

Research Article

Time-Scale Integral Inequalities of Copson with Steklov Operator in High Dimension

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The paper derives some new time-scale (TS) dynamic inequalities for multiple integrals. The obtained inequalities are special cases of Copson integral using Steklov operator in (TS) version with high dimension. We prove the inequalities with several formulas for the operator and in different cases $m > \mu + 1$ and $m < \mu + 1$ for every $\mu \geq 1$, using time-scales (TSs) setting for integral properties, chain rules, Fubini's theorem, and Hölder's inequality.

1. Introduction

Equations and inequalities are the core of scientific study and have a great influence on a huge number of applications. A large number of physical phenomena and engineering studies have been analyzed and explained through equations and inequalities. For this reason, the study in this field developed rapidly and many types of inequalities and equations appeared. Dynamic inequalities on (TS) are some of the important inequalities that were extended by a lot of researchers and have interesting applications. Furthermore, dynamic inequalities are used to study the behaviour of dynamic equations.

Mathematical analysis has been the most important study in mathematics for the past three decades. Integral inequalities are one of the main studies and the core of mathematical analysis. In the 20th century, a significant part of science was numerical inequalities as the first composition to be released in 1934, through the published study by P'olya et al. [1]. This framework of inequalities played a vital role in the improvement processes and various applications of mathematics.

A large number of essential studies of integral inequalities appeared in the twentieth century, including pure and applied mathematics study. In 1920, Hardy produced the discrete Hardy inequality [2]. This inequality was also

proved by himself in [3] (see also [4]), using the variations calculus to obtain the following inequality that is very valuable across both technological sciences and mathematics. If $p > 1$ and $h \geq 0$ in $(0, \infty)$ and

$$H(x) = \int_0^x h(t)dt < \infty,$$

then

$$\int_0^\infty \left(\frac{H}{x}\right)^p dx < \Lambda^p \int_0^\infty h^p(x)dx, \quad (1)$$

where $\Lambda = p(p-1)^{-1}$ is the best possible constant (BPC). Several important assessments and their implementation are done by inequality (1). Furthermore, the inequality is true in case $0 < a < b < \infty$,

$$\int_a^b \left(\frac{H}{x}\right)^p dx < \Lambda^p \int_a^b h^p(x)dx, \quad (2)$$

where $0 < \int_a^b h^p(x)dx < \infty$. The classical inequality of Hardy declares that if $p > 1$ and h is nonnegative and measurable on (a, b) , then (2) is valid except $h \equiv 0$ a.e. in (a, b) , considering the (BPC).

Integral inequalities (3) and (4) are established in 1928 by Hardy [5].

Let f be a nonnegative measurable function on $(0, \infty)$:

$$(Hh)(x) \leq \begin{cases} \int_0^x h(t)dt, & \text{for } a < p - 1, \\ \int_x^\infty h(t)dt, & \text{for } a > p - 1. \end{cases} \quad (3)$$

Then,

$$\int_0^\infty x^{a-p} (Hh)^p(x) dx \leq \left(\frac{p}{|p-a-1|} \right)^p \int_0^\infty x^a h^p(x) dx, \quad \text{for } p > 1. \quad (4)$$

Later, in 1976, Copson studied the integral inequalities ([6], Theorem 1, Theorem 3) as follows.

Let h and v be functions such that they are nonnegative measurable on $(0, \infty)$;

$$V(x) = \int_0^x v(t)dt, \\ (Ch)(x) \leq \begin{cases} \int_0^x h(t)v(t)dt, & \text{for } c > 1, \\ \int_x^\infty h(t)v(t)dt, & \text{for } c < 1. \end{cases}$$

Then,

$$\int_0^\infty V^{-c}(x)v(x)(Ch)^p(x)dx \\ \leq \left(\frac{p}{|c-1|} \right)^p \int_0^\infty V^{p-c}(x)v(x)dx, \quad \text{for } p \geq 1.$$

Many papers included new extensions and generalizations for the inequalities above in more general settings. For instance, in 1979, some generalizations of Hardy-type inequality were proved by Chan [7]. Then, in 1992, Pachpatte [8] generalized the inequalities that were produced by Chan [7]. In 2005, P. Rehak used (TS) setting to extend Hardy's inequalities [9]. In 2015, Pachpatte's inequalities [8] were extended by Saker and O'Regan [10], with setting of (TSs). Later, some extensions of (TSs) Hardy inequalities were done for functions with high dimensions (see, for example, [11–14]).

In 2021, Albalawi and Khan generalized the main integral of Hardy and Copson inequalities, using the Steklov operator. The operator is defined in the following formulas with considering conditions in two cases (for more details, see [15]).

The aim of this paper is extending the study in [16] that was used for some new Hardy-type inequalities to obtain new special Copson inequalities with the Steklov operator (see [15]) in (TS) versions with high dimension. The results below are proved in two cases $m > \mu + 1$ and $m < \mu + 1$ by considering some general conditions that can be applied for any variable in the integral. To achieve this paper, we use (TSs) settings in integrals properties, chain rules, Hölder's inequality, and Fubini's theorem.

The paper takes the following structure: After introduction, the main concepts of (TSs) are presented in Section

2. Then, in Section 3, we generalized a class of Copson inequalities pertaining the Steklov operator with (TS) in high dimension. Lastly, conclusion of our results is presented.

2. Preliminaries and Lemmas on Time Scales

We state the main concepts of (TSs) that are used in this paper (for more details about (TS) calculus, see [17, 18]).

(TS) calculus in continuous case and discrete analysis was introduced by Hilger [19] in 1988. We denote to a subset (TS) of the real numbers \mathbb{R} by \mathbb{T} . Hence, the sets of numbers \mathbb{R} , \mathbb{Z} , and \mathbb{N} can be considered as (TSs).

Let $\sigma: \mathbb{T} \rightarrow \mathbb{T}$ be a forward jump operator, such that $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, while $\varsigma: \mathbb{T} \rightarrow \mathbb{T}$ is the backward jump operator, given by $\varsigma(t) = \sup\{s \in \mathbb{T} : s < t\}$ for all $t \in \mathbb{T}$.

If $\sigma(t) > t$, then t is right-scattered, and if $\varsigma(t) < t$, t is left-scattered. In the case if points are right-scattered and left-scattered at the same time, then they will be isolated. The point t is right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, while t is left-dense if $t > \inf \mathbb{T}$ and $\varsigma(t) = t$.

Let $g: \mathbb{T} \rightarrow \mathbb{R}$ be a continuous function and if it satisfied the continuity at all right-dense points in \mathbb{T} and the limits of the left-sided exist (finite) at all left-dense points in \mathbb{T} , then g is known rd-continuous. We use $C_r(\mathbb{T}, \mathbb{R})$ to denote the space of all rd-continuous.

A function $g: \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable at $t \in \mathbb{T}$, if there is a real number $\beta = g^\Delta(t)$ and for all $\varepsilon > 0$, there exists a neighbor U of t satisfies

$$|g(\sigma(t)) - g(s) - \beta(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

The Δ -derivative of a function g in high order $n \in \mathbb{N}$ is given by

$$g^{\Delta^n}(t) = \left(g^{\Delta^{n-1}}(t) \right)^\Delta.$$

If the Δ -derivative of $g^{\Delta^{n-1}}(t)$ exists, the following examples show that the delta derivative for every number set of (TSs).

If $\mathbb{T} = \mathbb{R}$, then

$$g^\Delta(t) = g' = \lim_{\Delta t \rightarrow 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}, \quad \text{for all } t \in \mathbb{T}.$$

If $\mathbb{T} = \mathbb{N}$, then

$$g^\Delta(t) = g(t + 1) - g(t), \quad \text{for all } t \in \mathbb{T}.$$

Let $g: \mathbb{T} \rightarrow \mathbb{R}$; if g is continuous at right-scattered t , then it is delta-derivative of the function g , given by

$$g^\Delta(t) = \frac{g(\sigma(t)) - g(t)}{\sigma(t) - t}.$$

In the case of t is not right-scattered, then the derivative of g is given by

$$g^\Delta(t) = \lim_{s \rightarrow t} \frac{g(\sigma(t)) - g(s)}{t - s}$$

$$= \lim_{s \rightarrow \infty} \frac{g(t) - g(s)}{t - s}.$$

Here, the limit exists. Note that if $\mathbb{T} = \mathbb{R}$, we have

$$\sigma(t) = t,$$

$$g^\Delta(t) = g'(t).$$

If $\mathbb{T} = \mathbb{Z}$, we have

$$\sigma(t) = t + 1,$$

$$g^\Delta(t) = \Delta g(t),$$

$$\int_a^b g(t)\Delta t = \sum_{t=a}^{b-1} g(t).$$

Lemma 1. Let $h, g: \mathbb{T} \rightarrow \mathbb{R}$ be delta-differentiable. Then,

$$(hg)^\Delta = h^\Delta g + h^\sigma g^\Delta$$

$$= f g^\Delta + f^\Delta g^\sigma, \tag{5}$$

$$\left(\frac{h}{g}\right)^\Delta = \frac{h^\Delta g - h g^\Delta}{g g^\sigma}.$$

The Cauchy integral of a delta-differential function of $g(g^\Delta)$ is defined by

$$\int_a^d g^\Delta(t)\Delta t = g(d) - g(a), \quad \text{for } a, d \in \mathbb{T}.$$

The time-scale integration by parts formula is given by

$$\int_a^d h(t)g^\Delta(t)\Delta t = h(t)g(t)\Big|_a^d - \int_a^d h^\Delta(t)g^\sigma(t)\Delta t, \quad a, d \in \mathbb{T}. \tag{6}$$

The infinite integrals are defined by

$$\int_a^\infty g(t)\Delta t = \lim_{d \rightarrow \infty} \int_a^d g(t)\Delta t.$$

If $\mathbb{T} = \mathbb{R}$, we have

$$\int_a^d g(t)\Delta t = \int_a^d g(t)dt.$$

If $\mathbb{T} = \mathbb{Z}$, we get

$$\int_a^d g(t)\Delta t = \sum_{t=a}^{d-1} g(t).$$

Lemma 2 (chain rule [16]). Assume a continuous function, $w: \mathbb{R} \rightarrow \mathbb{R}$, a delta-differentiable, and $w: \mathbb{T} \rightarrow \mathbb{R}$, on \mathbb{T}^c

and a continuous differentiable $h: \mathbb{R} \rightarrow \mathbb{R}$. Then, there exists $c \in [t, \sigma(t)]$ with

$$(h \circ w)^\Delta(t) = h'(w(c))w^\Delta(t). \tag{7}$$

Lemma 3 (dynamic Hölder inequality). Let $a, d \in \mathbb{T}$ and $h, w \in C_{rd}([a, d]_{\mathbb{T}}, [0, \infty))$. If $p_1, p_2 > 1$ with $1/p_1 + 1/p_2 = 1$, then

$$\int_a^d h(t)w(t)\Delta t \leq \left(\int_a^d h^{p_1}(t)\Delta t\right)^{1/p_1} \left(\int_a^d w^{p_2}(t)\Delta t\right)^{1/p_2}. \tag{8}$$

Theorem 4 (Fubini's theorem [20]). Let (Y, N, μ_Δ) and $(\Sigma, L, \gamma_\Delta)$ be (TS) measure spaces with finite dimension. Consider $(Y \times \Sigma, N \times L, \mu_\Delta \times \gamma_\Delta)$ as the measure space, where $N \times L$ is the σ -algebra product that is generated by $\{E \times F: E \in N, F \in L\}$ and

$$(\mu_\Delta \times \gamma_\Delta)(E \times F) = \mu_\Delta(E)\gamma_\Delta(F).$$

Then, Fubini's theorem satisfied.

To be more accurate, if $\xi: Y \times \Sigma \rightarrow \mathbb{R}$ is $(\mu_\Delta \times \gamma_\Delta)$ -integrable,

$$\Psi(\gamma) = \int_\Sigma \xi(\gamma, \Pi)\Delta \Pi, \quad \text{exists for } \Pi \in Y,$$

and

$$\Psi(\Pi) = \int_Y \xi(\gamma, \Pi)\Delta \gamma, \quad \text{exists for } \gamma \in \Sigma.$$

Then,

$$\int_Y \Delta \gamma \int_\Sigma \xi(\gamma, \Pi)\Delta \Pi = \int_\Sigma \Delta \Pi \int_Y \xi(\gamma, \Pi)\Delta \gamma.$$

3. Main Results

A new (TS) version of Copson-type inequality with Steklov operator for multiple integrals is obtained in this section. We consider the nonnegative rd-continuous functions w_l, f_l, g_l , and v_l are Δ -integrable and defined integrals. Throughout this paper, we set $K(t_1, \dots, t_k)$ as the Copson–Steklov-type operator considering the existence of the integral and also finite.

Theorem 5. Let \mathbb{T}_l be a (TS) and $a \in [0, \infty)_{\mathbb{T}_l}$, for $1 \leq l \leq k$ with $l, k \in \mathbb{N}$. In addition, let w_l, f_l, g_l , and v_l be nonnegative and rd-continuous functions on $[a, \infty)_{\mathbb{T}_l}$. Furthermore, assume there exist $\mu, \lambda \geq 1$ such that

$$\frac{w_l^{\Delta_l}(t_l)}{w_l^\sigma(t_l)} \leq \mu \frac{V_l^{\Delta_l}(t_l)}{V_l(t_l)},$$

and

$$\frac{g_l^{\Delta_l}(t_l)}{g_l^\sigma(t_l)} \leq \lambda \frac{F^{\Delta_l}(t_1, \dots, t_k)}{F(t_1, \dots, t_k)},$$

where $\Delta_l = \partial/\partial t_l$ for every l ,

$$V_l(t_l) = \int_a^{t_l} v_l(s_l)\Delta s_l, \quad \text{with } V_l(\infty) = \infty, \text{ and } w_l(a) = 0,$$

and

$$F(t_1, \dots, t_k) := \int_a^{t_1} \dots \int_a^{t_k} \prod_{l=1}^k \frac{1}{g_l(s_l)} \frac{v_l(s_l)}{V_l(s_l)} f \cdot (s_1, \dots, s_k) \Delta s_1, \dots, \Delta s_k.$$

Define the operator

$$K(t_1, \dots, t_k) = \prod_{l=1}^k g_l(t_l) F(t_1, \dots, t_k), \tag{9}$$

Then

$$\begin{aligned} & \int_a^\infty \dots \int_a^\infty \prod_{l=1}^k w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} (K^\sigma(t_1, \dots, t_k))^p \Delta t_1, \dots, \Delta t_k \\ & \leq \left(\frac{p(\lambda + 1)}{m - (\mu + 1)} \right)^p \int_a^\infty \frac{v_k(t_k)}{V_k^m(t_k)} \frac{w_k^p(t_k)}{(w_k^\sigma(t_k))^{p-1}} \left(\frac{g_k^\sigma(t_k)}{g_k(t_k)} \right)^p \left(\int_a^\infty \dots \int_a^\infty \prod_{l=1}^{k-1} w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} f^p(t_1, \dots, t_k) \Delta t_1, \dots, \Delta t_{k-1} \right) \Delta t_k, \end{aligned} \tag{10}$$

where $p \geq 1$ and $m > \mu + 1$.

Proof. We write the left side of (10) as follows:

$$\int_a^\infty \dots \int_a^\infty \prod_{l=1}^{k-1} w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} \Gamma_k \Delta t_1, \dots, \Delta t_{k-1}, \tag{11}$$

where Γ_k is the k -term

$$\Gamma_k = \int_a^\infty w_k^\sigma(t_k) \frac{v_k(t_k)}{V_k^m(t_k)} (K^\sigma(t_1, \dots, t_k))^p \Delta t_k.$$

Using formula (6) for integration by parts to compute Γ_k , we have

$$\begin{aligned} \Gamma_k &= \int_a^\infty w_k^\sigma(t_k) \frac{v_k(t_k)}{V_k^m(t_k)} (K^\sigma(t_1, \dots, t_k))^p \Delta t_k \\ &= [z(t_k)u(t_k)]_a^\infty - \int_a^\infty u(t_k) (z(t_k))^{\Delta_k} \Delta t_k, \end{aligned} \tag{12}$$

where $u^{\Delta_k}(t_k) = (v_k(t_k)/V_k^m(t_k))$ and then $u(t_k) = (-m + 1)V_k^{-m+1}(t_k)$ and $z^\sigma(t_k) = w_k^\sigma(t_k)(K^\sigma(t_1, \dots, t_k))^p$, implying that $z(t_k) = w_k(t_k)(K(t_1, \dots, t_k))^p$, and hence,

$$\begin{aligned} & z^{\Delta_k}(t_k) \\ & \&9; \cdot [g_k^{\Delta_k}(t_k)F(t_1, \dots, t_k) + g_k^\sigma(t_k)F^{\Delta_k}(t_1, \dots, t_k)]. \end{aligned}$$

Assume $\lambda \geq 1$ such that

$$\frac{g_k^{\Delta_k}(t_k)}{g_k^\sigma(t_k)} \leq \lambda \frac{F^{\Delta_k}(t_1, \dots, t_k)}{F(t_1, \dots, t_k)},$$

where $F^{\Delta_k} = (\partial F/\partial t_k)$, and since $c_l \in [t_l, \sigma(t_l)]$, we have

$$\begin{aligned} z^{\Delta_k}(t_k) &\leq w_k^{\Delta_k}(t_k) (K^\sigma(t_1, \dots, t_k))^p + p(\lambda + 1)w_k(t_k) \\ &\cdot (K^\sigma(t_1, \dots, t_k))^{p-1} g_k^\sigma(t_k) F^{\Delta_k}(t_1, \dots, t_k). \end{aligned}$$

Substituting the previous quantities in (12) and since $V_l(\infty) = \infty$ and $w_l(a) = 0$, then we have

$$\begin{aligned} \int_a^\infty w_k^\sigma(t_k) \frac{v_k(t_k)}{V_k^m(t_k)} (K^\sigma(t_1, \dots, t_k))^p \Delta t_k &= \frac{1}{m-1} \int_a^\infty \frac{1}{V_k^{m-1}(t_k)} w_k^{\Delta_k}(t_k) (K^\sigma(t_1, \dots, t_k))^p \Delta t_k \\ &+ \frac{p(\lambda + 1)}{m-1} \int_a^\infty \frac{1}{V_k^{m-1}(t_k)} w_k(t_k) (K^\sigma(t_1, \dots, t_k))^{p-1} g_k^\sigma(t_k) F^{\Delta_k}(t_1, \dots, t_k) \Delta t_k. \end{aligned}$$

Assume $\mu \geq 1$ such that

$$\frac{w_l^{\Delta_l}(t_l)}{w_l^\sigma(t_l)} \leq \mu \frac{V_l^{\Delta_l}(t_l)}{V_l(t_l)}.$$

Then, we obtain

$$\int_a^\infty w_k^\sigma(t_k) \frac{v_k(t_k)}{V_k^m(t_k)} (K^\sigma(t_1, \dots, t_k))^p \Delta t_k = \frac{\mu}{m-1} \int_a^\infty w_k^\sigma(t_k) \frac{v_k(t_k)}{V_k^m(t_k)} (K^\sigma(t_1, \dots, t_k))^p \Delta t_k + \frac{p(\lambda+1)}{m-1} \int_a^\infty \frac{1}{V_k^{m-1}(t_k)} w_k(t_k) (K^\sigma(t_1, \dots, t_k))^{p-1} g_k^\sigma(t_k) F^{\Delta_k}(t_1, \dots, t_k) \Delta t_k.$$

Since $F^{\Delta_k}(t_1, \dots, t_k) = (f(t_1, \dots, t_k)/g_k(t_k)) (v_k(t_k)/V_k(t_k))$, then we have

$$\int_a^\infty w_k^\sigma(t_k) \frac{v_k(t_k)}{V_k^m(t_k)} (K^\sigma(t_1, \dots, t_k))^p \Delta t_k \leq \frac{p(\lambda+1)}{m-1-\mu} \int_a^\infty \frac{v_k(t_k)}{V_k^m(t_k)} w_k(t_k) (K^\sigma(t_1, \dots, t_k))^{p-1} g_k^\sigma(t_k) \frac{f(t_1, \dots, t_k)}{g_k(t_k)} \Delta t_k.$$

Then, Hölder's inequality (8) with indices p and $p/(p-1)$ can be applied:

$$\Gamma_k = \int_a^\infty w_k^\sigma(t_k) \frac{v_k(t_k)}{V_k^m(t_k)} (K^\sigma(t_1, \dots, t_k))^p \Delta t_k \leq \left(\frac{p(\lambda+1)}{m-1-\mu} \right)^p \int_a^\infty \frac{v_k(t_k)}{V_k^m(t_k)} \frac{w_k^p(t_k)}{(w_k^\sigma(t_k))^{p-1}} f^p(t_1, \dots, t_k) \left(\frac{g_k^\sigma(t_k)}{g_k(t_k)} \right)^p \Delta t_k. \tag{13}$$

Substituting Γ_k in (11) and applying Fubini's Theorem 4, then we obtain the inequality

$$\int_a^\infty \dots \int_a^\infty \prod_{l=1}^k w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} (K^\sigma(t_1, \dots, t_k))^p \Delta t_1 \dots \Delta t_k \leq \left(\frac{p(\lambda+1)}{m-(\mu+1)} \right)^p \int_a^\infty \frac{v_k(t_k)}{V_k^m(t_k)} \frac{w_k^p(t_k)}{(w_k^\sigma(t_k))^{p-1}} \left(\frac{g_k^\sigma(t_k)}{g_k(t_k)} \right)^p \left(\int_a^\infty \dots \int_a^\infty \prod_{l=1}^{k-1} w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} f^p(t_1, \dots, t_k) \Delta t_1 \dots \Delta t_{k-1} \right) \Delta t_k.$$

□

Corollary 6. If $l = 1$ in Theorem 5, inequality (10) becomes

$$\int_a^\infty w^\sigma(t) \frac{v(t)}{V^m(t)} (K^\sigma(t))^p \Delta t \leq \left(\frac{p(\lambda+1)}{m-(\mu+1)} \right)^p \int_a^\infty \frac{v(t)}{V^m(t)} \frac{w^p(t)}{(w^\sigma(t))^{p-1}} \left(\frac{g^\sigma(t)}{g(t)} \right)^p f^p(t) \Delta t. \tag{14}$$

Corollary 7. If $\mathbb{T} = \mathbb{R}$ in Corollary 6, we obtain

$$\int_a^\infty w(t) \frac{v(t)}{V^m(t)} K^P(t) dt \leq \left(\frac{p(\lambda + 1)}{m - (\mu + 1)} \right)^P \int_a^\infty \frac{v(t)}{V^m(t)} w(t) f^P(t) dt. \tag{15}$$

Remark 8. Assume $\mu = 0$ and $\beta > \lambda$ in Corollary 7; then, we have Corollary 3 in [15]

$$\begin{aligned} & \int_a^\infty w(t) \frac{v(t)}{V^m(t)} K^P(t) dt \\ & \leq \left(\frac{\beta p}{m - 1} \right)^P \int_a^\infty w(t) \frac{v(t)}{V^m(t)} f^P(t) dt. \end{aligned} \tag{16}$$

Theorem 9. Let \mathbb{T}_l be (TS) and $a \in [0, \infty)_{\mathbb{T}_l}$, for $1 \leq l \leq k$ with $l, k \in \mathbb{N}$. In addition, let $w_l, f_l, g_l,$ and v_l be nonnegative and

rd-continuous functions on $[a, \infty)_{\mathbb{T}_l}$. Furthermore, assume there exist $\lambda, \mu \geq 1$ such that

$$\frac{w_l^{\Delta_l}(t_l)}{w_l^\sigma(t_l)} \geq \mu \frac{V_l^{\Delta_l}(t_l)}{V_l(t_l)},$$

and

$$\frac{g_l^{\Delta_l}(t_l)}{g_l^\sigma(t_l)} \geq \lambda \frac{F^{\Delta_l}(t_1, \dots, t_k)}{F(t_1, \dots, t_k)},$$

where $F^{\Delta_l} = (\partial F / \partial t_l)$; for every $l,$

$$V_l(t_l) = \int_a^{t_l} v_l(s_l) \Delta s_l, \quad \text{with } V_l(\infty) = \infty, \text{ and } w_l(a) = 0,$$

and

$$F(t_1, \dots, t_k) := \int_{t_1}^\infty \dots \int_{t_k}^\infty \prod_{l=1}^k \frac{1}{g_l(s_l)} \frac{v_l(s_l)}{V_l(s_l)} f(s_1, \dots, s_k) \Delta s_1 \dots \Delta s_k.$$

Define the operator

$$K(t_1, \dots, t_k) = \prod_{l=1}^k g_l(t_l) F(t_1, \dots, t_k).$$

Then,

$$\begin{aligned} & \int_a^\infty \dots \int_a^\infty \prod_{l=1}^k w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} (K^\sigma(t_1, \dots, t_k))^P \Delta t_1 \dots \Delta t_k \\ & \% \\ & \leq \left(\frac{p(\lambda + 1)}{\mu + 1 - m} \right)^P \int_a^\infty \frac{v_k(t_k)}{V_k^m(t_k)} \frac{w_k^p(t_k)}{(w_k^\sigma(t_k))^{p-1}} \left(\frac{g_k^\sigma(t_k)}{g_k(t_k)} \right)^P \left(\int_a^\infty \dots \int_a^\infty \prod_{l=1}^{k-1} w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} f^P(t_1, \dots, t_k) \Delta t_1 \dots \Delta t_{k-1} \right) \Delta t_k, \end{aligned} \tag{17}$$

where $p \geq 1$ and $0 \leq m < \mu + 1$.

Proof. We write the left side of (17) as follows:

$$\int_a^\infty \dots \int_a^\infty \prod_{l=1}^{k-1} w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} \Gamma_k \Delta t_1 \dots \Delta t_{k-1}. \tag{18}$$

Use formula (6) to calculate the following k -term:

$$\begin{aligned} \Gamma_k &= \int_a^\infty w_k^\sigma(t_k) \frac{v_k(t_k)}{V_k^m(t_k)} (K^\sigma(t_1, \dots, t_k))^P \Delta t_k \\ &= [u(t_k)z(t_k)]_a^\infty + \int_a^\infty (u(t_k))(-z(t_k))^{\Delta_k} \Delta t_k, \end{aligned} \tag{19}$$

where $u^{\Delta_k}(t_k) = w_k^\sigma(t_k)(v_k(t_k)/(V_k(t_k))^m)$ and $z^\sigma(t_k) = (K^\sigma(t_1, \dots, t_k))^P$.

Using (7) and the product rule (5), there exists $c_k \in [s_k, \sigma(s_k)]$ such that

$$\begin{aligned} (w_k(s_k)V_k^{1-m}(s_k))^{\Delta_k} &= w_k^{\Delta_k}(s_k)V_k^{1-m}(s_k) \\ &+ w_k^\sigma(1-m)(V_k^{-m}(c_k))V_k^{\Delta_k}(s_k). \end{aligned}$$

Assume $\mu \geq 1$ such that

$$\frac{w_k^{\Delta_k}(t_k)}{w_k^\sigma(t_k)} \geq \mu \frac{V_k^{\Delta_k}(t_k)}{V_k(t_k)}.$$

Since $V_k^{\Delta_k}(s_k) = v_k(s_k) \geq 0, s_k \leq c_k \leq \sigma(s_k),$ and $0 \leq m < 1,$ then

$$w_k^\sigma V_k^{-m}(s_k)v_k(s_k) \leq \frac{1}{1-m+\mu} (w_k(s_k)V_k^{1-m}(s_k))^{\Delta_k}. \quad (20)$$

By integration, we have

$$\begin{aligned} & u(t_k) \\ & \leq \frac{1}{1-m+\mu} \int_a^\infty (w_k(s_k)V_k^{1-m}(s_k))^{\Delta_k} \Delta s_k. \end{aligned}$$

Now, we calculate $(-K^P(t_1, \dots, t_k))^{\Delta_k}$ and we obtain

$$\begin{aligned} & (K^P(t_1, \dots, t_k))^{\Delta_k} \\ & \cdot [g_k^{\Delta_k}(t_k)F(t_1, \dots, t_k) + g_k^\sigma(t_k)F^{\Delta_k}(t_1, \dots, t_k)]. \end{aligned}$$

Assume $\lambda \geq 1$ such that

$$\frac{g_k^{\Delta_k}(t_k)}{g_k^\sigma(t_k)} \geq \lambda \frac{F^{\Delta_k}(t_1, \dots, t_k)}{F(t_1, \dots, t_k)},$$

where

$$F(t_1, \dots, t_k) := g_k(t_k) \int_{t_k}^\infty \frac{1}{g_k(s_k)} \frac{v_k(s_k)}{V_k(s_k)} f(s_1, \dots, s_k) \Delta s_k, \quad (21)$$

and since $V_k(\infty) = \infty$ and $c_k \geq s_k$, then we have

$$\begin{aligned} & (K^P(t_1, \dots, t_k))^{\Delta_k} \geq pK^{P-1}(t_1, \dots, t_k)(\lambda + 1)g_k^\sigma(t_k)F^{\Delta_k}(t_1, \dots, t_k) \\ & \geq p(\lambda + 1)K^{P-1}(t_1, \dots, t_k) \left(\frac{g_k^\sigma(t_k)}{g_k(t_k)} \frac{v_k(t_k)}{V_k(t_k)} f(t_1, \dots, t_k) \right). \end{aligned}$$

Then,

$$\begin{aligned} & (-K^P(t_1, \dots, t_k))^{\Delta_k} \\ & \leq p(\lambda + 1)K^{P-1}(t_1, \dots, t_k) \frac{g_k^\sigma(t_k)}{g_k(t_k)} \frac{v_k(t_k)}{V_k(t_k)} f(t_1, \dots, t_k). \end{aligned}$$

Hence, we have

$$\int_a^\infty w_k^\sigma(t_k) \frac{v_k(t_k)}{V_k^m(t_k)} (K^\sigma(t_1, \dots, t_k))^p \Delta t \leq \frac{p(\lambda + 1)}{\mu + 1 - m} \int_a^\infty w_k(t_k) \frac{v_k(t_k)}{V_k^m(t_k)} f(t_1, \dots, t_k) \frac{g_k^\sigma(t_k)}{g_k(t_k)} (K^\sigma(t_1, \dots, t_k))^{p-1} \Delta t_k.$$

Using Hölder's inequality, where $p_1 = p$ and $p_2 = (p/(p-1))$, we obtain

$$\int_a^\infty w_k^\sigma(t_k) \frac{v_k(t_k)}{V_k^m(t_k)} (K^\sigma(t_1, \dots, t_k))^p \Delta t_k \leq \left(\frac{p(\lambda + 1)}{\mu + 1 - m} \right)^p \int_a^\infty \frac{v_k(t_k)}{V_k^m(t_k)} \frac{w_k^p(t_k)}{(w_k^\sigma(t_k))^{p-1}} f^p(t_1, \dots, t_k) \left(\frac{g_k^\sigma(t_k)}{g_k(t_k)} \right)^p \Delta t_k. \quad (22)$$

Substituting (22) in (17), we have

$$\int_a^\infty \dots \int_a^\infty \prod_{l=1}^k w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} (K^\sigma(t_1, \dots, t_k))^p \Delta t_1 \dots \Delta t_k$$

$$\leq \left(\frac{p(\lambda + 1)}{\mu + 1 - m} \right)^p \int_a^\infty \frac{v_k(t_k)}{V_k^m(t_k)} \frac{w_k^p(t_k)}{(w_k^\sigma(t_k))^{p-1}} \left(\frac{g_k^\sigma(t_k)}{g_k(t_k)} \right)^p \left(\int_a^\infty \dots \int_a^\infty \prod_{l=1}^{k-1} w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} f^p(t_1, \dots, t_k) \Delta t_1 \dots \Delta t_{k-1} \right) \Delta t_k.$$

Corollary 10. If $l = 1$ and $\mathbb{T} = \mathbb{R}$ in Theorem 9, we get

$$\int_a^\infty w(t) \frac{v(t)}{V^m(t)} K^p(t) dt$$

$$\leq \left(\frac{p(\lambda + 1)}{\mu + 1 - m} \right)^p \int_a^\infty \frac{v(t)}{V^m(t)} w(t) f^p(t) dt, \tag{23}$$

where

$$K(t) = g(t) \int_t^\infty \frac{1}{g(s)} \frac{v(s)}{V(s)} f(s) ds.$$

Remark 11. Assume $\mu = 0$ and $\theta > \lambda$ in Corollary 10; we have Corollary 5 in [15].

Theorem 12. Let \mathbb{T}_l be (TS) and $a \in [0, \infty)_{\mathbb{T}_l}$, for $1 \leq l \leq k$ with $l, k \in \mathbb{N}$. In addition, let w_l, f_l, g_l , and v_l be nonnegative rd-continuous functions on $[a, \infty)_{\mathbb{T}_l}$. Furthermore, assume there exist $\lambda, \mu \geq 1$ such that

$$\frac{w_l^{\Delta_l}(t_l)}{w_l^\sigma(t_l)} \geq \mu \frac{V_l^{\Delta_l}(t_l)}{V_l(t_l)},$$

and

$$\frac{g_l^{\Delta_l}(t)}{g_l^\sigma(t)} \geq \lambda \frac{F^{\Delta_l}(t_1, \dots, t_k)}{F(t_1, \dots, t_k)},$$

where for every l ,

$$V_l(t_l) = \int_a^{t_l} v_l(s_l) \Delta s_l, \quad \text{with } V_l(\infty) = \infty, \text{ and } w_l(a) = 0,$$

and

$$F(t_1, \dots, t_k) := \int_{t_1}^\infty \dots \int_{t_k}^\infty \prod_{l=1}^k g_l(s_l) \frac{v_l(s_l)}{V_l(s_l)} f(s_1, \dots, s_k) \Delta s_1 \dots \Delta s_k.$$

Define the operator

$$K(t_1, \dots, t_k) := \prod_{l=1}^k \frac{1}{g_l(t_l)} F(t_1, \dots, t_k).$$

Then,

$$\int_a^\infty \dots \int_a^\infty \prod_{l=1}^k w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} (K^\sigma(t_1, \dots, t_k))^p \Delta t_1 \dots \Delta t_k$$

$$\leq \left(\frac{p(\lambda - 1)}{\mu + 1 - m} \right)^p \int_a^\infty \frac{v_k(t_k)}{V_k^m(t_k)} \frac{w_k^p(t_k)}{(w_k^\sigma(t_k))^{p-1}} \left(\frac{g_k(t_k)}{g_k^\sigma(t_k)} \right)^p \left(\int_a^\infty \dots \int_a^\infty \prod_{l=1}^{k-1} w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} f^p(t_1, \dots, t_k) \Delta t_1, \dots, \Delta t_{k-1} \right) \Delta t_k, \tag{24}$$

where $p \geq 1$ and $0 \leq m < \mu + 1$.

Proof. We write the left side of (24) as follows:

$$\int_a^\infty \dots \int_a^\infty \prod_{l=1}^{k-1} w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} \Gamma_k \Delta t_1 \dots \Delta t_{k-1}. \tag{25}$$

Apply (6) to calculate the following k -term:

$$\Gamma_k = \int_a^\infty w_k^\sigma(t_k) \frac{v_k(t_k)}{V_k^m(t_k)} (K^\sigma(t_1, \dots, t_k))^p \Delta t_k$$

$$= [u(t_k)z(t_k)]_a^\infty + \int_a^\infty (u(t_k))(-z(t_k))^{\Delta_k} \Delta t_k,$$

where $u^{\Delta_k}(t_k) = w_k^\sigma(t_k)(v_k(t_k)/V_k^m(t_k))$. Using the chain rule on (TS) (7) and product rule (5), there exist $c_k \in [s_k, \sigma(s_k)]$ such that

$$\begin{aligned} & (w_k(s_k)V_k^{1-m}(s_k))^{\Delta_k} \\ & + (1-m)w_k^\sigma(s_k)V_k^{-m}(c_k)V_k^{\Delta_k}(s_k). \end{aligned}$$

Assume $\mu \geq 1$ such that

$$\frac{w_k^{\Delta_k}(t_k)}{w_k^\sigma(t_k)} \geq \mu \frac{V_k^{\Delta_k}(t_k)}{V_k(t_k)}.$$

Since $V_k^{\Delta_k}(s_k) = v_k(s_k) \geq 0$, $s_k \leq c_k \leq \sigma(s_k)$, and $0 \leq m < \mu + 1$, then

$$(w_k(s_k)V_k^{1-m}(s_k))^{\Delta_k} \geq (1-m+\mu)w_k^\sigma V_k^{-m}(s_k)v_k(s_k),$$

implying

$$\begin{aligned} & u(t_k) \\ & \leq \frac{1}{1-m+\mu} \int_a^{t_k} (w_k(s_k)V_k^{1-m}(s_k))^{\Delta_k} \Delta s_k \\ & \leq \frac{1}{1-m+\mu} w_k(t_k)V_k^{1-m}(t_k). \end{aligned}$$

We calculate $(-K^P(t_1, \dots, t_k))^{\Delta_k}$, and we obtain

$$(K^P(t_1, \dots, t_k))^{\Delta_k} = pK^{P-1}(t_1, \dots, c_k) \left[\frac{-g_k^{\Delta_k}}{g_k^\sigma(t_k)g(t_k)} F(t_1, \dots, t_k) + \frac{1}{g_k^\sigma(t_k)} F^{\Delta_k}(t_1, \dots, t_k) \right].$$

Assume $\lambda \geq 1$ such that

$$\frac{g_k^{\Delta_k}(t_k)}{g_k(t_k)} \geq \lambda \frac{F^{\Delta_k}(t_1, \dots, t_k)}{F(t_1, \dots, t_k)},$$

where

$$F(t_1, \dots, t_k) := \frac{1}{g_k(t_k)} \int_a^{t_k} g_k(s_k) \frac{v_k(s_k)}{V_k(s_k)} f(s_1, \dots, s_k) \Delta s_k, \tag{27}$$

and since $c_k \geq t_k$, then we have

$$\begin{aligned} (K^P(t_1, \dots, t_k))^{\Delta_k} & \geq pK^{P-1}(t_1, \dots, t_k) (1-\lambda) \frac{1}{g_k^\sigma(t_k)} F^{\Delta_k}(t_1, \dots, t_k) \\ & \geq p(\lambda-1)K^{P-1}(t_1, \dots, t_k) \frac{1}{g_k^\sigma(t_k)} g_k(t_k) \frac{v_k(t_k)}{V_k(t_k)} f(t_1, \dots, t_k). \end{aligned}$$

Since $V(\infty) = \infty$, then

$$\begin{aligned} (-K^P(t_1, \dots, t_k))^{\Delta_k} & \leq -p(\lambda-1)K^{P-1}(t_1, \dots, t_k) \frac{g_k(t_k)}{g_k^\sigma(t_k)} \frac{v_k(t_k)}{V_k(t_k)} f(t_1, \dots, t_k) \\ & \leq p(\lambda-1)K^{\sigma(P-1)}(t_1, \dots, t_k) \frac{g_k(t_k)}{g_k^\sigma(t_k)} \frac{v_k(t_k)}{V_k(t_k)} f(t_1, \dots, t_k). \end{aligned}$$

Hence, we have

$$\begin{aligned} & \int_a^\infty w_k^\sigma(t_k) \frac{v_k(t_k)}{V_k^m(t_k)} (K^\sigma(t_1, \dots, t_k))^p \Delta t_k \\ & \leq [u(\infty)z(\infty) - u(a)z(a)] + \frac{p(\lambda - 1)}{\mu + 1 - m} \int_a^\infty w_k(t_k) \frac{v_k(t_k)}{V_k^m(t_k)} f(t_1, \dots, t_k) \frac{g_k(t_k)}{g_k^\sigma(t_k)} (K^\sigma(t_1, \dots, t_k))^{p-1} \Delta t_k \\ & \leq \frac{p(\lambda - 1)}{\mu + 1 - m} \int_a^\infty w_k(t_k) \frac{v_k(t_k)}{V_k^m(t_k)} f(t_1, \dots, t_k) \frac{g_k(t_k)}{g_k^\sigma(t_k)} (K^\sigma(t_1, \dots, t_k))^{p-1} \Delta t_k. \end{aligned}$$

Then, Hölder’s inequality (8) can be applied with indices p and $p/(p - 1)$:

$$\int_a^\infty w_k^\sigma(t_k) \frac{v_k(t_k)}{V_k^m(t_k)} (K^\sigma(t_1, \dots, t_k))^p \Delta t_k \leq \left(\frac{p(\lambda - 1)}{\mu + 1 - m} \right)^p \int_a^\infty \frac{v_k(t_k)}{V_k^m(t_k)} \frac{w_k^p(t_k)}{(w_k^\sigma(t_k))^{p-1}} f^p(t_1, \dots, t_k) \left(\frac{g_k(t_k)}{g_k^\sigma(t_k)} \right)^p \Delta t_k. \tag{28}$$

Substituting (28) in (26), we have

$$\begin{aligned} & \int_a^\infty \dots \int_a^\infty \prod_{l=1}^k w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} (K^\sigma(t_1, \dots, t_k))^p \Delta t_1 \dots \Delta t_k \\ & \leq \left(\frac{p(\lambda - 1)}{\mu + 1 - m} \right)^p \int_a^\infty \frac{v_k(t_k)}{V_k^m(t_k)} \frac{w_k^p(t_k)}{(w_k^\sigma(t_k))^{p-1}} \left(\frac{g_k(t_k)}{g_k^\sigma(t_k)} \right)^p \left(\int_a^\infty \dots \int_a^\infty \prod_{l=1}^{k-1} w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} f^p(t_1, \dots, t_k) \Delta t_1 \dots \Delta t_{k-1} \right) \Delta t_k. \end{aligned}$$

Corollary 13. If $l = 1$ and $\mathbb{T} = \mathbb{R}$ in Theorem 12, we obtain

$$\begin{aligned} & \int_a^\infty w(t) \frac{v(t)}{V^m(t)} K^p(t) dt \\ & \leq \left(\frac{p(\lambda - 1)}{\mu + 1 - m} \right)^p \int_a^\infty \frac{v(t)}{V^m(t)} w(t) f^p(t) dt, \end{aligned} \tag{29}$$

where

$$K(t) = \frac{1}{g(s)} \int_a^t g(t) \frac{v(s)}{V(s)} f(s) ds.$$

Example 14. Choose $\mu = m, \lambda = 2$, and $p = 1$ in Theorem 12. Hence, we get

$$\begin{aligned} & \int_a^\infty \dots \int_a^\infty \prod_{l=1}^k w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} K^\sigma(t_1, \dots, t_k) \Delta t_1 \dots \Delta t_k \\ & \leq \int_a^\infty \frac{v_k(t_k)}{V_k^m(t_k)} \frac{g_k(t_k)}{g_k^\sigma(t_k)} w_k(t_k) \left(\int_a^\infty \dots \int_a^\infty \prod_{l=1}^{k-1} w_l^\sigma(t_l) \frac{v_l(t_l)}{V_l^m(t_l)} f(t_1, \dots, t_k) \Delta t_1 \dots \Delta t_{k-1} \right) \Delta t_k. \end{aligned} \tag{30}$$

4. Conclusions

(TSs) calculus is used in this paper to prove special cases of (TS) Copson–Steklov-type inequalities with several variables. The obtained inequalities would be interesting to

apply in different fields of mathematics (functional spaces, partial differential equations, mathematical modeling). Furthermore, the inequalities can be discussed in calculus, discrete calculus, and quantum calculus. As a perspective, we propose to study these results for other kinds of

operators and solve the singularity that appeared in Theorem 12 with case $m > \mu + 1$.

Data Availability

All data that support the findings of this study are included within the article.

Conflicts of Interest

The author declares no conflicts of interest.

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