) thus, \(\psi_1(x)\) is log-convex. Taking \(\eta_i = 12.6, \) \(\eta_i = 0.2,\) owing to \(\phi_2^o \psi_1(x) = (\eta_i - \eta_i) (\eta_i x + 1)^{1/\eta_i} \leq 0,\) which implies that \(\phi_2^o \psi_1(x)\) is not super additive, and the plot of the difference functions \(F_{Y_{1:n}}(x) - F_{Y^*_{1:n}}(x)\) in Figure 7, which neither non-negative nor non-positive, hence, the superset additive of \(\phi_2^o \psi_1\) of Theorem 9 cannot be dropped.

Theorem 10. Suppose \(X_i \sim W - G(\alpha_i, \beta_i, \gamma_i)\) \([X_i^* \sim W - G(\alpha_i, \beta_i, \gamma_i^*)]\) \((i = 1, 2, \ldots, n)\), and they have the associated Archimedean survival copula with the generator \(\psi_{1:3}\). Let \(I_{p_1}, I_{p_2}, \ldots, I_{p_n} [I_{p_1^*}, I_{p_2^*}, \ldots, I_{p_n^*}]\) be a set of independent Bernoulli random variables, independent of \(X_i^* \mid X_i\)'s, with \(E(I_{p_i}) = p_i[E(I_{p_i}^*) = p_i^*], \) \((i = 1, 2, \ldots, n)\). Further, let \(\phi_2^o \psi_{1:3}\) is super-additive, \(\psi_1\) or \(\psi_2\) is log-convex. Then, for any \(\prod_{i=1}^n p_i \leq \prod_{i=1}^n p_i^*\), and \((\gamma_1^*, \gamma_2^*, \ldots, \gamma_n^*) < m (\gamma_1, \gamma_2, \ldots, \gamma_n)\) implies
\[Y_{1:n} \preceq Y^*_{1:n}\] (81)

Proof. From (60), we have
And thus the desired result follows immediately from assumption \( \prod_{i=1}^{n} p_i \leq \prod_{i=1}^{n} p_i^* \) and Lemma 8.

\[ F_Y(x) = e^{-\sum_{i=1}^{n} \alpha_i (w(y_i))} \prod_{i=1}^{n} p_i \leq F_{Y^*(x)} = \prod_{i=1}^{n} p_i^* \leq \prod_{i=1}^{n} p_i^*, \]  

(82)

Remark 11. As per Theorem 10, we know that more heterogeneity among the distribution functions of claim severities leads to a smaller value of the smallest claim amount in the sense of the usual stochastic ordering. As one anonymous reviewer suggested, all the main results in Torrado [30] are for dependent portfolios. Actually, Theorem 9 and 10 are particular cases of Theorem 3.10(ii) in Torrado [30].

5. An Application

In this section, based on the real data of insurance, we apply the proposed methods to present some significantly insights for insurance industry. Eric and Ziegel [31] give the data of set of the third party motor insurance for 1977 for one of seven geographical zones. The data represent the total amount paid by the insurance company and the total number claims received by the insurance company. Raqab et al. [32] uses normal distribution

\[ F(x) = \int_{0}^{x} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(1/2)\left(t-\mu\right)^2}{\sigma^2}} \, dt, \]  

(83)

to fit the univariate payment data set, and obtained parameter’s estimate values \( \hat{\mu} = 31.1 \) and \( \hat{\sigma} = 24.062 \). Now, in order to help insurance industry to analysis the information of survival function on the smallest claim amounts under difference scale parameters, we select the previous normal distribution as the baseline distribution of the smallest amounts, and set \( \alpha_1 = (3, 3, 3), \alpha_2 = (4, 3.5, 3.6), \alpha_3 = (4.5, 4, 3.8), \alpha_4 = (5, 4.5, 4), \beta = (e^{-0.1}, e^{-0.1}, e^{-0.1}, e^{-0.1}) \). Obviously, \( \alpha_1 \prec_w \alpha_2 \prec_w \alpha_3 \prec_w \alpha_4 \).

Based on Theorem 9, we plot the curves of survival function in Figure 8, from which it see that
We establish the stochastic order between the largest claim amounts and the smallest claim amounts from sets of dependent heterogeneous insurance portfolios. Sufficient conditions are provided for the usual stochastic order between the largest claim amounts when the matrix of parameters \( (\mathbf{h}(p), \alpha) \) changes to another matrix in some certain mathematical sense. Meanwhile, based on the concepts of vector majorizations and related orders, we establish stochastic comparisons between the smallest claim amounts in the sense of the usual stochastic and hazard rate orders. The obtained findings present quite significantly perceptions and insights into determining the annual premium for an insurant.

As discussed in Zhang [14]; Zhang et al. [33]; it is of great importance to investigate some other stochastic orderings, such as the convex transform ordering, star ordering and right spread ordering for the extreme claim amounts arising from two sets of heterogeneous portfolios risk, which remain open and merit further discussion.

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare no conflicts of interest.

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