Research Article

Some Developments in the Field of Homological Algebra by Defining New Class of Modules over Nonassociative Rings

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The LA-module is a nonassociative structure that extends modules over a nonassociative ring known as left almost rings (LA-rings). Because of peculiar characteristics of LA-ring and its inception into noncommutative and nonassociative theory, drew the attention of many researchers over the last decade. In this study, the ideas of projective and injective LA-modules, LA-vector space, as well as examples and findings, are discussed. We construct a nontrivial example in which it is proved that if the LA-module is not free, then it cannot be a projective LA-module. We also construct free LA-modules, create a split sequence in LA-modules, and show several outcomes that are connected to them. We have proved the projective basis theorem for LA-modules. Also, split sequences in projective and injective LA-modules are discussed with the help of various propositions and theorems.

1. Introduction

Kazim and Naseeruddin came up with the idea of left almost semigroups [1]. If a groupoid S meets $(ab)c = (cb)a$ for all $a, b, c \in S$, that is, left invertive law, then it is termed as LA-semigroup. Abel-Grassmann groupoid (abbreviated as AG-groupoid) is another name for this structure [2, 3]. A structure that exists among a commutative semigroup and a groupoid is known as an AG-groupoid. LA-semigroup has been extended to left almost group (LA-group) by Kamran [4]. Groupoid G is referred to as a left almost group (LA-group) if there will be a left identity $e \in G$ (such that $ea = a$ for each $a \in G$), and for $a \in G$, there will be b in G implies that $ba = e$. Also, left invertive law is true in G. The left almost group, despite having a nonassociative structure, bears an interesting resemblance to a commutative group.

Many researchers produced various valuable results for LA-semigroups and LA-groups due to nonsmooth structure development. With these ideas, the theory of the left almost ring was introduced [5]. The byproduct of LA-semigroup and LA-group is the left almost ring (LA-ring). Due to its unique properties, it has gradually developed as a useful nonassociative class with a decent contribution to nonassociative ring theory. A nonempty set R with at least two elements is a LA-ring if $(R, +)$ is LA-group, $(R, \cdot)$ is a LA-semigroup, and both left and right distributive laws are satisfied. For example, we can get LA-ring $(R, \oplus, \cdot)$ from commutative ring $(R, +, \cdot)$ by declaring $a \oplus b = b - a$ for all $a, b \in R$ and $a \cdot b$ is same as it was in the ring.

Shah and Rehman extended the study of LA-rings in [6]. Shah and Rehman [7] examined certain features of LA-rings using their ideals, and as a result, the ideal theory can be a good place to start looking into fuzzy sets and intuitionistic fuzzy sets. Mace4 has been used for certain computational tasks, and interesting and useful LA-ring properties have been examined [8]. In [6], the concept of a commutative semigroup ring is generalized using both the LA-semigroup and the LA-ring. Moreover, Shah and Rehman also develop the concept of a LA-module, which is a nonabelian non-associative structure that is closer to an abelian group. As a result, studying this algebraic structure is quite similar to studying modules which are fundamentally abelian groups. Shah et al. [9] have done additional work in the subject of LA-modules, establishing various isomorphism theorems

and direct sum of LA-module results. Alghamdi and Sahraoui [10] developed and built a tensor product of two LA-modules lately, extending simple conclusions from ordinary tensor to the new scenario. In [11], by defining exact sequences, Asima Razzaque et al. added to the study of LA-modules. Shah et al. [12] presented a complete survey and advances of the existing literature of nonassociative and noncommutative rings, as well as a list of some of their varied applications in diverse fields. Recently, Rehman et al. [13] introduced the concept of neutrosophic LA-rings. In 2020, Razzaque et al. [14], worked on soft LA-modules by defining projective soft LA-modules, free soft LA-modules, split sequence in soft LA-modules, and establish various results on projective and injective soft LA-modules. Abulebda in [15] discussed the uniformly primal submodule over noncommutative ring and generalized the prime avoidance theorem for modules over noncommutative rings to the uniformly primal avoidance theorem for modules. In [16], Groenewald worked on weakly prime and weakly 2-absorbing modules over noncommutative rings. He introduced a weakly m-system and characterized the weakly prime radical in terms of weakly m-system. Putman and Sam in [17] introduced VIC-modules over noncommutative rings. They proved a twisted homology stability for GLn(R) with R a finite noncommutative ring. Nonassociative ring structure was enriched by introducing the hyperstructures. Rehman et al. in 2017 [18] have given the concept of LA-hyperrings. Through their hyperideals and hypersystems, they investigate various important characterizations of LA-hyperrings. Massouros and Yaqoob [19] presented the study of algebraic structures, left/right almost groups, and hypergroups equipped with the inverted associativity axiom, and they analyzed the algebraic properties of these special nonassociative hyperstructures. We refer readers to see if they want to learn more about LA-rings [9, 20–23].

Some further developments in the field of modules were done by Ansari and Habib in [24] by defining their graphs over rings. They investigate the relationship between the graph-theoretic properties and algebraic properties of modules. Moreover, Madhvi and Talebi defined the small intersection graph of submodules of modules [25]. In addition, Abbasi et al. presented a new graph connected with modules over commutative rings in [26]. They look at the connection between some algebraic features of modules and the graphs that go with them. For the completeness of the special subgraphs, they gave a topological characterization. Furthermore, in 2017, Rajkhowa and Saikia worked on the graphs of noncommutative rings by defining the total directed graphs of noncommutative rings [27]. For more study of graphs of rings and graphs of modules over rings, we advised the readers to study [28–34].

In this work, we introduce the concepts of projective and injective LA-modules, LA-vector space, as well as examples and findings, over nonassociative and noncommutative rings. We construct a nontrivial example in which it is proved that if the LA-module is not free, then it cannot be a projective LA-module. We also construct free LA-modules, create a split sequence in LA-modules, and show several outcomes that are connected to them. We have proved the

projective basis theorem for LA-modules. Also, split sequences in projective and injective LA-modules are discussed with the help of various propositions and theorems.

2. Background

In 2011, Shah et al., [9] promoted the notion of LA-module over an LA-ring defined in [6] and further established the substructures, operations on substructures, and quotient of an LA-module by its LA-submodule. They also indicated the nonsimilarity of an LA-module to the usual notion of a module over a commutative ring. Shah et al. [9] have done more work on LA-modules, proving numerous isomorphism theorems and establishing a direct sum of LA-module findings. Alghamdi and Sahraoui [10] recently developed and constructed a tensor product of two LA-modules, extending simple conclusions from ordinary tensor to the new scenario. In [11], Asima Razzaque et al. contributed to the study of LA-modules by defining exact sequences and split sequences.

In the following, we will go over some basic definitions and findings related to the LA-modules.

Definition 1 (see [6]). Let $(R, +, .)$ be LA-ring having left identity e. $(M, +)$ an LA-group is called LA-module over R, and a map $R \times M \longrightarrow M$ is defined $(a, m) \mapsto am \in M$, where $a \in R$, $m \in M$ satisfies

(i) $(a + b)m = am + bm$

(ii) $a(m + n) = am + an$

(iii) $a(bm) = b(am)$

(iv) $1.m = m$

For every $a, b \in R$, $m, n \in M$.

R^M or simply M is the abbreviation for the left R LA-module. M_R denotes the right R LA-module, which can be defined similarly.

Shah and Rehman [6] developed a nontrivial example of LA-module in the following example. The following example shows that every LA-module is not a module.

Example 1 (see [6]). Let commutative semigroup S and $(R, +, .)$ be LA-ring with left identity. Then, $R[S] = \left\{ \sum_{j=1}^{n} a_j s_j : a_j \in R, s_j \in S \right\}$, $R \times R[S] \mapsto R[S]$ as defined by $(a, \sum_{j=1}^{n} a_j s_j) \mapsto \sum_{j=1}^{n} (aa_j)s_j$ becomes an LA-module.

Definition 2 (see [9]). Consider the left R LA-module M. Then, an abelian LA-subgroup N over LA-ring R is left R LA-submodule, if the condition $RN \subseteq N$ holds, which means $rn \in N$ for each $r \in R$, $n \in N$.

Theorem 1 (see [19]). *$A \cap B$ is LA-submodule of M, where A and B are the LA-submodules of an LA-module M.*

Definition 3 (see [9]). $\varphi: M \longrightarrow N$ is called LA-module homomorphism if for all $r \in R$ and $m, n \in M$, where M and N are LA-modules over LA-ring R.

(i) $\varphi(m+n) = \varphi(m) + \varphi(n)$

(ii) $\varphi(rm) = r\varphi(m)$

Theorem 2 (see [9]). *The following statements hold if $\varphi\colon M \longrightarrow N$ is LA-module homomorphism:*

(1) *$\varphi(A)$ an LA-submodule of N, where A is LA-submodule of M*

(2) *$\varphi^{-1}(B)$ an LA-submodule of M, where B is LA-submodule of N*

Proposition 1 (see [35]). *$\cap_{i\in I}M_i$ and $\sum_{i\in I}M_i$ are submodules of M, where $\{M_i | i \in I\}$ is a nonempty family of submodules.*

Definition 4. In [35], readers can see the definition of a short exact sequence.

Proposition 2. *In [36], readers are referred to a proposition in which the relationship between exact sequence, monomorphism, epimorphism, and isomorphism of the modules is developed.*

Theorem 3. *[37] Every free left R module has a homomorphic image in every left R module.*

3. Main Results

We divide our work into two sections and look into a number of significant findings that are backed up with examples. Throughout the paper, R denotes an LA-ring.

3.1. Projective LA-Module. This section begins with a definition of the projective LA-module as well as an example.

Definition 5. Let M denotes the left LA-module over R. Then, M is projective LA-module. In Figure 1, if R LA-modules and LA-homomorphisms have an exact row, then there is an R LA-homomorphism $g\colon M \longrightarrow A$ which results in the completed diagram commutative which means $\alpha g = f$.

To construct the example of a projective LA-module, first, we need to define LA-vector space.

Definition 6. The triplet $(V, +, \cdot)$ is an LA-vector space over an LA-field F if $F \times V \longrightarrow V$ defined as $(f, v) \mapsto fv \in V$, where $f \in F$ and $v \in V$ satisfy the following conditions:

(i) $(V, +)$ is an LA-group

(ii) If $f, g \in F$ and $v \in V \Rightarrow (f + g)v = fv + gv$

(iii) If $f \in F$ and $v_1, v_2 \in V \Rightarrow f(v_1 + v_2) = fv_1 + fv_2$

(iv) If $f, g \in F$ and $v \in V \Rightarrow f(gv) = g(fv)$

(v) For $v \in V$, $1.v = v$

Example 2. Let a commutative semigroup S and $(F, +, .)$ is LA-field with left identity. Consider $F[S] = \{\sum_{i=1}^{n} f_i s_i\colon f_i \in F, s_i \in S\}$, which is obviously is an additive LA-group.
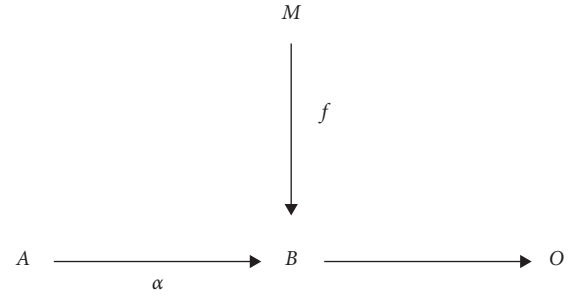

FIGURE 1: Projective LA-module.

Define the map $F \times F[S] \mapsto F[S]$ by $(f, \sum_{i=1}^{n} f_i s_i) \mapsto \sum_{i=1}^{n} (f f_i) s_i$ which is well-defined. It is easy to verify the (ii), (iii), and (v) property. Here, we only prove the (iv) property of LA-vector space. Consider $f(g \sum_{i=1}^{n} f_i s_i) = f(\sum_{i=1}^{n} (g f_i) s_i) = (\sum_{i=1}^{n} (f(g f_i)) s_i)$. Since F is an LA-field, then by ([38], Lemma 4), $a(bc) = b(ac)$ is true for all $a, b, c \in F$. Hence, $f(g \sum_{i=1}^{n} f_i s_i) = (\sum_{i=1}^{n} (g(f f_i)) s_i) = g(\sum_{i=1}^{n} (f f_i) s_i) = g(f \sum_{i=1}^{n} f_i s_i)$. Thus, $f(g \sum_{i=1}^{n} f_i s_i) = g(f \sum_{i=1}^{n} f_i s_i)$. Hence, it is proved.

Remark 1. Let $(S, .)$ be a commutative semigroup. It is easy to observe that S becomes free basis for $F[S]$ as LA-vector space over LA-field F. Indeed, consider $f_1 s_1 + f_2 s_2 + \cdots + f_n s_n = 0$. Since $s_1, s_2, \ldots, s_n \in S$ and $s_i \neq 0$ for each $i = 1, 2, \ldots, n$, so $f_i = 0$ for each $i = 1, 2, \ldots, n$. This implies $s_1, s_2, \ldots, s_n$ are linearly independent. Now let $f_1, f_2, \ldots, f_n \in F$ and $s_1, s_2, \ldots, s_n \in S$. Then, $f_1 s_1 + f_2 s_2 + \cdots + f_n s_n$ is linear combination of elements of S whose coefficients are from LA-field F. Therefore, S is free basis for $F[S]$ as an LA-vector space over LA-field F.

Example 3. An LA-vector space over an LA-field F is free F LA-module, so is a projective LA-module.

Definition 7. A left R LA-module F is called free left R LA-module on a basis $X \neq \phi$, if there will be a map $\alpha\colon X \longrightarrow F$ such that the given map $f\colon X \longrightarrow M$, where M is any left R LA-module, there exists a unique R LA-homomorphism $g\colon F \longrightarrow M$ such that $f = g\alpha$.

Unique LA-homomorphism $g\colon F \longrightarrow A$ is said to extend the map $f\colon X \longrightarrow A$.

R-homomorphism is referred to as R LA-homomorphism in this study.

Theorem 4. *Free R LA-module implies projective LA-module.*

Proof. Suppose F is a free LA-module having basis X.

In Figure 2, R LA-modules and R-homomorphism have the row exact. Let $x \in X$. Then, $f(x) \in B$ and as α is onto, so there exists $a \in A$ then $\alpha(a) = f(x)$. Define $g\colon X \longrightarrow A$ as $g(x) = a$ and extend the function $g\colon F \longrightarrow A$, and here, it is
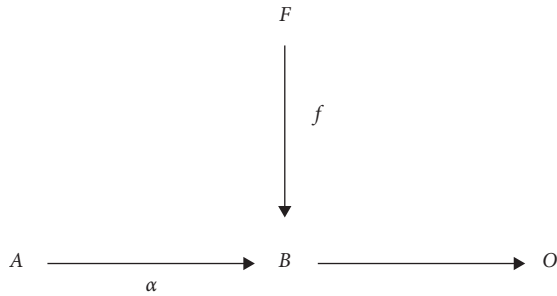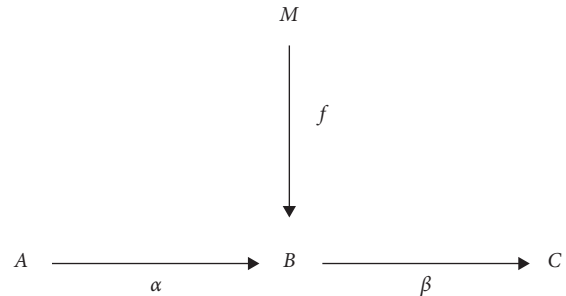
Figure 2: Free LA-module.

Figure 3: R-homomorphism.

clear $\alpha g(x) = \alpha(g(x)) = \alpha(a) = f(x)$. This implies $\alpha g = f$. □

Remark 2. On the other hand, if LA-module is not free, then it will not be projective LA-module. This remark is justified in the next example.

Example 4. Following is the Cayley tables of LA-module $(M, +)$ over R

$$\text{LA} - \text{ring} (R, +, .),$$
$$\text{LA} - \text{group} (M, +). \tag{1}$$

$$LA - ring \ (R, +, .)$$

+	0	1	2	3	4	5	6	7		.	0	1	2	3	4	5	6	7
0	2	4	0	6	1	7	3	5		0	2	5	2	5	5	2	5	2
1	3	2	6	5	0	4	7	1		1	0	1	2	3	4	5	6	7
2	0	1	2	3	4	5	6	7		2	2	2	2	2	2	2	2	2
3	1	5	4	2	7	6	0	3		3	0	4	2	6	1	5	3	7
4	6	0	3	7	2	1	5	4		4	0	3	2	1	6	5	4	7
5	7	3	5	1	6	2	4	0		5	2	0	2	0	0	2	0	2
6	4	7	1	0	5	3	2	6		6	0	6	2	4	3	5	1	7
7	5	6	7	4	3	0	1	2		7	2	7	2	7	7	2	7	2

+	0	1	2	3	4	5	6	7
0	2	4	0	6	1	7	3	5
1	3	2	6	5	0	4	7	1
2	0	1	2	3	4	5	6	7
3	1	5	4	2	7	6	0	3
4	6	0	3	7	2	1	5	4
5	7	3	5	1	6	2	4	0
6	4	7	1	0	5	3	2	6
7	5	6	7	4	3	0	1	2

$$LA - group \ (M, +)$$

From tables, it is can be seen that additive LA-group $(M, +)$ becomes LA-module. If n is the order of the finite LA-group M, then $n = |M| = 8$, and also, we can observe from the table that 2 is the zero element of M. Choose an element $1 \in M$ we see that $8(1) = 2_M$, also $8(2) = 2_M$, and likewise, it can be proved that for all $m \in M$, $nm = 0_M$. Where m can never be a part of basis. Thus, M is not a free LA-module. So is not a projective LA-module.

Proposition 3. *Figure 3R LA-modules and LA-homomorphisms have exact row, $\beta f = 0$. This gives the homomorphism $g: M \longrightarrow A$ so that $\alpha g = f$.*

Proof. If $\overline{B} = \text{Im}\alpha = \ker\beta$ and $\overline{\alpha}: A \longrightarrow \overline{B}$ be the induced homomorphism by α. From $\beta f = 0$, this follows that $\text{Im} f \subset \ker\beta = \text{Im}\alpha = \overline{B}$. So a homomorphism is induced by $f, \overline{f}: M \longrightarrow \overline{B}$ if $i: \overline{B} \longrightarrow B$ becomes inclusion map, so $\alpha = i\overline{\alpha}$ and also $f = i\overline{f}$. Figure 4 has an exact row. M is as projective LA-module, so there will be homomorphism $g: M \longrightarrow A$ so $\overline{\alpha}g = \overline{f}$. On the other hand, $\alpha g = i\overline{\alpha}g = i\overline{f} = f$. This implies $\alpha g = f$. □

Proposition 4. *M is projective LA-module iff M_j is projective LA-module for each $j \in J$.*

Proof. Assume for each $j \in J$, M_j is a projective LA-module. Figure 5 has an exact row. The homomorphism $f i_j: M_j \longrightarrow B$ for every $j \in J$ and M_j is the projective LA-module. Hence, there is a homomorphism $g_j: M_j \longrightarrow A$ such that $\alpha g_j = f i_j$. Now, define $g: M \longrightarrow A$ by $g(x) = \sum_j g_j \pi_j(x)$, for $x \in M$ (see Figure 6).

It is obvious the right side sum is finite. Therefore, g is the homomorphism. Let $x \in M$, $\alpha g(x) = \alpha(\sum_j g_j \pi_j(x)) = \sum_j \alpha g_j \pi_j(x) = \sum_j f i_j \pi_j(x) = f(\sum_j i_j \pi_j(x)) = f((\sum_j i_j \pi_j)(x)) = f(x)$. It shows $\alpha g = f$. It is clear M is projective LA-module. Conversely, let M a projective LA-module. We have Figure 7 having an exact row.

For any $j \in J$, a homomorphism $f \pi_j: M \longrightarrow B$ where M is a projective LA-module. There will be a homomorphism $g: M \longrightarrow A$ so that $\alpha g = f \pi_j$. Now, let take $g_j = g i_j$ which is a homomorphism from $M_j \longrightarrow A$, then $\alpha g_j = \alpha g i_j = f \pi_j i_j = f$. Hence, M_j is projective LA-module. □

Definition 8. A short exact sequence $O \longrightarrow P \longrightarrow {}_i Q \longrightarrow {}_j R \longrightarrow O$ of LA-modules and homomorphisms is called splits or split sequence of LA-modules. If any of the statement is true,

(i) A homomorphism $\gamma: Q \longrightarrow P$ exists if $\gamma i = \perp_P$

(ii) A homomorphism $\theta: R \longrightarrow Q$ exists if $j\theta = \perp_R$

(iii) Imi is a direct summand of Q

Lemma 1. *Homomorphic image of projective R LA-module is R LA-module.*

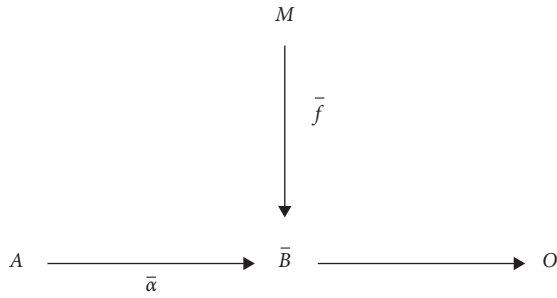*Proof.* Straightforward by Theorems 3 and 4. □

Figure 4: R-homomorphism.
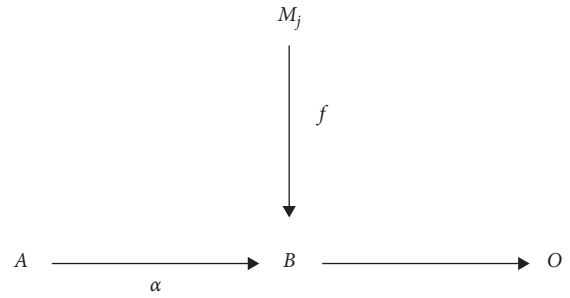

Figure 5: R-homomorphism.
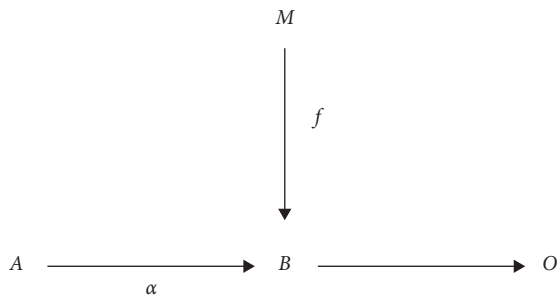

Figure 6: R-homomorphism.

Figure 7: R-homomorphism.
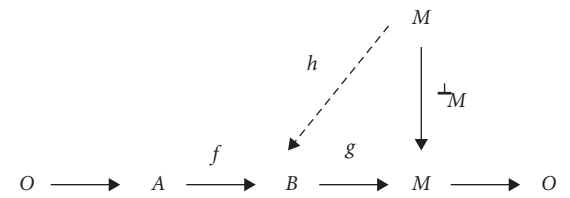

Figure 8: Exact sequences.
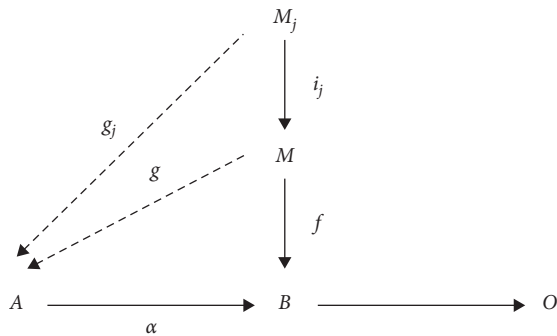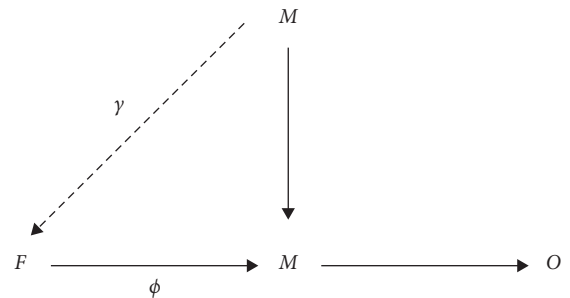

Figure 9: R-homomorphism.

(i) For any $m \in M$, $\phi_i(m) = 0$ for almost all $i \in I$.

(ii) For any $m \in M$, $m = \Sigma \phi_i(m)m_i$. Then, M is generated by $\{m_i : i \in I\}$.

Theorem 5. *M is projective LA-module iff every exact sequence $O \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow O$ splits.*

Proof. Consider an exact sequence (see Figure 8), by definition of projective LA-module, we have homomorphism $h: M \longrightarrow B$ implies $gh = \perp_M$ which results that the split sequence. Conversely, let the sequence $O \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow O$ splits as every LA-module is a homomorphic image of free R LA-module. $\alpha: F \longrightarrow M$ is epimorphism where F is free R LA-module. If A is the kernel α, we get exact sequence $O \longrightarrow A \longrightarrow^i F \longrightarrow^\alpha M \longrightarrow O$ which splits by the supposition. Hence, $F \cong M \oplus A$. F becomes free projective LA-module, and this implies M and A are projective. □

Proposition 5 (Projective basis theorem for LA-module). *An R LA-module M is projective iff there will be a subset $\{m_i : i \in I\}$ of M and $\{\phi_i : M \longrightarrow R, i \in I\}$ of R LA-homomorphisms, then*

Proof. First, M is projective LA-module, F is free R LA-module having the basis $\{x_i : i \in I\}$, and $\phi: F \longrightarrow M$ is epimorphism. As M a projective LA-module over R, there will be homomorphism $\gamma: M \longrightarrow F$ so that $\phi\gamma = \perp_M$. Now, for any $i \in I$, define $\phi_i: M \longrightarrow R$ by $\phi_i(m) = r_i$ where $\gamma(m) = \sum_{i \in I} r_i x_i, r_i \in R$ (see Figure 9). It is clear that ϕ_i are well-defined. As F is free LA-module on $\{x_i : i \in I\}$, ϕ_i are clearly LA-homomorphisms. Since $\gamma(m)$ is finite sum $\sum_{i \in I} r_i x_i$, $\phi_i(m) = 0$ for almost every i. For $i \in I$, define $m_i = \phi(x_i)$. If $m \in M$, then $m = \phi\gamma(m) = \phi(\gamma(m)) = \phi(\sum_{i \in I} r_i x_i) = \sum_{i \in I} r_i \phi(x_i) = \sum_{i \in I} r_i m_i = \sum_{i \in I} \phi_i(m)m_i$. Hence, (i) and (ii) are proved. Conversely, consider a subset $\{x_i : i \in I\}$ of M and a set $\{\phi_i : M \longrightarrow R, i \in I\}$ of R LA-homomorphisms such that the conditions (i) and (ii) holds. Now, a set $X = \{x_i : i \in I\}$ of symbols that are indexed by the same set I and let F be a free LA-module having the basis X. $\phi: X \longrightarrow M$ defined by $\phi(x_i) = m_i$ where $i \in I$ spreads to a homomorphism $\phi: F \longrightarrow M$. If $m \in M$, then $m = \sum_{i \in I} \phi_i(m)m_i = \sum_{i \in I} \phi_i(m)\phi(x_i) = \phi(\sum_{i \in I} \phi_i(m)x_i)$ which gives the result that ϕ is epimorphism. Define $\gamma: M \longrightarrow F$ by

$\gamma(m) = \sum_{i \in I} \phi_i(m) x_i \ldots\ldots (\ast)$. By condition (i) satisfied by ϕ_i, the right side of ($\ast$) is a finite sum. This implies γ is LA-homomorphism. For $m \in M$, $\phi\gamma(m) = \phi(\sum_{i \in I} \phi_i(m) x_i) = \sum_{i \in I} \phi_i(m)\phi(x_i) = \sum_{i \in I} \phi_i(m) m_i = m$ by condition (ii). Hence, $\phi\gamma = \perp_M$. Therefore, ϕ an LA-epimorphism splits which follows that M becomes direct summand of the free LA-module F. Therefore, M is projective LA-module. $\square$

3.2. Injective LA-Module. We define injective LA-modules and establish several essential conclusions in this part.

Definition 9. Let I denotes the left LA-module over R. Then, I is injective LA-module, if in the figure R LA-modules and LA-homomorphisms having exact row.

There will be R LA-homomorphism $g: B \longrightarrow I$ which results the completed diagram commutative which means $g\alpha = f$ (see Figure 10).

Alternatively, we can define injective LA-module as follows.

Definition 10. A left LA-module M over R is injective, if M is LA-submodule of some other left LA-module N. Then, there will be another LA-submodule K of N so the internal direct sum of M and K is N, that is, $M + K = N$ and $M \cap K = \{0\}$.

Example 5. In view of above definition, we have the following examples of injective LA-module.

(1) $\{0\}$ LA-module is trivially an injective LA-module
(2) Let K be an LA-field, then every K LA-vector space is an injective K LA-module

Proposition 6. *Let an injective R LA-module I. In the following figure having an exact row and $f\alpha = 0$, there will be a homomorphism $g: C \longrightarrow I$ which results the completed diagram commutative, that is, $g\beta = f$ (see Figure 11).*

Proof. Suppose $X = \ker\beta = \operatorname{Im}\alpha$. Then, β induces a monomorphism $\bar{\beta}: B/X \longrightarrow C$, given by $\bar{\beta}(b + X) = \beta(b)$, where $b \in B$. Let $\bar{\beta}(b_1 + X) = \bar{\beta}(b_2 + X) \Rightarrow \beta(b_1) = \beta(b_2) \Rightarrow \beta(b_1) - \beta(b_2) = 0 \Rightarrow \beta(b_1 - b_2) = 0 \Rightarrow b_1 - b_2 \in \ker\beta = X \Rightarrow b_1 - b_2 \in X \Rightarrow b_1 - b_2 + X = X \Rightarrow b_1 - b_2 + X = b_2 + X$, it follows $\bar{\beta}$ is monomorphism. Also, $f\alpha = 0 \Rightarrow f(\alpha(a)) = 0(a) = 0 \Rightarrow f(\alpha(a)) = 0 \Rightarrow \alpha(a) \in \ker f$ but $\alpha(a) \in \operatorname{Im}\alpha \Rightarrow \operatorname{Im}\alpha \subseteq \ker f \Rightarrow X = \operatorname{Im}\alpha \subseteq \ker f$; therefore, a homomorphism induced by f, $\bar{f}: B/X \longrightarrow I$ by $\bar{f}(b + X) = f(b)$, where $b \in B$. Let $\pi: B \longrightarrow B/X$ represents the natural projection. $\bar{f}\pi(b) = \bar{f}(\pi(b)) = \bar{f}(b + X) = f(b)$ for every $b \in B$; hence, $\bar{f}\pi = f$ and $\bar{\beta}\pi(b) = \bar{\beta}(\pi(b)) = \bar{\beta}(b + X) = \beta(b)$ for all $b \in B$; therefore, $\bar{\beta}\pi = \beta$. As I is an injective LA-module, there will be a homomorphism $g: C \longrightarrow I$ such that $\bar{f} = g\bar{\beta}$; then, $g\beta = g\bar{\beta}\pi = \bar{f}\pi = f$ is proved. $\square$
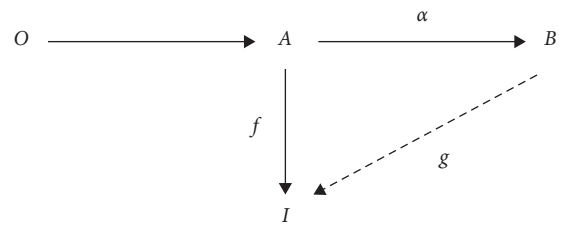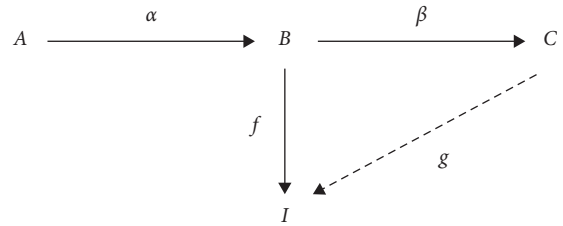

FIGURE 10: Injective LA-module.
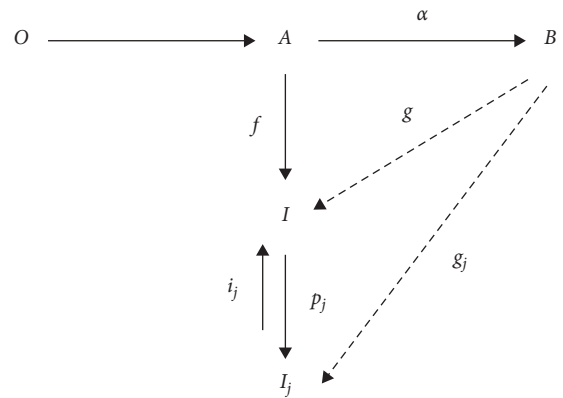

FIGURE 11: R-homomorphism.

FIGURE 12: R-homomorphism.

Proposition 7. *If $I = \prod_{j \in J} j$ (= direct product of I_j) and $\{I_j\}_{j \in J}$ is a family of R LA-modules, then I an injective LA-module iff I_j is injective.*

Proof. Since $I = \prod_{j \in J} I_j$, there will be a homomorphism $i_j: I_j \longrightarrow I$ and $p_j: I \longrightarrow I_j$ which implies $p_j i_j = \perp_{I_j}$ and $p_k i_j = 0$, the zero map if $j \neq k$. Consider a monomorphism of R LA-modules $\alpha: A \longrightarrow B$ and assume that each I_j is injective LA-module, considering diagram (see Figure 12)

Assume $f: A \longrightarrow I$ is homomorphism. Then, $p_j f: A \longrightarrow I_j$ is a homomorphism. As I_j is injective, there will be a homomorphism $g_j: B \longrightarrow I_j$ which implies $g_j \alpha = p_j f$. Now, define $g: B \longrightarrow I$ by $g(b) = (g_j(b)), b \in B$. It follows g is a homomorphism, and for $a \in A$, $g(\alpha(a)) = (g_j \alpha(a)) = p_j f(a) = f(a)$, which gives $g\alpha = f$. This results I is an injective LA-module. Now, conversely, let I be injective LA-module. Let $f_j: A \longrightarrow I_j$ be a homomorphism for any $j \in J$.

As I is injective LA-module, there will be a homomorphism $g: B \longrightarrow I$ which implies $g\alpha = i_j f_j$ (see
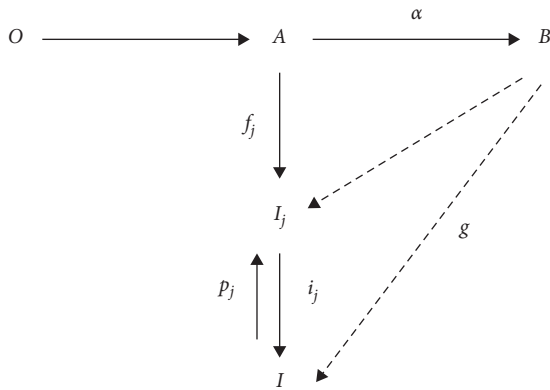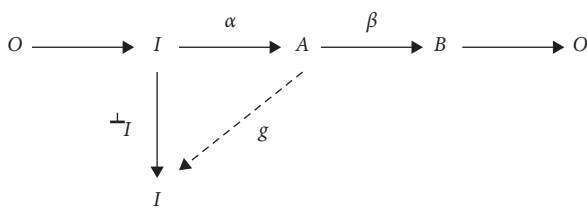
FIGURE 13: R-homomorphism.

FIGURE 14: Exact sequence.

Figure 13). It follows $p_j g: B \longrightarrow I_j$ will be a homomorphism which means $p_j g \alpha = p_j i_j f_j = f_j$. Hence, I_j is injective LA-module. □

Proposition 8. *Every exact sequence* $O \longrightarrow I \longrightarrow {}_\alpha A \longrightarrow {}_\beta B \longrightarrow O$ *splits if I is injective LA-module.*

Proof. In Figure 14, as I is injective LA-module, therefore, there will be a homomorphism $g: A \longrightarrow I$ which implies $g\alpha = \perp_I$. It follows that exact sequence $O \longrightarrow I \longrightarrow {}_\alpha A \longrightarrow {}_\beta B \longrightarrow O$ splits. □

4. Conclusion

Mathematics is becoming increasingly nonassociative and noncommutative. It is widely predicted that nonassociativity and noncommutativity will dominate mathematics and applied sciences in the coming years. In this paper, the study of LA-modules can be classified as a theoretical study in the development of nonassociative and noncommutative algebraic theory. The notions of split sequence, free LA-module, projective LA-modules, injective LA-modules, and their related features were discussed in relation to LA-modules. Further advancements in the study of LA-modules can be made by defining functors, pull back and push outs, and so on. In addition, LA-modules and its substructures can be defined in the study of neutrosophic sets and hyper structures. Also, neutrosophic graphs of these algebraic structures can be constructed. Moreover, graphs of LA-modules over nonassociative rings and nonassociative hyper structures can be defined. Furthermore, nonassociative rings and nonassociative hyper structures can be used in various decision-making procedures, and fuzzy theory and its applications can be extended to the medical sciences.

Data Availability

All data are available in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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