Research Article

Contribution for the Summation Formulas for the Generalized Hypergeometric Series $\text{6F}_5(-1)$

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In this paper, our aim is to provide as many as one hundred and ninety-two summation formulas for the generalized hypergeometric series $\text{6F}_5(-1)$ in terms of gamma functions. For this, we have established eight theorems containing general results. This is achieved by means of applying generalized Watson’s and Dixon’s summation formulas obtained earlier by Lavoie et al. into a known transformation formula available in the literature. Results obtained earlier by Zhao follows special cases of our main findings.

1. Introduction and Results Required

The well-known generalized hypergeometric function $\text{pF}_q$ which is a natural generalization of the Gauss’s hypergeometric function $\text{2F}_1$ is defined as follows [1]:

$$
\text{pF}_q \left[ \begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_q \end{array} ; z \right] = 1 + \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{z^n}{n!},
$$

(1)

where $(\alpha)_n$ is the well-known Pochhammer symbol defined by

$$
(\alpha)_n = \begin{cases} 
\alpha (\alpha + 1), \ldots, (\alpha + n - 1), & n \in \mathbb{N}, \\
1, & n = 0.
\end{cases}
$$

(2)

In terms of well-known gamma function, we have

$$
(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}.
$$

(3)

Here as usual, $p$ and $q$ are nonnegative integers and the parameters $\alpha_j (1 \leq j \leq p)$ and $\beta_j (1 \leq j \leq q)$ can have arbitrary complex values with zero or negative integer values of $\beta_j$ excluded. It is clear that the generalized hypergeometric function $\text{pF}_q(z)$ converges for $|z| < \infty (p \leq q)$, $|z| < 1 (p = q + 1)$, and $|z| = 1 (p = q + 1)$ and $\Re(\delta) > 0$ where $\delta$ is the parametric excess defined as

$$
\delta = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j.
$$

(4)

It is not out of place to mention here that the generalized hypergeometric function occurs in many theoretical and practical applications such as mathematics, theoretical physics, engineering, and statistics.

For more details about this function, we refer [1–6]. It should be remarked here that whenever a generalized hypergeometric function reduces to the gamma function, the results are very important from the applications point of view. Thus, the classical summation theorems such as those of Gauss, Gauss second, Kummer and Bailey for the series

$\text{6F}_5(-1)$...
Watson, Dixon, Whipple, and Saalschütz for the series \( _2F_1 \) and others play an important role. Recently, a good deal of progress has been made in the direction of generalizing and extending the abovementioned classical summation theorems. For this, we refer here the research papers by Chu [7], Kim et al. [8], Lavoie et al. [9–11], and Rakha and Rathie [12].

However, in our present investigation, we are interested in the following classical Watson and Dixon summation theorems. These are

\[
\begin{align*}
&_3F_2 \left[ \begin{array}{c} \alpha, \beta, \gamma \\ \alpha + \beta + 1/2, 2\gamma \\ \end{array} \right] = \frac{\Gamma(1/2)\Gamma(\gamma + 1/2)\Gamma(\alpha/2 + \beta/2 + 1/2)}{\Gamma(\alpha/2 + 1/2)\Gamma(\beta/2 + 1/2)}(y - a/2 - \beta/2 + 1/2)
\end{align*}
\] (5)

provided \( \Re (2y - a - \beta) > -1 \), and

\[
\begin{align*}
&_3F_2 \left[ \begin{array}{c} \alpha, \beta, \gamma \\ 1 + \alpha - \beta, 1 + \alpha - \gamma \\ \end{array} \right] = \frac{\Gamma(1 + \alpha - \beta)\Gamma(1 + \alpha - \gamma)\Gamma(1 + \alpha/2)\Gamma(1 + \alpha/2 - \beta - \gamma)}{\Gamma(1 + \alpha)\Gamma(1 + \alpha/2 - \beta)\Gamma(1 + \alpha/2 - \gamma)\Gamma(1 + \alpha - \beta - \gamma)}
\end{align*}
\] (6)

provided \( \Re (a - 2\beta - 2\gamma) > -2 \).

During 1992–1994, in a series of three research papers, Lavoie et al. [9, 10] generalized the abovementioned classical Watson and Dixon summation theorems and obtained twenty-five results closely related to Watson theorem and twenty-three results closely related to Dixon theorem in the form of two general formulas given as follows:

**Generalization of Watson summation theorem [9]**

\[
\begin{align*}
A_{ij} &= 2^{a+b+c-2} \frac{1}{2} a + \frac{1}{2} b + \frac{1}{2} c + \frac{1}{2}\Gamma(c + [j/2] + 1/2) \Gamma(c - 1/2a - 1/2b - [i + j]/2 - j/2 - 1/2)
\end{align*}
\] (7)

\[
\times \frac{\Gamma(c - 1/2a + 1/4(1 - (-1)^j))\Gamma(1/2b)}{\Gamma(1/2)\Gamma(a)\Gamma(b)}
\]

\[
\times \left\{ B_{ij} + C_{ij} \right\}
\]

where \( i, j = 0, \pm 1, \pm 2 \).

Also, here \([x]\) denotes the greatest integer less than or equal to \( x \) and its modulus is denoted by \(|x|\). The coefficients \( A_{ij}, B_{ij}, \) and \( C_{ij} \) are given in Tables 1–3 at the end of this paper.
Clearly, for $i, j = 0$, the result (7) reduces to the Watson theorem (5).

Generalization of Dixon summation theorem [10]

$$\frac{\binom{a}{b, c}}{3F_2}, \quad \text{where} \quad \binom{a}{b, c} = \left[ \begin{array}{cccc} a & b & c \\ 1+a-b+i & 1+a-c+i+j & \end{array} \right]$$

$$= 2^{-2c+i+j} \Gamma(1+a-b+i)\Gamma(1+a-c+i+j)$$

$$\times \frac{\Gamma(b-i/2-j/2)\Gamma(c-1/2(i+j)+i+j!)}{\Gamma(a-2c+i+j+1)\Gamma(a-b-c+i+j+1)\Gamma(b)\Gamma(c)}$$

$$\times \left\{ D_{ij} \frac{\Gamma(1/2a-c+1/2+i+j/2)\Gamma(1/2a-b-c+i+1+[j+1/2])}{\Gamma(1/2a+1/2)\Gamma(1/2a-b+1+[i/2])} \right\}$$

$$+ E_{ij} \left( \frac{\Gamma(1/2a-c+1+[i+j/2])\Gamma(1/2a-b-c+3/2+i+[j+1/2])}{\Gamma(1/2a)\Gamma(1/2a-b+1/2+[i+1/2])} \right).$$

provided that $\text{Re}(a-2b-2c)>-2-2i-j$, for $i = -3, -2, -1, 0, 1, 2$ and $j = 0, 1, 2, 3$.

Also, as usual, here $[x]$ denotes the greatest integer less than or equal to $x$ and its modulus is denoted by $|x|$.

The coefficients $D_{ij}$ and $E_{ij}$ appear in Tables 4 and 5 at the end of this paper.

Clearly, for $i, j = 0$, the result (8) reduces to the Dixon theorem (6).

By applying classical summation theorem (5) and Dixon summation theorem (6) into the following well-known and useful transformation formula available in the literature [2], viz.,

$$\frac{\binom{a}{b, c, d, e}}{6F_5}, \quad \text{where} \quad \binom{a}{b, c, d, e} = \left[ \begin{array}{cccc} a & a/2+1 & b & c \\ 1+a-b & 1+a-c & 1+a-d & 1+a-e \end{array} \right], \quad \text{where} \quad \binom{a}{b, c, d, e} = \left[ \begin{array}{cccc} 1+a-b-c & d \end{array} \right]$$

$$= \frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(a+1)\Gamma(a+d-e)} \left[ \begin{array}{cccc} 1+a-b-c, d \\ 1+a-b, 1+a-c \end{array} \right].$$
very recently Zhao [13] established among several results the following five interesting summation formulas for the series \(_6F_5(-1)\). These are

\[
\begin{align*}
P_{11} &= \_6F_5 \left[ \begin{array}{c}
\frac{a}{2} + 1, b, c, 1 - b, 1 - c \\
\frac{a}{2}, 1 + a - b, 1 + a - c, a + b, a + c
\end{array} \right] ; -1 \\
&= \frac{\pi 2^{1 - 2a} \Gamma (a - b + 1) \Gamma (a + b) \Gamma (a - c + 1) \Gamma (a + c)}{
\Gamma (a) \Gamma (a + 1) \Gamma (a/2 - b/2 - c/2 + 1) \Gamma (a/2 + b/2 - c/2 + 1/2) \Gamma (a/2 + c/2 - b/2 + 1/2) \Gamma (a/2 + b/2 + c/2)}.
\end{align*}
\]

\[
\begin{align*}
P_{12} &= \_6F_5 \left[ \begin{array}{c}
a, \frac{a}{2} + 1, 2b, c, 1 + a - 2c, 1 + 2a - 2b - 2c \\
\frac{a}{2}, 1 + a - 2b, 2c, 1 + a - c, -a + 2b + 2c
\end{array} \right] ; -1 \\
&= \frac{\sqrt{\pi} \Gamma (c + 1/2) \Gamma (-a - b + 1) \Gamma (-a + b + 2c)}{
\Gamma (a + 1) \Gamma (b + 1/2) \Gamma (-b + c + 1/2) \Gamma (a - b - c + 1) \Gamma (-a + b + 2c)}.
\end{align*}
\]

\[
\begin{align*}
P_{13} &= \_6F_5 \left[ \begin{array}{c}
a, \frac{a}{2} + 1, 2b, 2c, a - b - c + \frac{1}{2}, -a + 2b + 2c \\
\frac{a}{2}, 1 + a - 2b, 1 + a - 2c, 2a - 2b - 2c + 1, b + c + \frac{1}{2}
\end{array} \right] ; -1 \\
&= \frac{\sqrt{\pi} \Gamma (b + c + 1/2) \Gamma (a - 2b + 1) \Gamma (a - 2c + 1) \Gamma (a - b - c + 1) \Gamma (a - 2b - c + 1)}{
\Gamma (a + 1) \Gamma (b + 1/2) \Gamma (c + 1/2) \Gamma (a - b - c + 1) \Gamma (a - 2b + 1) \Gamma (a - 2b - c + 1)}.
\end{align*}
\]

\[
\begin{align*}
P_{14} &= \_6F_5 \left[ \begin{array}{c}
\frac{1}{2} a, a + \frac{3}{4}, 2a - \frac{1}{2}, 2a - b, b \\
a - \frac{1}{4}, 2a, a + \frac{1}{2}, 2a - b + \frac{1}{2}, b + \frac{1}{2}
\end{array} \right] ; -1 \\
&= \frac{\sqrt{\pi} \Gamma (a + 1/2)^2 \Gamma (b + 1/2)^2 \Gamma (2a - b + 1/2)}{
\Gamma (2a + 1/2) \Gamma (b/2 + 1/2)^2 \Gamma (a - b/2 + 1/2)^2}.
\end{align*}
\]
The following general summation formula holds true:

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} = e^a$$

Our aim of this paper is to provide as many as one-hundred and ninety-two summation formulas for the generalized hypergeometric series $_6F_5(-1)$. This is achieved by means of applying generalized Watson’s and Dixon’s summation formulas (7) and (8) into the transformation formula (9). Results (10) to (14) obtained earlier by Zhao follow special cases of our main findings.

In this section, we shall establish as many as one-hundred and ninety-two summation formulas in the form of eight general summation formulas asserted in the following theorems.

**Theorem 1.** For $i, j = 0, \pm 1, \pm 2$ and $\Re(2a + b + c + i) > 0$, the following general summation formula holds true:

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} = e^a$$

The coefficients $A_{ij}$, $B_{ij}$, and $C_{ij}$ can be obtained from the tables of $A_{ij}$, $B_{ij}$, and $C_{ij}$ by simply changing
For \(a \longrightarrow 2c - 2b - j, \quad b \longrightarrow 2a - 2b - 2c + 1 - i + j, \quad \text{and} \quad c \longrightarrow c - j\), respectively.

\[
\begin{align*}
\begin{bmatrix}
a, \frac{a}{2} + 1, 2b - i, 2c, \frac{1}{2} + a - b - c, -a + 2b + 2c + j \\
a, \frac{1}{2} + b + c, 1 + a - 2c, 1 + a - 2b + i, 1 + 2a - 2b - 2c - j
\end{bmatrix}
\end{align*}
\]

\[\frac{1}{\Gamma(1/2)\Gamma(1+a-2b+i)\Gamma(1+a-2c)\Gamma(1+a-2b+i)}\]

\[\times \frac{\Gamma(1/2 + b + c)\Gamma(1/2 + b + c)\Gamma(1/2 + b + c)}{\Gamma(2c)\Gamma(1 + \tan - q2hb_{x}x27c; i)}\]

\[
\frac{\Gamma(c + 1/2)\Gamma(b - i/2 + (1 - (1)^{j}/4))}{\Gamma(1/2 + a - 2b - 2c - j + |j + 1/2|)\Gamma(1/2 + a - 2b - c + i/2 - j + |j + 1/2| - (-1)^{j}/4(1 - (1)^{j})}]
\]

Theorem 2. For \(i, j = 0, \pm 1, \pm 2\) and \(\Re(a - 2b - 2c + i) > -1\), the following general summation formula holds true:

\[
\begin{align*}
\begin{bmatrix}
2a - \frac{1}{2} a + \frac{3}{4} b, a - j, \frac{1}{2} + j, 2a - b - i + j \\
a - \frac{1}{2} + 2a - b, 2a - j, \frac{1}{2} + a + j \frac{1}{2} + b + i - j
\end{bmatrix}
\end{align*}
\]

\[\frac{1}{\Gamma(1/2 + 2a - b)\Gamma(1/2 + j)\Gamma(1/2 + a - j + |j/2|)}\]

\[\times \frac{\Gamma(1/2 + b + i - j)\Gamma(-a + b + 1/2 + i/2 - j/2 - |i + j/2|)}{\Gamma(1/2 - a + b + i)\Gamma(2at - nbq - h_{i}, x_{j})}\]

\[
\frac{\Gamma(a - b/2 - j/2 + (1 - (1)^{j}/4))\Gamma(a - b/2 - i/2 + j/2)}{\Gamma(1/2 - 2a - b + i)\Gamma(2at - nbq - h_{i}, x_{j})}\]

\[
\frac{\Gamma(a - b/2 + 1/2 - i/2 + j/2)\Gamma(a - b/2 - j/2 + 1/2 - i/2 + j/2)}{\Gamma(1/2 - 2a - b + i)\Gamma(2at - nbq - h_{i}, x_{j})}\]

\[+ \frac{\Gamma(a - b/2 + 1/2 - i/2 + j/2)\Gamma(a - b/2 - j/2 + 1/2 - i/2 + j/2)}{\Gamma(1/2 - 2a - b + i)\Gamma(2at - nbq - h_{i}, x_{j})}\]
Theorem 3. For \( i, j = 0, \pm 1, \pm 2 \) and \( \Re(b - a + i) > -1/2 \), the following general summation formula holds true:

\[
\begin{aligned}
\left( a + 1, a, b, a - b + \frac{1}{2}, 1 + a - 2b + j \right)_{6F_5} = \\
\frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( 1 - \frac{i}{2} \right) \Gamma \left( a + \frac{1}{2} \right) \Gamma \left( a - b + \frac{1}{2} \right)}{\Gamma \left( 1 + a \right) \Gamma \left( 1 + a - b \right) \Gamma \left( 1 + a - 2b + j \right)} \\
\times \\
\frac{\Gamma \left( \frac{3}{2} \right) \Gamma \left( a/2 - b/2 + 1/4 - j + \left\lfloor j/2 \right\rfloor \right) \Gamma \left( a/2 - b/2 + 1/4 - i/2 + \left( -1 \right)^j \right)}{\Gamma \left( 1 \right) \Gamma \left( 1 - i \right) \Gamma \left( 1 - i - j \right)} \\
\cdot \\
\frac{\Gamma \left( a/2 - b/2 + 3/4 \right) \Gamma \left( a/2 - b/2 - 1/4 - j + \left\lfloor j/2 \right\rfloor \right) \Gamma \left( a/2 - b/2 - 1/4 - i/2 + \left( -1 \right)^j \right)}{\Gamma \left( 1 \right) \Gamma \left( 1 - i \right) \Gamma \left( 1 - i - j \right)} \\
\end{aligned}
\]

(18)

Theorem 4. For \( i, j = 0, \pm 1, \pm 2 \) and \( \Re(2b - a + i) > 0 \), the following general summation formula holds true:

\[
\begin{aligned}
\left( \frac{a}{2} + 1, a, b, 1 - b + i + j, 1 - c + i \right)_{6F_5} = \\
\frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( 1 + a - b \right) \Gamma \left( 1 + a - c \right) \Gamma \left( a + b - i - j \right) \Gamma \left( a - b + i + j \right) \Gamma \left( a + c - i \right)}{\Gamma \left( a + 1 \right) \Gamma \left( a - i \right) \Gamma \left( 1 - b + i + j \right) \Gamma \left( 1 - c + i \right)} \\
\times \\
\frac{\Gamma \left( 1 - c + i / 2 - j / 2 \right) \Gamma \left( 1 - b + i / 2 + j / 2 - j / 2 \right)}{\Gamma \left( a - b + c - i - j \right) \Gamma \left( a + b + c - 1 - 2j - j \right)} \\
\cdot \\
\frac{\Gamma \left( a/2 + b/2 - c/2 - i - j + \left\lfloor j + 1 / 2 \right\rfloor \right) \Gamma \left( a/2 + b/2 + c/2 - 1/2 - i - j + \left\lfloor j + 1 / 2 \right\rfloor \right)}{\Gamma \left( a/2 - b/2 - c/2 + 1/2 \right) \Gamma \left( a/2 - b/2 + c/2 + 1/2 - i + \left\lfloor j / 2 \right\rfloor \right)} \\
\end{aligned}
\]

(19)

The coefficients \( A_{ij} \), \( B_{ij} \), and \( C_{ij} \) can be obtained from the tables of \( A_{ij} \), \( B_{ij} \), and \( C_{ij} \) by simply changing \( a \rightarrow 2a - b - j \), \( b \rightarrow 2a - b + i + j \), and \( c \rightarrow a - j \), respectively.
Theorem 5. For \( i = -3, -2, -1, 0, 1, 2 \) and \( j = 0, 1, 2, 3 \), the following general summation formula holds true:

\[
\begin{align*}
\sum_{k} F_5^{a, b, c, 1 + a - b - 2c - i - j, 1 + 2a - 2b - 2c - i - j} & = 2^{-2a+b+2c-2+i+j} \frac{\Gamma (1 + a - 2b) \Gamma (1 + a - c) \Gamma (2c + j)}{\Gamma (a + 1) \Gamma (2b) \Gamma (1 + a - 2b - c)} \times \\
& \times \frac{\Gamma (-a + 2b + 2c + i + j) \Gamma (1 + a - 2c + i/2 - j + |i/2|) \Gamma (1 + a - 2b - c - i/2 - j/2 - |i + j|/2)}{\Gamma (c + j) \Gamma (1 + a - 2c - j) \Gamma (-2a + 2b + 4c - 1 - i + 2j)} \\
& \times \frac{\Gamma (b + 1/2 - i/2 - j/2 + |i + j + 1/2|) \Gamma (-a + b + 2c - 1/2 + i/2 + j/2 + |j + 1/2|)}{\Gamma (1 + a - b - c - i/2 - j/2 + |i|/2)} \\
& + \frac{\Gamma (b + 1/2 - i/2 - j/2 + |i + j|/2) \Gamma (-a + b + 2c + i/2 + j/2 + |j + 1/2|)}{\Gamma (a + b - c + 1/2 - i/2 - j/2) \Gamma (-b + c - i/2 + j/2 + |i + j|/2)}
\end{align*}
\]

(20)

The coefficients \( _1D_{ij} \) and \( _1E_{ij} \) can be obtained from the tables of \( D_{ij} \) and \( E_{ij} \) by simply changing \( a \rightarrow 1 + 2a - 2b - 2c - i - j \), \( b \rightarrow 1 + a - 2c - j \), and \( c \rightarrow 1 + a - 2b - c \), respectively.

Theorem 6. For \( i = -3, -2, -1, 0, 1, 2 \) and \( j = 0, 1, 2, 3 \), the following general summation formula holds true:

\[
\begin{align*}
\sum_{k} F_5^{a, b, c, a + 3/4, b, 2a - \frac{1}{2}, 2a - b - i, 1/2 + j} & = 2^{j-1} \frac{\Gamma (a + 1/2) \Gamma (1/2 + 2a - b) \Gamma (2a - j) \Gamma (1/2 + b + i)}{\Gamma (a) \Gamma (2a + 1/2) \Gamma (1/2 + a - b)} \times \\
& \times \frac{\Gamma (1/2 + a - b - i/2 - |i/2|) \Gamma (1/2 - i/2 + j/2 - |i + j|/2)}{\Gamma (1/2 + j) \Gamma (b + i - j) \Gamma (2a - b - j)} \\
& \times \frac{\Gamma (a - b/2 - i/2 - j + |i + j + 1/2|) \Gamma (b/2 + i/2 - j + |j + 1/2|)}{\Gamma (a - b/2 + 1/2 - i/2) \Gamma (b/2 + 1/2 - i/2 + |i/2|)} \\
& + \frac{\Gamma (a - b/2 + 1/2 - i/2 - j + |i + j|/2) \Gamma (b/2 + 1/2 - i/2 - j + |j + 1/2|)}{\Gamma (a - b/2 - i/2) \Gamma (b/2 - i/2 + |i + 1/2|)}
\end{align*}
\]

(21)

The coefficients \( _2D_{ij} \) and \( _2E_{ij} \) can be obtained from the tables of \( D_{ij} \) and \( E_{ij} \) by simply changing \( a \rightarrow 1 + 2a - 2b - 2c - i - j \), \( b \rightarrow 1 + a - 2c - j \), and \( c \rightarrow 1 + a - 2b - c \), respectively.
Theorem 7. For \( i = -3, -2, -1, 0, 1, 2 \) and \( j = 0, 1, 2, 3 \), the following general summation formula holds true:

The coefficients \( D_{ij} \) and \( E_{ij} \) can be obtained from the tables of \( D_{ij} \) and \( E_{ij} \) by simply changing \( a \rightarrow 2a - b - i \), \( b \rightarrow 1/2 + a - b \), and \( c \rightarrow 1/2 + j \), respectively.

\[
\begin{align*}
\text{For Theorem 7.} & \quad \frac{a, b, a}{2} + 1, a - b + \frac{1}{2}, a - b + \frac{1}{2}, -i, 1 + a - 2b + j, e_{6F_5} ; -1
\end{align*}
\]

\[
\begin{align*}
\text{Proof.} & \quad \text{In order to establish (15) asserted in Theorem 1, we proceed as follows. In the transformation formula (9), if we take } d = 1 + a - 2c + j \\
& \quad \text{and } e = 1 + 2a - 2b - 2c - i + j, \text{ then for } i, j = -1, 2, 1, 0, \text{ it takes the following form:}
\end{align*}
\]

Theorem 8. For \( i = -3, -2, -1, 0, 1, 2 \) and \( j = 0, 1, 2, 3 \), the following general summation formula holds true:

The coefficients \( D_{ij} \) and \( E_{ij} \) can be obtained from the tables of \( D_{ij} \) and \( E_{ij} \) by simply changing \( a \rightarrow a - b + 1/2 - i \), \( b \rightarrow 1/2 \), and \( c \rightarrow a - 2b + 1 + j \), respectively.

\[
\begin{align*}
\text{For Theorem 8.} & \quad \begin{bmatrix} a, a, b, a \cdot 2 + 1, 2b, c - j, 1 + a - 2c + j, 1 + 2a - 2b - 2c - i + j, \end{bmatrix}_{6F_5} ; -1
\end{align*}
\]

\[
\begin{align*}
\text{Proof.} & \quad \text{In order to establish (15) asserted in Theorem 1, we proceed as follows. In the transformation formula (9), if we take } d = 1 + a - 2c + j \\
& \quad \text{and } e = 1 + 2a - 2b - 2c - i + j, \text{ then for } i, j = -2, -1, 0, 1, 2, \text{ it takes the following form:}
\end{align*}
\]
We now observed that the $\binom{3}{2}$ appearing on the right hand side of (20) can be evaluated with the help of generalized Watson's summation theorem (7) by taking $a \rightarrow 2c - 2b - j$, $b \rightarrow 2a - 2b - 2c + 1 - i + j$, and $c \rightarrow c - j$ and after some algebras, we easily arrive at the desired result (15).

In exactly the same manner, the generalized summation formulas (16) to (22) asserted in Theorems 2 to 8 can be established. We, however, prefer to omit the details.

3. Corollaries

In this section, we shall mention some of the known as well as unknown results of our main findings.
Corollary 1. In Theorem 1, if we take (i) \( i = 0, j = 0 \), (ii) \( i = 0, j = 1 \), (iii) \( i = 1, j = 0 \), and (iv) \( i = 1, j = 1 \), we get the following interesting results:

The result (24) is the known result (11) due to Zhao [13] written in a different form.

Similarly, other results can be obtained.

\[
\begin{bmatrix}
a, \frac{a}{2} + 1, 2b, 2c, a - b - c + \frac{1}{2} - a + 2b + 2c \\
a, \frac{a}{2} + 1, 2b, 2c, a - b - c, 1 - a + 2b + 2c \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
a, \frac{1}{2} + b + c, 1 + a - 2b, 1 + a - 2c, a - 2b - 2c \\
a, \frac{1}{2} + b + c, 1 + a - 2b, 1 + a - 2c, 2a - 2b - 2c \\
\end{bmatrix}
\]

\[
= 4^{b+c-1} \frac{\Gamma(b)\Gamma(c)}{\Gamma(1 + a - 2b - c)\Gamma(1 + a - b - 2c)} \times \frac{\Gamma(1/2 + b + c)\Gamma(1 + a - 2b)\Gamma(1 + a - 2c)\Gamma(1 + a - b - c+)}{\Gamma(1/2)\Gamma(a + 1)\Gamma(2b)\Gamma(2c)}
\]

\[
\begin{bmatrix}
a, \frac{a}{2} + 1, 2b, 2c, \frac{1}{2} + a - b - c, 1 - a + 2b + 2c \\
a, \frac{1}{2} + b + c, 1 + a - 2b, 1 + a - 2c, 1 + 2a - 2b - 2c \\
\end{bmatrix}
\]

\[
= 4^{b+c-1} \frac{\Gamma(1/2 + b + c)\Gamma(2 + a - 2b)\Gamma(1 + a - 2c)\Gamma(1 + a - b - c+)}{\Gamma(1/2)\Gamma(a + 1)\Gamma(2b - 2c)\Gamma(2a - 2b - 2c)}
\]

\[
\begin{bmatrix}
a, \frac{a}{2} + 1, 2b - 1, 2c, \frac{1}{2} + a - b - c, - a + 2b + 2c \\
a, \frac{1}{2} + b + c, 2 + a - 2b, 1 + a - 2c, 1 + 2a - 2b - 2c \\
\end{bmatrix}
\]

\[
= 4^{b+c-1} \frac{\Gamma(1/2 + b + c)\Gamma(2 + a - 2b)\Gamma(1 + a - 2c)\Gamma(1 + a - b - c+)}{\Gamma(1/2)\Gamma(a + 1)\Gamma(2b - 2c)\Gamma(2a - 2b - 2c)}
\]

\[
\begin{bmatrix}
a, \frac{a}{2} + 1, 2b - 1, 2c, \frac{1}{2} + a - b - c, 1 - a + 2b + 2c \\
a, \frac{1}{2} + b + c, 2 + a - 2b, 1 + a - 2c, 2a - 2b - 2c \\
\end{bmatrix}
\]

\[
= 4^{b+c-1} \frac{\Gamma(1/2 + b + c)\Gamma(2 + a - 2b)\Gamma(1 + a - 2c)\Gamma(1 + a - b - c+)}{\Gamma(1/2)\Gamma(a + 1)\Gamma(2b - 2c)\Gamma(2a - 2b - 2c)}
\]
Corollary 2. In Theorem 2, if we take (i) \( i = 0, j = 0 \),
(ii) \( i = 0, j = 1 \), (iii) \( i = 1, j = 0 \), and (iv) \( i = 1, j = 1 \), we get the following interesting results:

The result (28) is the known result (12) due to Zhao [13] written in a different form.

Similarly, other results can be obtained.
Corollary 3. In Theorem 3, if we take (i) \( i = 0, j = 0 \), (ii) \( i = 0, j = 1 \), (iii) \( i = 1, j = 0 \), and (iv) \( i = 1, j = 1 \), we get the following interesting results:

The result (32) is the known result (13) due to Zhao [13] written in a different form.

Similarly, other results can be obtained.
Corollary 4. In Theorem 4, if we take (i) $i = 0, j = 0$, (ii) $i = 0, j = 1$, (iii) $i = 1, j = 0$, and (iv) $i = 1, j = 1$, we get the following interesting results:

The result (36) is the known result (14) due to Zhao [13] written in a different form.

Similarly, other results can be obtained.
Corollary 5. In Theorem 5, if we take (i) \( i = 0, j = 0 \), (ii) \( i = 0, j = 1 \), (iii) \( i = 1, j = 0 \), and (iv) \( i = 1, j = 1 \), we get the following interesting results:

The result (40) is the known result (10) due to Zhao [13] written in a different form.

Similarly, other results can be obtained.
In Theorem 6, if we take Corollary 6.

\[
\begin{align*}
&\text{(i)} i = 0, j = 0, \\
&\text{(ii)} i = 0, j = 1, \\
&\text{(iii)} i = 1, j = 0, \\
&\text{(iv)} i = 1, j = 1,
\end{align*}
\]
and (iv) \(i = 1, j = 1\), we get the following interesting results:

\[
\begin{align*}
&\text{Corollary 6. In Theorem 6, if we take (i) } i = 0, j = 0, \\
&\quad (ii) i = 0, j = 1, (iii) i = 1, j = 0, \\
&\quad \text{and (iv) } i = 1, j = 1, \text{ we get the following interesting results:}
\end{align*}
\]
Corollary 7. In Theorem 7, if we take (i) \( i = 0, j = 0 \), (ii) \( i = 0, j = 1 \), (iii) \( i = 1, j = 0 \), and (iv) \( i = 1, j = 1 \), we get the following interesting results:

\[
\frac{\Gamma (2a - b + 1) \Gamma (a + b - 1)}{\Gamma (a + b + 1)} \frac{\Gamma (a + b + 1)}{\Gamma (a + b + 2)} \frac{\Gamma (a + b + 2)}{\Gamma (a + b + 3)} \frac{\Gamma (a + b + 3)}{\Gamma (a + b + 4)} \ldots \frac{\Gamma (\frac{\alpha}{2} + \frac{\beta}{2} + \frac{1}{2})}{\Gamma (\frac{\alpha}{2} + \frac{\beta}{2} + 1)} \frac{\Gamma (\frac{\alpha}{2} + \frac{\beta}{2} + 1)}{\Gamma (\frac{\alpha}{2} + \frac{\beta}{2} + 2)} \ldots
\]

The result (48) is the known result (13) due to Zhao [13] written in a different form.

Similarly, other results can be obtained.
\[
{\begin{array}{c}
{_{6}F_{5}}
\left[
\begin{array}{c}
(a, \frac{a}{2} + 1, b, a - b + \left(\frac{1}{2}\right), a - b - \left(\frac{1}{2}\right), a - 2b + 2
\end{array}
\right]
\end{array}
\right] ; -1
\]

\[
= \frac{4[\Gamma (b - (1/2))]^{2}\Gamma (2b - 1)\Gamma (1 + a - b)}{3[\Gamma (a + 1)\Gamma (b)\Gamma (a - 2b + 2)\Gamma (a - b + (3/2))\Gamma (-a + 3b - (5/2))}
\left[
\frac{(b - a)\Gamma ((a/2) - (b/2) + (1/4))\Gamma (-a/2) + (3b/2) + (1/4))}{\Gamma ((a/2) - (b/2) + (1/4))\Gamma (-a/2) + (3b/2) - (5/4))}
\right] - \frac{(a - 3b + 2)\Gamma ((a/2) - (b/2) + (5/4))\Gamma (-a/2) + (3b/2) - (1/4))}{\Gamma ((a/2) - (b/2) + (1/4))\Gamma (-a/2) + (3b/2) - (3/4))}.
\]

**Corollary 8.** In Theorem 8, if we take (i) \(i = 0, j = 0\), (ii) \(i = 0, j = 1\), (iii) \(i = 1, j = 0\), and (iv) \(i = 1, j = 1\), we get the following interesting results:

The result (52) is the known result (14) due to Zhao [13] written in a different form.

Similarly, other results can be obtained.

4. **Concluding Remark**

In this paper, we have provided as many as one-hundred and ninety-two summation formulas for the generalized hypergeometric series \(_{6}F_{5}\) \((-1)\) in terms of gamma functions. This is achieved by means of applying generalized Watson's and Dixon's summation formulas obtained earlier by Lavoie et al. into a known transformation formula available in the literature.

We believe that the results established in this paper have not been appeared in the literature before and represent a definite contribution to the theory of generalized hypergeometric function. Since the results established are in terms of gamma function, it is hoped that the results could be of potential use in the area of mathematics, statistics, and mathematical physics.

Here,

\[
\begin{align*}
A_{2,-2} &= \frac{1}{2(c - 1)(a - b - 1)(a - b + 1)} \\
A_{2,-1} &= \frac{1}{2(a - b - 1)(a - b + 1)} \\
A_{2,0} &= \frac{1}{4(a - b - 1)(a - b + 1)} \\
A_{2,1} &= \frac{1}{4(a - b - 1)(a - b + 1)} \\
A_{2,2} &= \frac{1}{8(c + 1)(a - b - 1)(a - b + 1)} \\
A_{1,-2} &= \frac{1}{(c - 1)(a - b)} \\
A_{1,2} &= \frac{1}{2(c + 1)(a - b)}
\end{align*}
\]
Here,

\[ B_{2-2} = c(a + b - 1) - (a + 1)(b + 1) + 2, \]
\[ B_{2-1} = a + b + 1 + b(2c - b) - 2c + 1, \]
\[ B_{20} = a(2c - a) - (a - b)^2 + 1, \]
\[ B_{22} = 2c(c + 1)((2c + 1)(a + b - 1) - a(a - 1) - b(b - 1)) \]
\[ - 3cm - (a - b - 1)((a - b + 1)(2c - a - b + 1) + ab), \]
\[ B_{11} = 2c - a + b - (a - b)(c - b + 1), \]
\[ B_{0-2} = (c - a + 1)(c - b + 1) + c(c + 1), \]
\[ B_{-1-2} = 2(c - 1)(c - 2) - (a - b)(c - b - 1), \]
\[ B_{-2-2} = 2(c - 1)(c - 2)((2c - 1)(a + b - 1) \]
\[ - a(a + 1) - b(b + 1) + 2, \]
\[ - (a - b - 1)(a - b + 1)((c - 1)(2c - a - b - 3) + ab), \]
\[ B_{-2-1} = 2(c - 1)(a + b - 1) - (a - b)^2 + 1, \]
\[ B_{-2} = a(2c - a) + b(2c - b) - 2c - 1, \]
\[ B_{-2} = c(a + b - 1) - (a - 1)(b - 1). \]

(57)

Here,

\[ C_{21} = -\{8c^2 - 2c(a + b - 1) - (a - b)^2 + 1\}, \]
\[ C_{22} = -4(2c + a + b + 1)(2c - a + b + 1) \]
\[ - 4(2c + a + b + 1)(2c - a + b + 1), \]
\[ C_{12} = -(2c(c + 1) + (a - b)(c - a + 1)), \]
\[ C_{-1-2} = 2(c - 1)(c - 2) + (a - b)(c - b - 1), \]
\[ C_{-2-2} = 4(2c - a + b - 3)(2c + a - b - 3), \]
\[ C_{-2-1} = 8c^2 - 2(c - 1)(a + b + 7) - (a - b)^2 - 7. \]

Here,

\[ D_{30} = 5a - b^2 + (a + 1)^2 - (2a - b + 1)(b + c), \]
\[ D_{20} = \frac{1}{2}(a - 1)(a - 4) - (b^2 - 5a + 1) - (a - b + 1)(b + c), \]
\[ D_{21} = (b - 1)(b - 2) - (a - b + 1)(a - b - c + 3), \]
\[ D_{22} = \frac{1}{2}(a - c + 2)(a - 2b - c + 5)(a - c + 2)(a - 2b + 2) - a(c - 3), \]
\[ - (b - 1)(b - 2)(c - 2)(c - 3), \]
\[ D_{12} = a(a - 1) + (b + c - 3)(c - 2a - 1), \]
\[ D_{0-2} = \frac{1}{2}(a - b - c + 1)^2 + (c - 1)(c - 3) - b^2 + a \]
\[ + 3ab + c(a - b - c + 4) - (a + 1)(a + 2) - (a - 1)(b - 1), \]
\[ D_{-13} = (c - 1)(c - 2) - b(a - c + 1), \]
\[ D_{-20} = \frac{1}{2}(a - 1)(a - 2b - 2) - c(a - b - 1), \]
\[ D_{-22} = \frac{1}{2}(a - 1)(a - 2b - 2c) + b(b + c), \]
\[ D_{-23} = (a - b - 1)(c - 1) - b(b + 1), \]
\[ D_{-30} = (a - 1)(a - 2b - 2c - 4) + bc, \]
\[ D_{-31} = (a - b - 2)(a - c - 1) - ac, \]
\[ D_{-32} = (a - b - 1)(a - b - 2c - 2) - bc, \]
\[ D_{-33} = b(b + 1) + (a - 1)(a - b) - c(2a - b - 2). \]
Here,
\[ E_{30} = -a + 3b^2 - (a + 3)^2 + (2a - 3b + 5)(b + c), \]
\[ E_{21} = -(b - 1)(b - 2) + (a - b - 2c + 5)(a - b - c + 3), \]
\[ E_{22} = -2(a - c + 2)(a - 2b - c + 5), \]
\[ E_{12} = -(a - b - c + 2)(a - b - c + 3) + (b - 1)(b - c + 1), \]
\[ E_{03} = (a + 2)(a + 4) - b(2a + 5) - 3c(a - b - c + 4) + 3, \]
\[ E_{13} = (c - 1)(c - 2) + b(a - 2b - c + 1), \]
\[ E_{-23} = b(a - 2c + 2) - (b - c + 1)(a - b - 2c + 1), \]
\[ E_{-30} = (a - 2)(a - 2b - 2c - 3) + 3bc, \]
\[ E_{-31} = (a - b - 2)(a - 2b - 2c - 3) + bc, \]
\[ E_{-32} = (a - b - 2)(a - b - 2c - 1) + bc, \]
\[ E_{-33} = (a - 1)(a - 2) - 3b(a - b - 2) - c(2a - 3b - 4). \]

(60)

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflicts of Interest

The authors declare that they have no conflicts interest.

Authors’ Contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

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