Generalized Multiquartic Mappings, Stability, and Nonstability

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In this article, a generalized form of \( n \)-quartic mappings is introduced. The structure of such mappings is studied, and in fact, it is shown that every multiquartic mapping can be described as an equation, namely, the (generalized) multiquartic functional equation. Moreover, by applying two fixed point techniques, some results corresponding to known stability outcomes including Hyers, Rassias, and Găvruţa stabilities are presented. Using a characterization result, an appropriate counterexample is supplied to invalidate the results in the case of singularity.

1. Introduction

When we are speaking about the stability of one variable mappings or an equation, we remember the name of Ulam who proposed the first stability problem regarding of group homomorphisms [1] whether such mappings are stable. Roughly speaking, it means that (for instance) any approximate solution of an additive functional equation \( \mathcal{F} \), namely, a mapping \( h \) with the property \( \| h(x+y) - h(x) - h(y) \| \leq \varepsilon \) is near to an exact solution of \( \mathcal{F} \). In other words, does there exists a unique additive mapping \( A \) for example on commutative groups such that \( \| h(x) - A(x) \| \leq \delta \varepsilon \)? Note that \( \delta \varepsilon \) is depended on \( \varepsilon \). This fact is called the stability of additive or Cauchy functional equation. In the case that \( f \) is additive, we say the functional equation is hyperstable. The Ulam stability problem was partially answered by Hyers [2] for linear mappings on normed spaces and then extended by Aoki [3] (for additive mappings), T. M. Rassias [4] (for linear mappings by considering an unbounded Cauchy difference), and Găvruţa [5] (by a general control function instead of the sum of powers of norms).

In recent decades, the Ulam stability problem has received attention from the authors to study it for multiple variable mappings. Let us state the definition of some of them as follows.

Let \( (V,+) \) be a commutative group, \( W \) be a linear space over rationals, and \( n \) be a natural number greater than 1. Let \( z_{i}, z_{i}' \in V \), where \( i \in \{1, \ldots, n\} \). A mapping \( f : V^{n} \rightarrow W \) is called

(i) Multiadditive if it is additive in each of components [6], that is,

\[
f(z_{1}, \ldots, z_{i-1}, z_{i} + z_{i}', z_{i+1}, \ldots, z_{n}) = f(z_{1}, \ldots, z_{i-1}, z_{i}, z_{i+1}, \ldots, z_{n}) + f(z_{1}, \ldots, z_{i-1}, z_{i}', z_{i+1}, \ldots, z_{n}) - 2f(z_{1}, \ldots, z_{i-1}, z_{i}', z_{i+1}, \ldots, z_{n})
\]

(ii) Multiquadratic if it is quadratic in all variables [7, 8], namely

\[
f(z_{1}, \ldots, z_{i-1}, z_{i} + z_{i}', z_{i+1}, \ldots, z_{n}) + f(z_{1}, \ldots, z_{i-1}, z_{i}, z_{i+1}, \ldots, z_{n})
\]

\[
= 2f(z_{1}, \ldots, z_{i-1}, z_{i}, z_{i+1}, \ldots, z_{n}) + 2f(z_{1}, \ldots, z_{i-1}, z_{i}', z_{i+1}, \ldots, z_{n})
\]

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Let \( (V,+) \) be a commutative group, \( W \) be a linear space over rationals, and \( n \) be a natural number greater than 1. Let \( z_{i}, z_{i}' \in V \), where \( i \in \{1, \ldots, n\} \). A mapping \( f : V^{n} \rightarrow W \) is called

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\]

\[
= 2f(z_{1}, \ldots, z_{i-1}, z_{i}, z_{i+1}, \ldots, z_{n}) + 2f(z_{1}, \ldots, z_{i-1}, z_{i}', z_{i+1}, \ldots, z_{n})
\]

(2)
(iii) Multicubic if it is cubic in each of its \( n \) arguments
[9], that is,

\[
\begin{align*}
&f(z_1, \ldots, z_{i-1}, 2z_i + z'_i, z_{i+1}, \ldots, z_n) + f(z_1, \ldots, z_{i-1}, 2z_i, z'_{i+1}, \ldots, z_n) + f(z_1, \ldots, z_{i-1}, 2z_i - z'_{i+1}, \ldots, z_n) \\
&+ 12f(z_1, \ldots, z_{i-1}, z'_i, z_{i+1}, \ldots, z_n).
\end{align*}
\]

Here, this is a question in which equations can describe the mappings as mentioned earlier. Ciepliński [6], Zhao et al. [8], and Bodaghi et al. [9] showed that each of the mappings above can be shown and unified as an equation. Moreover, the stability of multiadditive, multiquadratic, and multicubic mappings is available for instance in [6-10].

Let us recall from [11, 12] two quartic functional equations as follows:

\[
\begin{align*}
h(x + 2y) + h(x - 2y) &= 4[h(x + y) + h(x - y)] - 6h(x) + 24h(y), \\
h(2x + y) + h(2x - y) &= 4[h(x + y) + h(x - y)] + 24h(x) - 6h(y).
\end{align*}
\]

Kang [13] generalized equations (4) and (5) as follows:

\[
h(ax + by) + h(ax - by) = a^2b^2[h(x + y) + h(x - y)] + Ka^2h(x) - Kb^2h(y),
\]

(6)

for fixed integers \( a, b \) with \( a, b \neq 0, \pm 1 \), where \( K = 2(a^2 - b^2) \) and for the rest of the paper as well.

Motivated by equations (4) and (5), in [14, 15], the authors introduced two multiquartic mappings and characterized their structures. In fact, for a commutative group \( G \), a linear space \( V \), and an integer \( n \) with \( n \geq 2 \), a mapping \( f: \mathbb{G}^n \to V \) is called multiquartic if it is quartic (fulfills one of quartic functional equations (4) and (5)) in each variable. In [14, 15], the authors proved that every multiquartic mapping can be described as a single equation, and moreover, such equations define the multiquadratic mappings.

In this study, we introduce an extended class of multiquartic mappings which fulfill quartic equation (6) in each component. We also characterize the structure of such mappings and show that every multiquartic mapping can be unified as a single equation (namely, the multiquartic functional equation) and vice versa (under the quartic condition in each variable). Finally, we prove some Hyers-Ulam and Găvrutușa stability results for multiquartic functional equations in quasi-\( \beta \) normed spaces by applying a fixed point method. Using a characterization result, we indicate an example for a nonstable multiquartic functional equation.

2. The Structure of Generalized Multiquartic Mappings

Throughout, the set of positive integers and the set of rational numbers are denoted by \( \mathbb{N} \) and \( \mathbb{Q} \), respectively, and also \( \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\} \). For any \( p \in \mathbb{N}_0, k \in \mathbb{N}, r = (r_1, \ldots, r_k) \in k\text{-times} \{-1, 1\}^k \) in which \( \{-1, 1\}^k \equiv \{-1, 1\} \times \cdots \times \{-1, 1\} \) and \( x = (x_1, \ldots, x_k) \in V^k \), we write \( px := (px_1, \ldots, px_k) \) and \( rx := (r_1x_1, \ldots, r_kx_k) \), where \( px \) denotes for the \( p \) th power of an element \( x \) of the commutative group \( V \).

Definition 1. Let \( V \) and \( W \) be vector spaces over \( \mathbb{Q} \). A mapping \( f: \mathbb{V}^n \to W \) is called \( n \)-quartic or generalized multiquartic if it is quartic in each of its \( n \) arguments; that is,

\[
\begin{align*}
f(z_1, \ldots, z_{i-1}, ax_i + b'_i, z_{i+1}, \ldots, z_n) + f(z_1, \ldots, z_{i-1}, ax_i, z'_{i+1}, \ldots, z_n) + f(z_1, \ldots, z_{i-1}, ax_i - b'_i, z_{i+1}, \ldots, z_n) \\
+ 12f(z_1, \ldots, z_{i-1}, z'_i, z_{i+1}, \ldots, z_n)
\end{align*}
\]

\[
= a^2b^2[f(z_1, \ldots, z_{i-1}, z_i + z'_i, z_{i+1}, \ldots, z_n) + f(z_1, \ldots, z_{i-1}, z_i - z'_i, z_{i+1}, \ldots, z_n)] + Ka^2f(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_n) - Kb^2f(z_1, \ldots, z_{i-1}, z'_i, z_{i+1}, \ldots, z_n).
\]

Suppose that \( n \in \mathbb{N} \) such that \( n \geq 2 \) and \( x^n_i = (x_{1i}, \ldots, x_{ni}) \in V^n \) in which \( i \in [1, 2] \). We wish to denote \( x^n_i \) by \( x_i \) when no confusion can arise. Let \( x_1, x_2 \in V^n \) and \( p_i \in \{0, 1\} \), put \( \mathcal{N}^n = \{\mathcal{N}_n = (N_1, \ldots, N_n)\} \)

\[
N_j \in \{x_{ij} \pm x_{2j}, x_{1j}, x_{3j}\}, \quad j \in \{0, 1, \ldots, n\}.
\]

Put \( \mathcal{N}^n_{(p_1, p_2)} \) as a subset of \( \mathcal{N} \) including the elements of \( \mathcal{N} \) such that the cardinality of \( \{N_j: N_j = x_{ij}\} \) is \( p_i \) when \( i \in [1, 2] \).
For the mappings defined in Definition 1, put some conventions as follows:

\[ f(\mathcal{M}^n(p_1, p_2)) := \sum_{y_{\alpha} \in F_{(p_1, p_2)}} f(y_{\alpha}), \quad (8) \]

\[ f(\mathcal{N}^n(p_1, p_2), w) := \sum_{y_{\alpha} \in F_{(p_1, p_2)}} f(y_{\alpha}, w) (w \in V). \quad (9) \]

It was shown in [15], Theorem 2.2 that a mapping \( f: V^n \rightarrow W \) is multiquartic (in sense of (4)) if and only if it satisfies the following equation:

\[ \sum_{t \in [-1, 1]^n} f(x_1 + 2tx_2) = \sum_{p_1=0}^{n-p_1} \sum_{p_2=0}^{n-p_2} 4^{n-p_1-p_2} (-6)^p_2 2^{4p_2} f(\mathcal{N}^n(p_1, p_2)). \quad (10) \]

Moreover, it has been proven in Proposition 1 from [14] that if \( f: V^n \rightarrow W \) is a multiquartic mapping (in sense of (5)), then it fulfills the following equation:

\[ \sum_{t \in [-1, 1]^n} f(ax_1^{p_1} + tby_2^{p_2}) \]

\[ = a^2b^2 \sum_{t \in [-1, 1]^n} \sum_{s \in [-1, 1]} f(ax_1^n + tby_2^n, x_1^{p_1} + s2x_2^{p_2}) \]

\[ + Ka^2 \sum_{t \in [-1, 1]^n} f(ax_1^n + tby_2^n, x_1^{p_1} + s2x_2^{p_2}) \]

\[ = a^2b^2 \sum_{p_1=0}^{n-p_1} \sum_{p_2=0}^{n-p_2} (ab)^2(n-p_1-p_2)(Ka^2)^{p_1}(-Kb^2)^{p_2} f(\mathcal{M}^n(p_1, p_2), x_1^{p_1} + t2x_2^{p_2}) \]

\[ + Ka^2 \sum_{p_1=0}^{n-p_1} \sum_{p_2=0}^{n-p_2} (ab)^2(n-p_1-p_2)(Ka^2)^{p_1}(-Kb^2)^{p_2} f(\mathcal{M}^n(p_1, p_2), x_1^{p_1} + t2x_2^{p_2}) \]

\[ + Kb^2 \sum_{p_1=0}^{n-p_1} \sum_{p_2=0}^{n-p_2} (ab)^2(n-p_1-p_2)(Ka^2)^{p_1}(-Kb^2)^{p_2} f(\mathcal{M}^n(p_1, p_2), x_1^{p_1} + t2x_2^{p_2}) \]

\[ = \sum_{p_1=0}^{n-p_1} \sum_{p_2=0}^{n-p_2} (ab)^2(n-p_1-p_2)(Ka^2)^{p_1}(-Kb^2)^{p_2} f(\mathcal{M}^n(p_1, p_2), x_1^{p_1} + t2x_2^{p_2}). \]

This shows that (12) holds for \( n + 1 \).

By means of Proposition 1, we find that the mapping \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) defined through \( f(r_1, \ldots, r_n) = cr_1^4 \ldots, r_n^4 \) satisfies (12) and so one can call the mentioned equation as multiquartic functional equation.

**Definition 2.** Let \( m \in \mathbb{Q} \) and \( r \in \mathbb{N} \). Consider a mapping \( f: V^n \rightarrow W \) and \( v_1, \ldots, v_n \in V^n \). Then,

(i) \( f \) has the \( (m, r) \)-power condition if for any arbitrary variable \( v_j \),

\[ f(v_1, \ldots, v_{j-1}, mv_j, v_{j+1}, \ldots, v_n) = m^r f(v_1, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, v_n). \quad (14) \]

(ii) \( f \) is even if for any arbitrary variable \( v_j \),

\[ f(v_1, \ldots, v_{j-1}, -v_j, v_{j+1}, \ldots, v_n) = f(v_1, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, v_n). \quad (15) \]

(iii) \( f \) has zero condition if \( f(v) = 0 \) for any \( v \in V^n \) for which at least one component is zero

**Proposition 1.** Every generalized multi-quartic mapping \( f: V^n \rightarrow W \) fulfills the following equation:

\[ \sum_{t \in [-1, 1]^n} f(ax_1 + tby_2) = \sum_{p_1=0}^{n-p_1} \sum_{p_2=0}^{n-p_2} (ab)^2(n-p_1-p_2)(Ka^2)^{p_1}(-Kb^2)^{p_2} f(\mathcal{M}^n(p_1, p_2)), \]

where \( x_1, x_2 \in V^n \).

**Proof.** The proof is based on induction. Form \( n = 1 \), it is clear that \( f \) satisfies equation (6). Suppose that (12) holds for some \( n \in \mathbb{N} \) with \( n > 1 \). We have

\[ \sum_{t \in [-1, 1]^n} f(2x_1 + tx_2) = \sum_{p_1=0}^{n-p_1} \sum_{p_2=0}^{n-p_2} 4^{n-p_1-p_2} 2^{4p_2} f(\mathcal{N}^n(p_1, p_2)). \]

Thus, for \( n + 1 \), it is also true.
(H2) \( f \) has the \((b, 4)\)-power condition and is even. The proof of the next lemma is similar to the proof of [14], Lemma 2.5, and we bring some parts of the proof for the sake of completeness.

\[ 2^n f(0, \ldots, 0) = \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} \binom{n}{p_1} \binom{n-p_1}{p_2} (ab)^2 \binom{n}{p_1} (Ka^2)^{p_1} (-Kb^2)^{p_2} 2^{n-p_1-p_2} f(0, \ldots, 0) \]

\[ = \sum_{p_1=0}^{n} \binom{n}{p_1} (ab)^2 (Ka^2)^{p_1} \left( a^2 b^2 - \frac{Kb^2}{2} \right)^{n-p_1} f(0, \ldots, 0) \]

\[ = (2b^4 + Ka^2)^n f(0, \ldots, 0). \]

Relation (16) implies that \( f(0, \ldots, 0) = 0 \). Similar to the proof of Proposition 1 from [14], one can continue in this fashion showing that \( f \) has zero condition. The second part can be obtained similarly. \( \square \)

**Lemma 1.** Under one of the hypotheses (H1) and (H2), every mapping \( f: V^n \rightarrow W \) fulfilling equation (12) has zero condition.

**Proof.** Assume that (H1) holds. Putting \( x_1 = x_2 = (0, \ldots, 0) \) in (12), we have

\[ f^*(x_{1j}, x_{2j}) = f(x_{11}, \ldots, x_{1,j-1}, x_{1j} + x_{2j}, x_{1,j+1}, \ldots, x_{1n}), \]

\[ + f(x_{11}, \ldots, x_{1,j-1}, x_{1j} - x_{2j}, x_{1,j+1}, \ldots, x_{1n}) \]

\[ f^*(x_{1j}) = f(x_{11}, \ldots, x_{1n}), f^*(x_{2j}) = f(x_{11}, \ldots, x_{1,j-1}, x_{2j}, x_{1,j+1}, \ldots, x_{1n}). \]

Putting \( x_{2k} = 0 \) for all \( k = 1, \ldots, n \setminus \{j\} \) in (12) and applying our assumption, we get

\[ 2^{n-1} \times a^{4(n-1)} \left[ f(x_{11}, \ldots, x_{1,j-1}, ax_{1j} + bx_{2j}, x_{1,j+1}, \ldots, x_{1n}) + f(x_{11}, \ldots, x_{1,j-1}, ax_{1j} - bx_{2j}, x_{1,j+1}, \ldots, x_{1n}) \right] \]

\[ = 2^{n-1} \left[ f(ax_{11}, \ldots, ax_{1,j-1}, ax_{1j} + bx_{2j}, ax_{1,j+1}, \ldots, ax_{1n}) + f(ax_{11}, \ldots, ax_{1,j-1}, ax_{1j} - bx_{2j}, ax_{1,j+1}, \ldots, ax_{1n}) \right] \]

\[ = \sum_{p_1=1}^{n-1} \binom{n-1}{p_1} (ab)^2 (n-p_1)(Ka^2)^{p_1} 2^{n-1-p_1} f^*(x_{1j}) + \sum_{p_1=1}^{n-1} \binom{n-1}{p_1-1} (ab)^2 (n-p_1)(Ka^2)^{p_1} 2^{n-p_1} f^*(x_{2j}) \]

\[ - Kb^2 \sum_{p_1=1}^{n-1} \binom{n-1}{p_1-1} (ab)^2 (n-p_1)(Ka^2)^{p_1} 2^{n-p_1} f^*(x_{2j}) \]
\[
\begin{align*}
&= (ab)^2 \sum_{p_l=0}^{n-1} \binom{n-1}{p_l} (Ka^2)^{p_l} 2^{n-1-p_l} f^*(x_{1j}, x_{2j}) \\
&+ Ka^2 \sum_{p_l=0}^{n-1} \binom{n-1}{p_l} (2a^2b^2)^{n-1-p_l} (Ka^2)^{p_l} f^*(x_{1j}) \\
&- Kb^2 \sum_{p_l=0}^{n-1} \binom{n-1}{p_l} (2a^2b^2)^{n-1-p_l} (Ka^2)^{p_l} f^*(x_{2j}) \\
&= (ab)^2 (2a^2b^2 + Ka^2)^{-n} f^*(x_{1j}, x_{2j}) + Ka^2 (2a^2b^2 + Ka^2)^{-n} f^*(x_{1j}) - Kb^2 (2a^2b^2 + Ka^2)^{-n} f^*(x_{2j}) \\
&= (ab)^2 2^{n-1} a^{4(n-1)} f^*(x_{1j}, x_{2j}) + Ka^2 2^{n-1} a^{4(n-1)} f^*(x_{1j}) - Kb^2 2^{n-1} a^{4(n-1)} f^*(x_{2j}).
\end{align*}
\]

(18)

It follows from (18) that

\[
\begin{align*}
&f(x_{11}, \ldots, x_{1j-1}, ax_{1j} + x_{2j}, x_{1,j+1}, \ldots, x_{1n}) + f(x_{11}, \ldots, x_{1,j-1}, ax_{1j} - x_{2j}, x_{1,j+1}, \ldots, x_{1n}) \\
&= a^2b^2f^*(x_{1j}, x_{2j}) + Ka^2 f^*(x_{1j}) - Kb^2 f^*(x_{2j}).
\end{align*}
\]

(19)

3. Some Results for the Stability of (12)

This section is devoted to prove the generalized Hyers–Ulam stability of equation (12) in the setting of quasi-\(p\)-normed spaces. The next theorem of fixed point theory was proved in [16]. This result can be useful to reach the desired aims.

**Theorem 2.** Suppose that \((X, d)\) is a complete generalized metric space, and \(\mathcal{S}: X \rightarrow X\) is a mapping with the Lipschitz constant \(L < 1\). For each element \(x \in X\), one of the following cases can happen.

(i) \(d(\mathcal{S}^n x, \mathcal{S}^{n+1} x) = \infty\) for all \(n \in \mathbb{N}_0\), or.

(ii) there exists \(n_0 \in \mathbb{N}\) such that.

(iii) \(d(\mathcal{S}^n x, \mathcal{S}^{n+1} x) < \infty\) for all \(n \geq n_0\);

(iv) there is a fixed point \(x^*\) of \(\mathcal{S}\) such that the sequence \(\{\mathcal{S}^n x\}\) converges to \(x^*\) in which it belongs to the set \(X^* = \{x \in X: d(\mathcal{S}^n x, x) < \infty\}\). (20)

(v) \(d(x, x^*) < (1/1 - L)d(x, \mathcal{S} x)\) for all \(x \in X^*\).

Let \(W\) be a vector space and \((G, +)\) be a commutative group. In the rest of this section, for a mapping \(f: G^n \rightarrow W\), we define the operator \(D f: G^n \times G^n \rightarrow W\) through

\[
D f(x_1, x_2) = \sum_{t \in \{-1, 1\}^n} f(ax_1 + tbx_2) - \sum_{p_l=0}^{n-1} \sum_{p_l=0}^{n-1} (ab)^{2(n-1-p_l)} (Ka^2)^{p_l} 2^{n-1-p_l} f^*(x_{1j}, x_{2j}).
\]

(21)

\[\text{for } x_1, x_2 \in G^n, \text{ where } f(N^n_{(p,q)}) \text{ is defined in (8)}.\]
From now on, we assume that all mappings $f: G^n \to W$ have zero conditions. With this assumption, we have Hyers' stability for equation (12) as follows.

**Theorem 3.** Let $\varepsilon > 0$. Suppose that $W$ is a complete normed space and $(G, +)$ is a commutative group. If a mapping $f: G^n \to W$ fulfilling the inequality

$$\|Df(x_1, x_2)\| \leq \varepsilon,$$

(22)

for all $x_1, x_2 \in G^n$, then there exists a unique solution $\mathcal{F}: G^n \to W$ of (12) such that

$$\|f(x) - \mathcal{F}(x)\| \leq \frac{1}{2^n(|a|^{4n} - 1)} \varepsilon,$$

(23)

for all $x \in G^n$. Moreover,

$$\mathcal{F}(x) = \lim_{l \to \infty} \left( \frac{1}{|a|^{4n}} \right)^l f(a^l x),$$

(24)

for all $x \in G^n$.

**Proof.** Considering $x = x_1$ and $x_2 = 0$ in (22) and applying the hypotheses, we have

$$\left\| 2^n f(ax) - \sum_{p_i=0}^n \left( \begin{array}{c} n \\ p_i \end{array} \right) (ab)^{p_1}(Kd)^{p_2}a^{-p_1}f(x) \right\| \leq \varepsilon,$$

(25)

for all $x \in G^n$, and hence,

$$\|2^n f(ax) - a^{4n} f(x)\| \leq \varepsilon.$$

(26)

Thus,

$$\| f(ax) - a^{4n} f(x) \| \leq \frac{1}{2^n} \varepsilon.$$ 

(27)

Set $X = \{ f \in W^G : f \text{ has zero condition} \}$, where $W^G$ is the set of all mappings from $G^n$ to $W$. Define the generalized metric $d$ on $X$.

$$d(g_1, g_2) = \inf \{ C \in [0, \infty) : \| g_1(x) - g_2(x) \| \leq C_{g_1, g_2} \varepsilon, \quad (x \in G^n) \}.$$ 

(28)

Similar to the proof of [17], Theorem 2.2, it is easily verified that $(X, d)$ is a complete generalized metric space. Consider the operator $\mathcal{F}: X \to X$ defined via

$$\mathcal{F} f(x) = \frac{1}{|a|^{4n}} f(ax),$$

(29)

for all $x \in G^n$. Take $g_1, g_2 \in \Omega$ and $C_{g_1, g_2} \in [0, \infty]$ with $d(g_1, g_2) \leq C_{g_1, g_2}$. Then, $\| g_1(x) - g_2(x) \| \leq C_{g_1, g_2} \varepsilon$, and hence,

$$\| \mathcal{F} g_1(x) - \mathcal{F} g_2(x) \| \leq \frac{1}{|a|^{4n}} \| g_1(ax) - g_2(ax) \|,$$

(30)

$$\leq \frac{1}{|a|^{4n}} C_{g_1, g_2} \varepsilon.$$

Therefore, $d(\mathcal{F} g_1, \mathcal{F} g_2) \leq (1/|a|^{4n})C_{g_1, g_2}$. This shows that $d(\mathcal{F} g_1, \mathcal{F} g_2) \leq (1/|a|^{4n})d(g_1, g_2)$. This means that $\mathcal{F}$ is a strictly contractive with the Lipschitz constant $1/|a|^{4n}$. Relation (27) necessitates that

$$\| \mathcal{F} f(x) - f(x) \| \leq \frac{1}{2^n} \| f(ax) - f(x) \| \leq \frac{1}{2^n} \frac{1}{|a|^{4n}} \varepsilon.$$ 

(31)

Thus,

$$d(\mathcal{F} f, f) \leq \frac{1}{2^n} \frac{1}{|a|^{4n}} < \infty.$$ 

(32)

Applying Theorem 3 for the space $(X, d)$, the operator $\mathcal{F}$ and $n_0 = 0$, we can deduce that the sequence $(\mathcal{F}^l f)_{l \in \mathbb{N}}$ converges to $\mathcal{F}$ in $(X, d)$ which is a fixed point of $\mathcal{F}$. Indeed, $\mathcal{F}(x) = \lim_{l \to \infty} \mathcal{F}^l f(x)$, and

$$\mathcal{F}(x) = \frac{1}{|a|^{4n}} f(x).$$

(33)

On the other hand, by induction on $l$, it can be shown for all $x \in G^n$ that

$$\mathcal{F}^l f(x) = \left( \frac{1}{|a|^{4n}} \right)^l f(a^l x),$$

(34)

and moreover, (24) follows. Obviously, $f \in X^*$. In addition, it follows from (32) and part (iii) of Theorem 2 that

$$d(f, \mathcal{F}) \leq \frac{1}{1 - \frac{1}{|a|^{4n}}} d(f, \mathcal{F}) \leq \frac{1}{2^n} \frac{1}{|a|^{4n}} - 1.$$ 

(35)

which proves (23). Furthermore,

$$\|D \mathcal{F} (x_1, x_2)\| = \lim_{l \to \infty} \left( \frac{1}{|a|^{4n}} \right)^l \| Df(ax_1, ax_2) \|,$$

(36)

$$\leq \lim_{l \to \infty} \left( \frac{1}{|a|^{4n}} \right)^l \varepsilon = 0,$$

for all $x_1, x_2 \in G^n$. The last relation shows that $D \mathcal{F} (x_1, x_2) = 0$ for all $x_1, x_2 \in G^n$, and therefore, $\mathcal{F}$ satisfies equation (12). For the uniqueness part, assume that $\mathcal{G}: G^n \to W$ is another solution of equation (12) with zero condition, and moreover, inequality (23) holds for it. Then, $\mathcal{G}$ satisfies (33), and hence, it is a fixed point of the operator $\mathcal{F}$. Besides, by (23), we obtain

$$d(f, \mathcal{G}) \leq \frac{1}{2^n} \frac{1}{|a|^{4n}} - 1 < \infty,$$

(37)

and consequently, $\mathcal{G} \in X^*$. Using part (ii) of Theorem 2, we find $\mathcal{G} = \mathcal{F}$. This finishes the proof.

Here, we recall some basic facts regarding the setting of quasi-$\beta$-normed space.

**Definition 3.** Let $V$ be a linear space over $\mathbb{R}$, $\beta$ be a fixed real number with $0 < \beta < 1$. A quasi-$\beta$-norm is a real-valued function on $V$ fulfilling the statements

(i) $\|v\| \geq 0$ for all $v \in V$
(ii) $\|v\| = 0$ if and only if $v = 0$

(iii) $\|rv\| = |r|\|v\|$ for all $v \in V$ and $r \in \mathbb{R}$

(iv) There is a constant $K \geq 1$ such that $\|u + v\| \leq K (\|u\| + \|v\|)$ for all $u, v \in V$

If $\cdot \cdot \cdot$ is a quasi-$\beta$-norm on $V$, then the pair $(V, \cdot \cdot \cdot)$ is said to be a quasi-$\beta$-normed space. The smallest possible $K$ is called the modulus of concavity of the norm $\cdot \cdot \cdot$. Moreover, a quasi-$\beta$-normed space is a complete quasi-$\beta$-normed space. If

$$\|u + v\| \leq \|u\| + \|v\|, \quad (38)$$

for all $u, v \in V$, then a quasi-$\beta$-norm $\cdot \cdot \cdot$ is called a $(\beta, \rho)$-norm, and indeed, a quasi-$\beta$-Banach space is said to be a $(\beta, \rho)$-Banach space. On the other hand, for a $\rho$-norm, the metric $d(u, v) = \|u - v\|$ leads us to a translation-invariant metric on $V$. Furthermore, it follows from the Aoki-Roelencz Theorem [18] that every quasi-norm is equivalent to some $\rho$-norm.

The upcoming fixed-point lemma is presented in [19], Lemma 3.1.

**Lemma 2.** Let $f \in \{-1, 1\}$ be fixed, $a, s \in \mathbb{N}$ with $a \geq 2$. Let also $V$ be a linear space, $W$ be a $(\beta, \rho)$-Banach space with $(\beta, \rho)$-norm $\cdot \cdot \cdot$. Suppose that for the function $\psi: \rho \rightarrow [0, \infty)$, there exists an $L < 1$ with $\psi(a^\beta v) < La^\beta \psi(v)$ for all $v \in V$. If $f: V \rightarrow W$ is a mapping fulfilling

$$\|f(\psi) - a^\beta f(v)\|_W \leq \frac{1}{a^\beta - 1} \psi(v), \quad (39)$$

for all $v \in V$, then there exists a unique mapping $F: V \rightarrow W$ with $F(\psi) = a^\beta F(v)$ and

$$\|f(v) - F(v)\|_W \leq \frac{1}{a^\beta - 1} \psi(v), \quad (40)$$

for all $v \in V$. Moreover, $F(v) = \lim_{L \rightarrow \infty} (f(a^\beta v))$ for all $v \in V$.

Next, we prove the Găvruta stability of (12) in quasi-$\beta$-normed spaces by applying Lemma 2.

**Theorem 4.** Let $f \in \{-1, 1\}$ be fixed, $V$ a linear space and $W$ be a $(\beta, \rho)$-Banach space and $\varphi: V^n \times V^n \rightarrow \mathbb{R}_+$ be a function for which there exists an $0 < L < 1$ with $\varphi(a^\alpha x_1, a^\alpha x_2) \leq |a|^\alpha \varphi(x_1, x_2)$ for all $x_1, x_2 \in V^n$. Suppose that a mapping $f: V^n \rightarrow W$ fulfilling the inequality

$$\|Df(x_1, x_2)\| \leq \varphi(x_1, x_2), \quad (41)$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique solution $g: V^n \rightarrow W$ of (12) such that

$$\|f(x) - g(x)\|_W \leq \frac{\theta}{2^{n^\beta} |a|^{n\rho} - |a|^{n\rho}} + \frac{|a|^{n\rho} \theta}{2^{n^\beta} |a|^{n\rho} (|a|^{n\rho} - |a|^{n\rho})},$$

for all $x \in V^n$. In particular, if $g$ has either the property (H1) or (H2), then it is a generalized multiquartic mapping.

**Proof.** Similar to the proof of Theorem 3, by putting $x_2 = 0$ in (41) and using our hypotheses, we get

$$\|f(x) - a^\rho f(x)\| \leq \frac{1}{2^{n\rho} |a|^{n\rho}} \varphi(x, 0), \quad (42)$$

for all $x = x_1 \in V^n$ and so

$$\|f(x) - a^\rho f(x)\| \leq \frac{1}{2^{n\rho} |a|^{n\rho}} \varphi(x, 0), \quad (43)$$

for all $x \in V^n$. Lemma 2 implies that there exists a unique mapping $\varphi: V^n \rightarrow W$ such that $\varphi(ax) = a^\rho \varphi(x)$ and

$$\|f(x) - \varphi(x)\| \leq \frac{1}{2^{n\rho} |a|^{n\rho}} \varphi(x, 0), \quad (44)$$

for all $x \in V^n$. We claim that $\varphi$ is a multiquartic map. Here, we note from Lemma 2 that for all $x \in V^n$, $\varphi(x) = \lim_{L \rightarrow \infty} (f(a^\rho x))$. Now, by (41), we have

$$\|Df(x_1, x_2)\| \leq |a|^{-\alpha} \varphi(x_1, x_2) \quad (45)$$

$$\leq |a|^{-\alpha} \varphi(x_1, x_2) = L \varphi(x_1, x_2),$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}$. Letting $l \rightarrow \infty$ in the above inequality, we find $D\varphi(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$, and hence, $\varphi$ satisfies (12). The last assertion follows from Theorem 1.

In upcoming corollaries, it is assumed that $V$ is a quasi-$\alpha$-normed space with quasi-$\alpha$-norm $\cdot \cdot \cdot$, and $W$ is a $(\beta, \rho)$-Banach space with $(\beta, \rho)$-norm $\cdot \cdot \cdot$. The following result is a direct consequence of Lemma 3 concerning the stability of (12) when the norm of $Df(x_1, x_2)$ is controlled by the sum of variables norms of $x_1$ and $x_2$ with positive powers.

**Corollary 2.** Let $\theta$ and $\lambda$ be positive numbers with $\lambda \neq 4n(\beta/\alpha)$. If a mapping $f: V^n \rightarrow W$ fulfilling the inequality

$$\|Df(x_1, x_2)\|_W \leq \theta \sum_{j=1}^{2n} (x_j)_{\alpha/\beta}^\beta, \quad (46)$$

for all $x_1, x_2 \in V^n$, then there exists a unique solution $g: V^n \rightarrow W$ of (12) such that

$$\|x_1 - g(x_1)\|_W \leq \frac{n}{\alpha} \lambda \left( \begin{array}{c} n \lambda \\end{array} \right) \left( \begin{array}{c} n \lambda \end{array} \right) \lambda \in \left( 0, \frac{4n^\beta}{\alpha} \right),$$

$$\|x_1 - g(x_1)\|_W \leq \frac{n}{\alpha} \lambda \left( \begin{array}{c} n \lambda \\end{array} \right) \left( \begin{array}{c} n \lambda \end{array} \right) \lambda \in \left( 0, \frac{4n^\beta}{\alpha} \right).$$

(47)

(48)
for all \( x = x_1 \in V^n \). Moreover, if \( \Omega \) has either the property (H1) or (H2), then it is generalized multi-quartic.

The hyperstability of equation (12) is presented as follows.

**Corollary 3.** Suppose \( \lambda_{ij} > 0 \) for \( i \in \{1, 2\} \) and \( j \in \{1, \ldots, n\} \) with \( \sum_{i=1}^{2} \sum_{j=1}^{n} \lambda_{ij} \neq 4n \lambda_{ij} \). If a mapping \( f : V^n \rightarrow W \) fulfilling the inequality

\[
\| Df(x_1, x_2) \|_W \leq \theta \prod_{j=1}^{n} x_{ij}^{\lambda_{ij}},
\]

for all \( x_1, x_2 \in V^n \), then it satisfies (12). In particular, if \( f \) has one of the hypotheses (H1) and (H2), then \( f \) is a generalized multi-quartic mapping.

The next proposition was proved in [20], Proposition 4.8.

**Proposition 2.** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a continuous \( n \)-quartic function. Then, there exists a constant \( c \in \mathbb{R} \) such that

\[
f(r_1, \ldots, r_n) = c \prod_{j=1}^{n} r_j^{4},
\]

for all \( r_1, \ldots, r_n \in \mathbb{R} \).

We end the paper with the following counterexample for the multi-quartic mappings on \( \mathbb{R}^n \) that its idea is taken from [21]. In fact, we show the assumption \( \lambda \neq 4n \) cannot be removed in Corollary 2 when \( V = W = \mathbb{R} \) in the case that \( \alpha = \beta = 1 \).

**Example 1.** Let \( \delta > 0 \) and \( n \in \mathbb{N} \). Put \( \mu = (2^{4n} - 1 - 2^{8n} \delta) \), where \( t = \max(|\alpha|, |b|) \)

\[
S = 2^n + \sum_{p_1 \neq 0, p_2 \neq 0}^{n} |ab|^2 (2^{n} - 1)^2 |K a^2| |K b^2| p_1 .
\]

Note that \( t \geq 2 \). Define the function \( \psi : \mathbb{R}^n \rightarrow \mathbb{R} \) through

\[
\psi(r_1, \ldots, r_n) = \begin{cases} 
\mu & \text{for all } r_j \text{ with } |r_j| < 1, \\
\mu & \text{otherwise}.
\end{cases}
\]

Hence, \( \psi(r_1, \ldots, r_n) \leq \mu \) for all \( (r_1, \ldots, r_n) \in \mathbb{R}^n \) and \( \psi \) is continuous as well. Using the function \( \psi \), define the function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) by

\[
f(r_1, \ldots, r_n) = \prod_{j=1}^{n} \frac{\psi(2^{r_1}, \ldots, 2^{r_n})}{2}, \quad (r_j \in \mathbb{R}).
\]

It is obvious that \( f \) is an even function in each variable and nonnegative. The series given in the last equality is uniformly convergent of continuous functions, and thus, \( f \) is continuous and bounded. In other words, for each \( (r_1, \ldots, r_n) \in \mathbb{R}^n \), we have \( f(r_1, \ldots, r_n) \leq \left(2^{4n/2^{2n}} - 1 \right) \mu \).

Put \( x_i = (x_{i1}, \ldots, x_{im}) \), where \( i \in \{1, 2\} \). We claim that

\[
|Df(x_1, x_2)| \leq \delta \sum_{j=1}^{n} x_{ij}^{4n},
\]

for all \( x_1, x_2 \in \mathbb{R}^n \). It is clear that (54) is valid for \( x_1 = x_2 = 0 \). Assume that \( x_1, x_2 \in \mathbb{R}^n \) with \( \sum_{j=1}^{2} \sum_{j=1}^{n} x_{ij}^{4n} < 1/2^{4n} \). Thus, there exists a positive integer \( N \) such that

\[
\frac{1}{2^{4n}(N+1)^{4n}} \leq \sum_{j=1}^{n} x_{ij}^{4n} \leq \frac{1}{2^{4n}N^{4n}}.
\]

Hence, \( x_{ij}^{4n} \leq \sum_{j=1}^{n} x_{ij}^{4n} < 1/2^{4n} \) and so \( 2^{N} |x_{ij}| < 1/4 \) for all \( i \in \{1, 2\} \) and \( j \in \{1, \ldots, n\} \). Therefore, \( 2^{N-1} |x_{ij}| < 1 \). If \( \gamma_1, \gamma_2 \in \{x_{ij} \mid i \in \{1, 2\}, j \in \{1, \ldots, n\} \} \), then \( \{2^{N-1} |\gamma_1 \pm \gamma_2|, 2^{N-1} |\gamma_1 \pm 2b\gamma_2| \} \subseteq (-1, 1) \).

Since \( \psi \) is quartiarc function on \((-1, 1)^n \), \( D\psi(2x_1, 2x_2) = 0 \) for all \( l \in \{0, 1, 2, \ldots, N-1\} \). It follows from the last equality and (55) that

\[
\frac{|Df(2x_1, 2x_2)|}{\sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{4n}} \leq \frac{|D\psi(2x_1, 2x_2)|}{\sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{4n}} \leq \frac{\mu S}{2^{4n} - 1} \delta,
\]

for all \( x_1, x_2 \in \mathbb{R}^n \). If \( \sum_{j=1}^{n} x_{ij}^{4n} \geq 1/2^{4n} \), then

\[
\frac{|Df(2x_1, 2x_2)|}{\sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{4n}} \leq \frac{\mu S}{2^{4n} - 1} \delta.
\]

Therefore, \( f \) fulfills (54) for all \( x_1, x_2 \in \mathbb{R}^n \). On the contrary, suppose that there is a number \( b \in [0, \infty) \) and a quartiarc function \( \Theta : \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( f(r_1, \ldots, r_n) - \Theta(r_1, \ldots, r_n) \leq c |b|^{\sum_{j=1}^{n} r_j^{4n}} \) for all \( (r_1, \ldots, r_n) \in \mathbb{R}^n \). By Proposition 2, there exists \( c \in \mathbb{R} \) such that \( \Theta(r_1, \ldots, r_n) = c \prod_{j=1}^{n} r_j^{4} \), and thus,

\[
f(r_1, \ldots, r_n) \leq (|c| + b) \prod_{j=1}^{n} r_j^{4},
\]

for all \( (r_1, \ldots, r_n) \in \mathbb{R}^n \). Here, we choose \( p \in \mathbb{N} \) such that \((p+1)\mu > |c| + b \). If \( r = (r_1, \ldots, r_n) \) belongs to \( \mathbb{R}^n \) such that \( r_j \in (0, 1/2) \) for all \( j \in \{1, \ldots, n\} \), then \( 2^{l} r_j \in (0, 1) \) for all \( l = 0, 1, \ldots, p \). Thus, we get

\[
f(r_1, \ldots, r_n) = \sum_{l=0}^{p} \frac{\psi(2^{l} r_1, \ldots, 2^{l} r_n)}{2^{4n}} = \sum_{l=0}^{p} \frac{\mu^{2^{4n}}}{2^{4n}} \prod_{j=1}^{n} r_j^{4} = (p+1)\mu \prod_{j=1}^{n} r_j^{4},
\]

where \( \mu \) is a constant. Consequently, \( f \) is not a quartiarc function.
The relation above leads us to a contradiction with (59).

4. Conclusion

In this paper, we have introduced the multiquartic mappings and have investigated the structure of such mappings. In other words, we have shown that the system of functional equations defining a multiquartic mapping can be unified as an equation. We have applied some fixed point theorems to establish the generalized Hyers–Ulam and Găvruta stability of multiquartic functional equations in Banach and quasi-$\beta$ normed spaces. Finally, by using a characterization result, we have presented a nonstable example to invalidate the results in the special case [21].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

The author wrote the first draft, read and approved the final manuscript.

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