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Research Article

Existence of Solutions for a p-Kirchhoff-Type Problem with Critical Exponent

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In this study, we prove the existence of a positive solution for a p-Kirchhoff-type problem with Sobolev exponent.

1. Introduction and Main Results

In this study, we are concerned with the following p-Kirchhoff-type problem:

$$\begin{cases} -\left(a\|u\|^{(\theta-1)p}+b\right)\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)=|u|^{p^*-2}u+\lambda|u|^{p-2}u, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \quad (P_\lambda) \tag{1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain, $1<p<N$, a, $b>0$, $\theta>1$, $\|.\|$ is the usual norm in $W_0^{1,p}(\Omega)$ given by $\|u\|^p=\int_\Omega|\nabla u|^p dx$, λ is a parameter, and $p^*=pN/(N-p)$ is the critical Sobolev exponent corresponding to the noncompact embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$.

Since equation (P_λ) contains an integral over Ω, it is no longer a pointwise identity; therefore, it is often called a nonlocal problem. It is called also nondegenerate if $b>0$ and $a\geq 0$, while it is named degenerate if $b=0$ and $a>0$.

Such nonlocal elliptic problem such as (P_λ) is related to the original Kirchhoff's equation in [1] which was first introduced by Kirchhoff as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the strings produced by transverse vibrations.

Much interest has grown on problems involving critical exponents, starting from the celebrated paper by Brézis and Nirenberg [2]. They considered problem (P_λ) with $a=0$, $b=1$, and $p=2$. From their results, it came out that the space dimension N was going to play a crucial role. They established existence results in dimension $N=3$ when Ω is a ball, namely, they ensure the existence of a positive constant λ_0 such that problem $(\widetilde{P})_{0,1}$ admits a positive solution for $\lambda\in(\lambda_0/4,\lambda_1)$, where λ_1 is the first eigenvalue of the operator $-\Delta$. In higher dimensions, $N\geq 4$, they proved the existence of a positive solution for $\lambda\in(0,\lambda_1)$ and no positive solution for $\lambda>\lambda_1$ or $\lambda\leq 0$ and Ω is a starshaped domain.

Moreover, by using the concentration compactness principle [3], the results of [2] were extended to the quasi-linear cases by Guedda and Veron [4]. Precisely, they proved that if $N \geq p^2$, then the quasi-linear Brezis–Nirenberg problem,

$$\begin{cases} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p^*-2}u + \lambda |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (2)$$

has a positive solution if $\lambda \in (0, \lambda_1)$, where λ_1 is the first eigenvalue of

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda |u|^{p-2}u, \quad \text{in } \Omega, \quad (3)$$

with Dirichlet boundary condition and no positive solution for $\lambda \geq \lambda_1$ or and Ω star shaped.

In the last few years, great attention has been paid to the study of Kirchhoff problems involving critical nonlinearities. This problems create many difficulties in applying variational methods, we refer the readers to [5–13] and the references therein. More precisely, Naimen in [10] generalized the results of [2] for 3- and 4-dimensional Kirchhoff-type equations. For larger dimensional case, Figueiredo [7] considers the case $N \geq 3$ if $\lambda > 0$ is sufficiently large.

The main results in the present paper can be considered as the extension of the work of [4] for a p-Kirchhoff problem with large range of N. The competing effect of the nonlocal term with the critical nonlinearity and the lack of compactness of the embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$, prevents us from using the variational methods in a standard way. So, we need more delicate estimates.

To the best of our knowledge, many of the results are new for $p > 1$ and even in the case $\theta = 2$. Our results and setting are more general and delicate, it is difficult to obtain the solution in the degenerate case when $\theta < (p^*/p)$.

Our technique is based on variational methods and concentration compactness argument [3], and we need to estimate the energy levels.

In this paper, we define the best Sobolev constant for the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ as

$$S := \inf_{u \in W_0^{1,p}(\Omega)\{0\}} \frac{\|u\|^p}{\left(\int_\Omega |u|^{p^*} \mathrm{d}x\right)^{p/p^*}}. \quad (4)$$

Then, we obtain the following existence result.

Theorem 1. *Let $b > 0$, $N \geq p^2$, and $\lambda \in (0, b\lambda_1)$. If $\theta = N/(N-p)$ and $0 < a < S^{-\theta}$ or $\theta < N/(N-p)$ and $a > 0$, then (P_λ) has a positive solution.*

Remark 1. If u is a solution of (P_λ), we obtain

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = g(x,u), \quad (5)$$

where $g(x,u) = M(\|u\|) f(x,u)$ with $M(\|u\|) = (a\|u\|^{(\theta-1)p} + b)^{-1}$ and $f(x,u) = |u|^{p^*-2}u + \lambda |u|^{p-2}u$. By the extended Pohozaev identity in [4], we can get a nonexistence solutions for $\lambda \leq 0$ and Ω is starshaped.

2. Preliminary Results

In this study, we use the following notation: $\longrightarrow$ (resp $\rightharpoonup$) denotes strong (resp. weak) convergence, $o_n(1)$ denotes $o_n(1) \longrightarrow 0$ as $n \longrightarrow +\infty$, λ_1 is the first eigenvalue of

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad (6)$$

with Dirichlet boundary condition, and $B_R(x_0)$ is the ball centered at x_0 and of radius R, $u^- = \max\{-u, 0\}$.

Recall that the infimum S is attained in $\mathbb{R}^N$ by the functions of the form

$$v_\varepsilon(x) := \left(N\varepsilon\left(\frac{N-p}{p-1}\right)^{p-1}\right)^{(N-p)/p^2} \left(\varepsilon + |x|^{p/(p-1)}\right)^{(p-N)/p}, \quad \varepsilon > 0. \quad (7)$$

Moreover, v_ε satisfies

$$\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^p \mathrm{d}x = \int_{\mathbb{R}^N} |v_\varepsilon|^{p^*} \mathrm{d}x = S^{p^*/(p^*-p)}. \quad (8)$$

Let R be a positive constant and set $\varphi \in C_0^\infty(\Omega)$ such that $0 \leq \varphi(x) \leq 1$ for $|x| \leq R$ and $\varphi(x) \equiv 1$ for $|x| \leq (R/2)$ and $B_R(0) \subset \Omega$. Set $\tilde{v}_\varepsilon(x) = \varphi(x) v_\varepsilon(x)$, by taking $z_\varepsilon = \tilde{v}_\varepsilon (\int_\Omega |\tilde{v}_\varepsilon|^{p^*} \mathrm{d}x)^{-1/p^*}$ so that $\int_\Omega |z_\varepsilon|^{p^*} \mathrm{d}x = 1$.

We have the well-known estimates as $\varepsilon \longrightarrow 0$:

$$\begin{cases} \|z_\varepsilon\|^p = S + O\left(\varepsilon^{(N-p)/p}\right) \\ \int_\Omega |z_\varepsilon|^p \mathrm{d}x \geq \begin{cases} C\varepsilon^{p-1}, & \text{if } N > p^2, \\ C\varepsilon^{N-p}|\ln \varepsilon|, & \text{if } N = p^2, \\ C\varepsilon^{(N-p)/p}, & \text{if } N < p^2, \end{cases} \end{cases} \quad (9)$$

(see [14]).

The energy functional $I_\lambda : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$, corresponding to the problem (P_λ), is given by

$$I_\lambda(u) = \frac{a}{\theta p}\|u\|^{\theta p} + \frac{b}{p}\|u\|^p - \frac{1}{p^*}\int_\Omega |u|^{p^*} \mathrm{d}x - \frac{\lambda}{p}\int_\Omega |u|^p \mathrm{d}x, \quad \forall u \in W_0^{1,p}(\Omega). \quad (10)$$

Notice that I_λ is well defined in $W_0^{1,p}(\Omega)$ and belongs to $C^1(W_0^{1,p}(\Omega),\mathbb{R})$. We say that $u\in W_0^{1,p}(\Omega)\backslash\{0\}$ is a weak solution of (Y_λ) if, for any $v\in W_0^{1,p}(\Omega)$, there holds

$$\begin{aligned}\langle I_\lambda'(u),v\rangle &= \left(a\|u\|^{(\theta-1)p}+b\right)\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\mathrm{d}x-\int_\Omega |u|^{p^*-2}uv\mathrm{d}x-\lambda\int_\Omega |u|^{p-2}uv\mathrm{d}x\\ &=0.\end{aligned}\tag{11}$$

Hence, a critical point of functional I_λ is a weak solution of problem (P_λ).

Definition 1. Let $c\in\mathbb{R}$; a sequence $(u_n)\subset W_0^{1,p}(\Omega)$ is called a $(PS)_c$ sequence (Palais–Smale sequence at level c) if

$$I_\lambda(u_n)\longrightarrow c \text{ and } I_\lambda'(u_n)\longrightarrow 0 \text{ as } n\longrightarrow +\infty. \tag{12}$$

Let $c\in\mathbb{R}$. We say that I_λ satisfies the Palais–Smale condition at level c if any $(PS)_c$ sequence contains a convergent subsequence in $W_0^{1,p}(\Omega)$.

Lemma 1. *Let $a,b>0$, $\theta>1$, $\sigma\geq 0$, and $\tilde{y}=((a/\sigma)S^\theta)^{1/(\sigma-1)}$, for $\sigma>1$. For $y\geq 0$, we consider the function $\Psi\colon \mathbb{R}^+\longrightarrow\mathbb{R}^*$, given by*

$$\Psi(y)=S^{-1}y^\sigma-aS^{\theta-1}y-b. \tag{13}$$

Then,

(1) If $\sigma=1$, $0\leq a<S^{-\theta}$, and $b>0$, then the equation $\Psi(y)=0$ has a unique positive solution,

$$y_1=\frac{b}{S^{\theta-1}\left(S^{-\theta}-a\right)}, \tag{14}$$

and $\Psi(y)\geq 0$, for all $y\geq y_1$.

(2) If $\sigma>1$, then the equation $\Psi(y)=0$ has a unique positive solution $y_2>\tilde{y}$ and $\Psi(y)\geq 0$, for all $y\geq y_2$.

Proof

(1) For $\sigma=1$, $0<a<S^{-\theta}$, and $b>0$, we have

$$\Psi(y)=S^{\theta-1}\left(S^{-\theta}-a\right)y-b, \tag{15}$$

that is, the equation $\Psi(y)=0$ has a unique positive solution,

$$y_1=\frac{b}{\left(S^{-\theta}-a\right)S^{\theta-1}}, \tag{16}$$

and $\Psi(y)\geq 0$, for all $y\geq y_1$.

(2) For $\sigma>1$, we have $\Psi'(y)=\sigma S^{-1}y^{\sigma-1}-aS^{\theta-1}$ and

$$\Psi''(y)=\sigma(\sigma-1)S^{-1}y^{\sigma-2}>0,\quad \forall y>0. \tag{17}$$

Then, $\Psi'(\tilde{y})=0$, $\Psi'(y)<0$, for $y<\tilde{y}$, and $\Psi'(y)>0$, for $y>\tilde{y}$. Hence, Ψ is concave function and

$$\min_{y\geq 0}\Psi(y)=\Psi(\tilde{y})=-(\sigma-1)S^{-1}\left(\frac{a}{\sigma}S^\theta\right)^{\sigma/(\sigma-1)}<0. \tag{18}$$

Moreover, we have $\Psi(\tilde{y})<0$ and $\lim_{y\longrightarrow+\infty}\Psi(y)=+\infty$; thus, from (18) and the concavity of Ψ, we can conclude that the equation $\Psi(y)=0$ has a unique positive solution $y_2>\tilde{y}$ and $\Psi(y)\geq 0$, for all $y\geq y_2$.

Now, we will verify that the functional I_λ exhibits the Mountain Pass geometry. □

Lemma 2. *Assume $N>p>1$, $b>0$, and $\lambda\in(0,b\lambda_1)$. Suppose that $\theta=N/(N-p)$ and $0<a<S^{-\theta}$ or $1<\theta<N/(N-p)$ and $a>0$. Then, there exist positive numbers δ_1 and ρ_1 such that, for all $\lambda\in(0,b\lambda_1)$, we have*

(1) $I_\lambda(u)\geq\delta_1>0$, with $\|u\|=\rho_1$

(2) There exists $e\in W_0^{1,p}(\Omega)\backslash\{0\}$ such that $\|e\|>\rho_1$ and $I_\lambda(e)<0$

Proof

(1) Let $u\in W_0^{1,p}(\Omega)\backslash\{0\}$; by Sobolev and Young inequalities, it holds that

$$\begin{aligned}I_\lambda(u)&=\frac{a}{\theta p}\|u\|^{\theta p}+\frac{b}{p}\|u\|^p-\frac{1}{p^*}\int_\Omega |u|^{p^*}\mathrm{d}x-\frac{\lambda}{p}\int_\Omega |u|^p\mathrm{d}x\\ &\geq -\frac{1}{p^*}S^{-p^*/p}\|u\|^{p^*}+\frac{a}{\theta p}\|u\|^{\theta p}+\frac{b}{p}\|u\|^p-\frac{\lambda}{p\lambda_1}\|u\|^p\\ &=-\frac{1}{p^*}S^{-p^*/p}\|u\|^{p^*}+\frac{a}{\theta p}\|u\|^{\theta p}+\frac{b\lambda_1-\lambda}{p\lambda_1}\|u\|^p.\end{aligned}\tag{19}$$

Let $\rho=\|u\|$; since $a>0$, $b>0$, and $\lambda<b\lambda_1$ and from (19), one has

$$I_\lambda(u)\geq -\frac{1}{p^*}S^{-p^*/p}\rho^{p^*}+\frac{a}{\theta p}\rho^{\theta p}+\frac{b\lambda_1-\lambda}{p\lambda_1}\rho^p. \tag{20}$$

Let us define

$$\begin{aligned}h(\rho)&=-\frac{1}{p^*}S^{-p^*/p}\rho^{p^*}+\frac{a}{\theta p}\rho^{\theta p}+\frac{b\lambda_1-\lambda}{p\lambda_1}\rho^p,\\ g(\rho)&=-S^{-p^*/p}\rho^{p^*-p}+a\rho^{(\theta-1)p}+\frac{b\lambda_1-\lambda}{\lambda_1}.\end{aligned}\tag{21}$$

Then,

$$h'(\rho) = -S^{-p^*/p}\rho^{p^*-1} + a\rho^{\theta p-1} + \frac{b\lambda_1-\lambda}{\lambda_1}\rho^{p-1}$$
$$= \left(-S^{-p^*/p}\rho^{p^*-p} + a\rho^{(\theta-1)p} + \frac{b\lambda_1-\lambda}{\lambda_1}\right)\rho^{p-1}. \quad (22)$$

(i) Let $1<\theta<N/(N-p)$ and $a>0$. We have g is strictly increasing on the interval:

$$\left[0, \left(\frac{a(\theta-1)p}{(p^*-p)}S^{p^*/p}\right)^{1/(p^*-\theta p)}\right], \quad (23)$$

and it is strictly decreasing on the interval

$$\left[\left(\frac{a(\theta-1)p}{p^*-p}S^{p^*/p}\right)^{1/(p^*-\theta p)}, +\infty\right[, \quad (24)$$

with $g(0) = (b\lambda_1-\lambda)/\lambda_1$, $\lim_{x\longrightarrow+\infty} g(x) = -\infty$, and

$$g\left(\left(\frac{a(\theta-1)p}{p^*-p}S^{p^*/p}\right)^{1/(p^*-\theta p)}\right) = \frac{p^*-\theta p}{(\theta-1)p}S^{-p^*/p}$$
$$\left(\frac{a(\theta-1)p}{p^*-p}S^{p^*/p}\right)^{(p^*-p)/(p^*-\theta p)} + \frac{b\lambda_1-\lambda}{\lambda_1} > 0. \quad (25)$$

Then, direct calculation shows that

$$\max_{\rho\geq 0} h(\rho) = h(\rho_1) > 0,$$
$$h(\rho)\geq 0 \text{ for all } \rho\leq\rho_1 \text{ with } \rho_1 > \left(\frac{a(\theta-1)p}{p^*-p}S^{p^*/p}\right)^{1/(p^*-\theta p)}. \quad (26)$$

So, for all $\lambda\in(0, b\lambda_1)$, we have

$$I_\lambda(u)\geq h(\rho_1) = \delta_1 > 0, \quad \text{with } \|u\| = \rho_1. \quad (27)$$

(ii) When $N = \theta p/(\theta-1)$, one obtains $p^* = \theta p$. Let $\rho = \|u\|$; from (20), one has

If $0<a<S^{-\theta}$, similar to (i), there exist $\rho_1, \delta_1>0$ such that

$$\rho_1 = \left[\frac{b\lambda_1-\lambda}{\lambda_1(S^{-\theta}-a)}\right]^{1/(\theta-1)p},$$
$$\delta_1 = h(\rho_1) = \frac{\theta-1}{\theta p}\left(S^{-\theta}-a\right)^{-p/(\theta-1)p}\left(\frac{b\lambda_1-\lambda}{\lambda_1}\right)^{\theta p/(\theta-1)p}. \quad (29)$$

(2) For $u\in W_0^{1,p}(\Omega)\backslash\{0\}$, $t>0$, we have

$$I_\lambda(tu)\leq -\frac{t^{p^*}}{p^*}\int_\Omega |u|^{p^*}dx + \frac{at^{\theta p}}{\theta p}\|u\|^{\theta p} + \frac{bt^p}{p}\|u\|^p. \quad (30)$$

(i) If $\theta<N/(N-p)$ and $t\longrightarrow+\infty$, then $I_\lambda(tu)\longrightarrow-\infty$. So, we can easily find $e\in W_0^{1,p}(\Omega)\backslash\{0\}$ with $\|e\|>\rho_1$, such that $I_\lambda(e)<0$.

(ii) If $\theta = N/(N-p)$ and $0<a<S^{-\theta}$, using (9) and taking $\varepsilon_1>0$ small enough, then

$$I_\lambda(tz_\varepsilon)\leq \frac{at^{\theta p}}{\theta p}\|z_\varepsilon\|^{\theta p} + \frac{bt^p}{p}\|z_\varepsilon\|^p - \frac{t^{p^*}}{p^*}\int_\Omega |z_\varepsilon|^{p^*}dx$$
$$\leq \frac{1}{\theta p}\left(a\|z_\varepsilon\|^{\theta p} - \int_\Omega |z_\varepsilon|^{p^*}dx\right)t^{\theta p} + \frac{bt^p}{p}\|z_\varepsilon\|^p$$
$$\leq \frac{1}{\theta p}\left[a\left(S^{p^*/(p^*-p)} + O\left(\varepsilon^{(N-p)/(p-1)}\right)\right)^\theta - \left(S^{p^*/(p^*-p)} + O\left(\varepsilon^{N/(p-1)}\right)\right)\right]t^{\theta p}$$
$$+ \frac{bt^p}{p}\left(S^{p^*/(p^*-p)} + O\left(\varepsilon^{(N-p)/(p-1)}\right)\right)$$
$$\leq \frac{1}{\theta p}\left(aS^{\theta p^*/(p^*-p)} - S^{p^*/(p^*-p)}\right)t^{\theta p} + \frac{b}{p}S^{p^*/(p^*-p)}t^p + O\left(\varepsilon^{(N-p)/(p-1)}\right)$$
$$\leq \frac{1}{\theta p}\left(a-S^{-\theta}\right)S^{\theta p^*/(p^*-p)}t^{\theta p} + \frac{b}{p}S^{p^*/(p^*-p)}t^p + O\left(\varepsilon^{(N-p)/(p-1)}\right), \quad (31)$$

for all $\varepsilon \in (0, \varepsilon_1)$. Then, it follows from the above inequality, $I_\lambda(tz_\varepsilon) \longrightarrow -\infty$ as $t \longrightarrow +\infty$. Thus, choosing $t_0 > 0$ sufficiently large and putting $e: = t_0 z_\varepsilon$, we have a function $e \in W_0^{1,p}(\Omega)\backslash\{0\}$ satisfying $\|e\| > \rho_1$, such that $I_\lambda(e) < 0$. □

Proof. The proof is complete. □

Lemma 3. *Assume $N > p > 1$, $a, b > 0$, $\lambda \in (0, b\lambda_1)$, and $\theta \leq N/(N-p)$. Let $c \in \mathbb{R}^+$ and $(u_n) \subset W_0^{1,p}(\Omega)$ be a $(PS)_c$ sequence for I_λ; then, there exists a subsequence of (u_n) which we still denote by (u_n) and $u \in W_0^{1,p}(\Omega)$ such that*

$$u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega), \tag{32}$$

with $I_\lambda'(u) = 0$.

Proof. We have

$$\begin{aligned} I_\lambda(u_n) &\longrightarrow c, \\ I_\lambda'(u_n) &\longrightarrow 0, \end{aligned} \tag{33}$$

that is,

$$\begin{aligned} c + o_n(1) &= I_\lambda(u_n), \\ o_n(1)\|v\| &= \langle I_\lambda'(u_n), v\rangle, \end{aligned} \tag{34}$$

for any $v \in W_0^{1,p}(\Omega)$.

Then, as $n \longrightarrow \infty$, it follows that

$$\begin{aligned} c + o_n(1) - \frac{1}{p^*} o_n(1)\|u_n\| &= I_\lambda(u_n) - \frac{1}{p^*}\langle I_\lambda'(u_n), u_n\rangle \\ &= a\frac{p^* - \theta p}{\theta p p^*}\|u_n\|^{\theta p} + b\frac{p^* - p}{pp^*}\|u_n\|^p - \lambda\frac{p^* - p}{pp^*}\int_\Omega |u_n|^p \mathrm{d}x, \\ &\geq a\frac{p^* - \theta p}{\theta p p^*}\|u_n\|^{\theta p} + b\frac{p^* - p}{pp^*}\|u_n\|^p - \frac{\lambda}{\lambda_1}\frac{p^* - p}{pp^*}\|u_n\|^p \\ &\geq a\frac{p^* - \theta p}{\theta p p^*}\|u_n\|^{\theta p} + \left(b - \frac{\lambda}{\lambda_1}\right)\frac{p^* - p}{pp^*}\|u_n\|^p. \end{aligned} \tag{35}$$

As $\lambda < b\lambda_1$, we obtain that (u_n) is bounded in $W_0^{1,p}(\Omega)$. Up to a subsequence if necessary, there exists a function $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega), \\ &u_n \rightharpoonup u \text{ in } L^{p^*}(\Omega), \\ &u_n \longrightarrow u \text{ in } L^r(\Omega), \text{for all } r < p^* \\ &u_n \longrightarrow u \text{ a.e on } \Omega. \end{aligned} \tag{36}$$

Then,

$$\langle I_\lambda'(u_n), v\rangle = 0, \text{for all } v \in C_0^\infty(\Omega), \tag{37}$$

and thus, $I_\lambda'(u) = 0$. This completes the proof of Lemma 4.

Now, we prove an important lemma which ensures the local compactness of the Palais–Smale sequence for I_λ.

For $i = \overline{1,2}$, y_i is defined in Lemma 2, and we define

$$C_i := a\left(\frac{1}{\theta p} - \frac{1}{p^*}\right)\left(Sy_i^{1/(\theta-1)}\right)^\theta + b\left(\frac{1}{p} - \frac{1}{p^*}\right)Sy_i^{1/(\theta-1)}, \quad a > 0, b > 0, \tag{38}$$

$$C^* := \begin{cases} C_1, & \text{if } \theta = \dfrac{N}{N-p} \text{ and } 0 < a < S^{-\theta}, \\ C_2, & \text{if } \theta < \dfrac{N}{N-p} \text{ and } a > 0. \end{cases} \tag{39}$$

□

Lemma 4. *Let $b > 0$ and $\lambda < b\lambda_1$, and suppose $\theta = N/(N-p)$ and $0 \leq a < S^{-\theta}$ or $\theta < N/(N-p)$ and $a > 0$. If $\{u_n\} \subset W_0^{1,p}(\Omega)$ is a $(PS)_c$ sequence for I_λ with $c < C^*$, then $\{u_n\}$ contains a subsequence converging strongly in $W_0^{1,p}(\Omega)$.*

Proof. By the proof of Lemma 3, we have $\{u_n\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$. Hence, by the concentration compactness principle due to Lions [3], there exists a subsequence, still denoted by $\{u_n\}$, such that

$$\begin{aligned}&\left|\nabla u_n\right|^p \rightharpoonup \mathrm{d}\eta \geq |\nabla u|^p + \sum_{j\in J}\eta_j\tilde{\delta}_j,\\ &\left|u_n\right|^{p^*} \rightharpoonup \mathrm{d}\tilde{\gamma} = |u|^{p^*} + \sum_{j\in J}\tilde{\gamma}_j\tilde{\delta}_j,\end{aligned} \tag{40}$$

where J is an at most countable index set, $\tilde{\delta}_j$ is the Dirac mass at x_j and $x_j \in \overline{\Omega}$, and $\{\eta_j\}_{j\in J\cup\{0\}}$ and $\{\tilde{\gamma}_j\}_{j\in J\cup\{0\}}$ are sets of nonnegative real numbers. Moreover,

$$\eta_j \geq S\tilde{\gamma}_j^{p/p^*} \text{ for all } j \in J. \tag{41}$$

For $\varepsilon > 0$, let $\phi_{\varepsilon,j}(x)$ be a smooth cut-off function centered at x_j such that $0 \leq \phi_{\varepsilon,j}(x) \leq 1$, and

$$\begin{aligned}&\phi_{\varepsilon,j}(x) = \begin{cases}1, & \text{in } B(x_j, \varepsilon),\\ 0, & \text{in } \Omega\backslash B(x_j, 2\varepsilon),\end{cases}\\ &\left|\nabla\phi_{\varepsilon,j}(x)\right| \leq \frac{2}{\varepsilon}.\end{aligned} \tag{42}$$

Since $\{\phi_{\varepsilon,j}u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ and $I_\lambda'(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$, it holds by Hölder's inequality:

$$\begin{aligned}0 &= \lim_{\varepsilon\longrightarrow 0}\lim_{n\longrightarrow\infty}\langle I_\lambda'(u_n), \phi_{\varepsilon,j}u_n\rangle\\ &= \lim_{\varepsilon\longrightarrow 0}\lim_{n\longrightarrow\infty}\left[\left(a\|u_n\|^{(\theta-1)p} + b\right)\int_\Omega\left|\nabla u_n\right|^{p-2}\nabla u_n\nabla\left(\phi_{\varepsilon,j}u_n\right)\mathrm{d}x\right]\\ &\quad -\int_\Omega\left|u_n\right|^{p^*}u_n\left(\phi_{\varepsilon,j}u_n\right)\mathrm{d}x - \lambda\int_\Omega\left|u_n\right|^{p-1}\left(\phi_{\varepsilon,j}u_n\right)\mathrm{d}x\Big]\\ &\geq \left(b + a\eta_j^{\theta-1}\right)\eta_j - \tilde{\gamma}_j.\end{aligned} \tag{43}$$

Then, $\tilde{\gamma}_j \geq b\eta_j + a\eta_j^\theta$. Therefore, by (41), we deduce that

$$\tilde{\gamma}_j = 0 \text{ or } S^{-1}\left(\tilde{\gamma}_j\right)^{(p^*-p)/p^*} - aS^{\theta-1}\left(\tilde{\gamma}_j\right)^{(p/p^*)(\theta-1)} - b \geq 0. \tag{44}$$

Assume by contradiction that there exists $j_0 \in J$ such that $\tilde{\gamma}_{j_0} \neq 0$. Set $y = (\tilde{\gamma}_{j_0})^{(p/p^*)(\theta-1)}$ and $\sigma = (p^* - p)/(\theta - 1)p$; then, by (44), we obtain

$$S^{-1}y^\sigma - aS^{\theta-1}y - b \geq 0. \tag{45}$$

It is clear that $\sigma \geq 1$, thanks to $\theta \leq N/(N-p)$. So, from (45) and the definition of Ψ in Lemma 1, we obtain

$$\Psi(y) = S^{-1}y^\sigma - aS^{\theta-1}y - b \geq 0. \tag{46}$$

We will discuss it in two cases:

Case 1: $\theta = N/(N-p)$ and $0 < a < S^{-\theta}$.

According to Lemma 1, we have $\Psi(y_1) = 0$ and $\Psi(y) \geq 0$ if $y \geq y_1$ with

$$y_1 = \frac{b}{\left(S^{-\theta} - a\right)S^{\theta-1}}, \tag{47}$$

which implies that

$$S\left(\tilde{\gamma}_{j_0}\right)^{(p/p^*)} \geq Sy_1^{1/(\theta-1)} =: K_1. \tag{48}$$

Case 2: $\theta < N/(N-p)$ and $a > 0$. In this case, from Lemma 2, we get $\Psi(y_2) = 0$ and $\Psi(y) \geq 0$ if $y \geq y_2$ with

$$y_2 > \left(\frac{a(\theta-1)p}{p^* - p}S^\theta\right)^{(\theta-1)p/(p^*-\theta p)}, \tag{49}$$

which implies that

$$S\left(\tilde{\gamma}_{j_0}\right)^{p/p^*} \geq Sy_2^{1/(\theta-1)} =: K_2. \tag{50}$$

Hence, using (41), we deduce

$$\eta_{j_0} \geq S\tilde{\gamma}_{j_0}^{p/p^*} \geq \begin{cases}K_1, & \text{if } \theta = \dfrac{N}{N-p} \text{ and } 0 < a < S^{-\theta},\\ K_2, & \text{if } \theta < \dfrac{N}{N-p} \text{ and } a > 0.\end{cases} \tag{51}$$

By Young inequality, we have

$$\begin{aligned}c &= \lim_{n\longrightarrow\infty} I_\lambda(u_n) - \frac{1}{\theta p}\langle I_\lambda'(u_n), u_n\rangle\\ &= \lim_{n\longrightarrow\infty}\frac{\theta-1}{\theta p}b\|u_n\|^p + \frac{p^*-\theta p}{\theta pp^*}\int_\Omega\left|u_n\right|^{p^*} - \lambda\frac{\theta-1}{\theta p}\int_\Omega\left|u_n\right|^p\mathrm{d}x\\ &\geq \frac{\theta-1}{\theta p}b\left(\|u\|^p + \eta_{j_0}\right) + \frac{p^*-\theta p}{\theta pp^*}\left(\int_\Omega|u|^{p^*} + \gamma_{j_0}\right) - \lambda\frac{\theta-1}{\theta p}\int_\Omega|u|^p\mathrm{d}x\\ &\geq \frac{\theta-1}{\theta p}\left(b - \frac{\lambda}{\lambda_1}\right)\|u\|^p + \frac{p^*-\theta p}{\theta pp^*}\int_\Omega|u|^{p^*} + \frac{\theta-1}{\theta p}b\eta_{j_0} + \frac{p^*-\theta p}{\theta pp^*}\tilde{\gamma}_{j_0}.\end{aligned} \tag{52}$$

We observe that $(\theta-1)/\theta p(b - (\lambda/\lambda_1)) > 0$, $p^* - \theta p \geq 0$, and thus, for $i \in \{1, 2\}$, we obtain

$$
\begin{aligned}
c &\geq \frac{\theta-1}{\theta p} b\eta_{j_0} + \frac{p^*-\theta p}{\theta p p^*}\widetilde{\gamma}_{j_0} \\
&\geq \left(\frac{1}{p}-\frac{1}{\theta p}\right) bK_i + \frac{p^*-\theta p}{\theta p p^*} K_i^{(p^*/p)} S^{-(p^*/p)} \\
&\geq \left(\frac{1}{p}-\frac{1}{\theta p}\right) bK_i + \frac{p^*-\theta p}{\theta p p^*} K_i^{(p^*/p)} S^{-(p^*/p)} + \\
&\quad + \frac{p^*-\theta p}{\theta p p^*} aK_i^{\theta} - \frac{p^*-\theta p}{\theta p p^*} aK_i^{\theta} + \frac{1}{p^*} bK_i - \frac{1}{p^*} bK_i \\
&\geq \left[\frac{p^*-\theta p}{\theta p p^*} aK_i^{\theta} + \left(\frac{1}{p}-\frac{1}{p^*}\right) bK_i\right] - \frac{p^*-\theta p}{\theta p p^*} bK_i + \\
&\quad + \frac{p^*-\theta p}{\theta p p^*} K_i^{(p^*/p)} S^{-(p^*/p)} - \frac{p^*-\theta p}{\theta p p^*} aK_i^{\theta} \\
&\geq \left[\frac{p^*-\theta p}{\theta p p^*} aK_i^{\theta} + \left(\frac{1}{p}-\frac{1}{p^*}\right) bK_i\right] + \\
&\quad + \frac{p^*-\theta p}{\theta p p^*} K_i\left(K_i^{(p^*-p)/p} S^{-(p^*/p)} - aK_i^{\theta-1} - b\right) \\
&\geq \left[\frac{p^*-\theta p}{\theta p p^*} a\left(Sy_i^{1/(\theta-1)}\right)^{\theta} + \left(\frac{1}{p}-\frac{1}{p^*}\right) bSy_i^{1/(\theta-1)}\right] + \\
&\quad + \frac{p^*-\theta p}{\theta p p^*} Sy_i^{1/(\theta-1)}\left(S^{-1} y_i^{(p^*-p)/p(\theta-1)} - aS^{\theta-1} y_i - b\right) \\
&\geq C^* + \frac{p^*-\theta p}{\theta p p^*} Sy_i^{1/(\theta-1)} \Psi(y_i) \\
&\geq C^*,
\end{aligned} \tag{53}
$$

since $\Psi(y_i) = 0$ for $i \in \{1,2\}$ and C^* is defined in (39). It is a contradiction with $c < C^*$. Then, J is empty, which implies that

$$\int_{\Omega} |u_n|^{p^*} \mathrm{d}x \longrightarrow \int_{\Omega} |u|^{p^*} \mathrm{d}x. \tag{54}$$

Now, set $l = \lim\|u_n\|$ as $n \longrightarrow +\infty$; then, we have

$$
\begin{aligned}
\langle I_\lambda'(u_n), u_n\rangle &= \left(a\|u_n\|^{(\theta-1)p} + b\right)\|u_n\|^p - \int_{\Omega}|u_n|^{p^*}\mathrm{d}x \\
&\quad - \lambda\int_{\Omega}|u_n|^p \mathrm{d}x = o_n(1),
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
\langle I_\lambda'(u_n), v\rangle &= \left(a\|u_n\|^{(\theta-1)p} + b\right)\left(\int_{\Omega}|\nabla u_n|^{p-2}\nabla u_n \nabla v \mathrm{d}x\right) \\
&\quad + \sqrt{2},
\end{aligned} \tag{56}
$$

$$-\int_{\Omega}|u_n|^{p^*} u_n v \mathrm{d}x - \lambda\int_{\Omega}|u_n|^{p-2} u_n v \mathrm{d}x = o_n(1), \tag{57}$$

for any $v \in W^{1,p}(\Omega)$.

Let $n \longrightarrow +\infty$; then, from (55) and (56), we deduce that

$$\left(al^{(\theta-1)p} + b\right) l^p - \int_{\Omega}|u|^{p^*}\mathrm{d}x - \lambda\int_{\Omega}|u|^p \mathrm{d}x = 0, \tag{58}$$

$$
\begin{aligned}
&\left(al^{(\theta-1)p} + b\right)\left(\int_{\Omega}|\nabla u|^{p-2}\nabla u \nabla v \mathrm{d}x\right) - \int_{\Omega}|u|^{p^*-2} u v \mathrm{d}x \\
&\quad - \lambda\int_{\Omega}|u|^{p-2} u v \mathrm{d}x = 0.
\end{aligned} \tag{59}
$$

Taking the test function $v = u$ in (59), we obtain

$$\left(al^{(\theta-1)p} + b\right)\|u\|^p - \int_{\Omega}|u|^{p^*}\mathrm{d}x - \lambda\int_{\Omega}|u|^p \mathrm{d}x = 0. \tag{60}$$

Therefore, equalities (58) and (59) imply that $\|u\| = l$. Consequently $\{u_n\}$ converges strongly in $W^{1,p}(\Omega)$, which is the desired result. □

2.1. Proof of the Main Result. By Lemma 4, I_λ satisfies the Palais–Smale condition at level c for any $c < C^*$. So, the existence of the positive solution follows immediately from the following estimates.

Lemma 5. *Let $N \geq p^2$, $b > 0$, and $\lambda < b\lambda_1$. Suppose that $\theta = N/(N-p)$ and $0 < a < S^{-\theta}$ or $\theta < N/(N-p)$ and $a > 0$. Then,*

$$\sup_{t\geq 0} I_\lambda(tz_\varepsilon) < C^*. \tag{61}$$

Proof. Employing estimate (9), we define the following functions:

$$
\begin{aligned}
g(t) &= I_\lambda(tz_\varepsilon) \\
&= \frac{a}{\theta p} t^{\theta p}\|z_\varepsilon\|^{\theta p} + \frac{b}{p} t^p \|z_\varepsilon\|^p - \frac{1}{p^*} t^{p^*}\int_{\Omega}|z_\varepsilon|^{p^*}\mathrm{d}x - \frac{\lambda}{p} t^p \int_{\Omega}|z_\varepsilon|^p \mathrm{d}x \\
&= \frac{a}{\theta p} t^{\theta p}\|z_\varepsilon\|^{\theta p} + \frac{b}{p} t^p \|z_\varepsilon\|^p - \frac{1}{p^*} S^{-(p^*/p)}\|z_\varepsilon\|^{p^*} t^{p^*} \\
&\quad - \frac{1}{p^*}\left(\int_{\Omega}|z_\varepsilon|^{p^*}\mathrm{d}x - S^{-(p^*/p)}\|z_\varepsilon\|^{p^*}\right) t^{p^*} - \frac{\lambda}{p} t^p \int_{\Omega}|z_\varepsilon|^p \mathrm{d}x, \\
h(t) &= -\frac{1}{p^*} S^{-(p^*/p)}\|z_\varepsilon\|^{p^*} t^{p^*} + \frac{a}{\theta p} t^{\theta p}\|z_\varepsilon\|^{\theta p} + \frac{b}{p} t^p \|z_\varepsilon\|^p.
\end{aligned} \tag{62}
$$

Note that $\lim_{t\longrightarrow\infty} g(t) = -\infty$ and $g(t) > 0$ when t is close to 0 so that $\sup_{t\geq 0} g(t)$ is attained for some $T_\varepsilon > 0$. Furthermore, from $g'(T_\varepsilon) = 0$, it follows that

$$-T_\varepsilon^{p^*-p}\int_{\Omega}|z_\varepsilon|^{p^*}\mathrm{d}x + aT_\varepsilon^{(\theta-1)p}\|z_\varepsilon\|^{\theta p} + b\|z_\varepsilon\|^p - \lambda\int_{\Omega}|z_\varepsilon|^p \mathrm{d}x = 0, \tag{63}$$

and therefore,

$$T_\varepsilon^{p^*-p}\int_\Omega |z_\varepsilon|^{p^*}\mathrm{d}x = aT_\varepsilon^{(\theta-1)p}\|z_\varepsilon\|^{\theta p} + b\|z_\varepsilon\|^p - \lambda\int_\Omega |z_\varepsilon|^p \mathrm{d}x \geq \left(b-\frac{\lambda}{\lambda_1}\right)\|z_\varepsilon\|^p. \tag{64}$$

Choose ε small enough so that, by (9), we have $T_\varepsilon \geq t_0$, for some $t_0 > 0$.

Besides, it holds

$$T_\varepsilon^{p^*-\theta p}\int_\Omega |z_\varepsilon|^{p^*}\mathrm{d}x = a\|z_\varepsilon\|^{\theta p} + \frac{b}{T_\varepsilon^{(\theta-1)p}}\|z_\varepsilon\|^p - \frac{\lambda}{T_\varepsilon^{(\theta-1)p}}\int_\Omega |z_\varepsilon|^p \mathrm{d}x \leq a\|z_\varepsilon\|^{\theta p} + \frac{b}{T_\varepsilon^{(\theta-1)p}}\|z_\varepsilon\|^p. \tag{65}$$

For $\theta < p^*/p$, $a > 0$, and $b > 0$, we have by (9)

$$T_\varepsilon^{p^*-\theta p}\int_\Omega |z_\varepsilon|^{p^*}\mathrm{d}x \leq a\|z_\varepsilon\|^{\theta p} + \frac{b\|z_\varepsilon\|^p}{t_0^{(\theta-1)p}}. \tag{66}$$

Then, for ε small enough, the above estimates yield $T_\varepsilon < t_1$ for some $t_1 > 0$ (independently of ε).

For $\theta = p^*/p$, $0 < a < S^{-\theta}$, and $b > 0$ and for ε small enough, we have, by (63),

$$T_\varepsilon^{(\theta-1)p} = \frac{\left(b\|z_\varepsilon\|^p - \lambda\int_\Omega |z_\varepsilon|^p \mathrm{d}x\right)}{\left(\int_\Omega |z_\varepsilon|^{p^*}\mathrm{d}x - a\|z_\varepsilon\|^{\theta p}\right)}, \tag{67}$$

which implies that T_ε is bounded above, for all $\varepsilon > 0$, that is, there exists a positive real number $t_2 > 0$ (independently of ε).

Now, we estimate $g(T_\varepsilon)$. It follows from $h'(t) = 0$ that

$$-S^{-(p^*/p)}\|z_\varepsilon\|^{p^*}t^{p^*-1} + at^{\theta p-1}\|z_\varepsilon\|^{\theta p} + bt^{p-1}\|z_\varepsilon\|^p = 0, \tag{68}$$

that is,

$$-\left[S^{-(p^*/p)}\|z_\varepsilon\|^{p^*-p}t^{p^*-p} - at^{(\theta-1)p}\|z_\varepsilon\|^{(\theta-1)p} - b\right] = 0. \tag{69}$$

Set $y = t^{(\theta-1)p}S^{1-\theta}\|z_\varepsilon\|^{(\theta-1)p}$, $\sigma = (p^*-p)/(\theta-1)p \geq 1$, and

$$y_* = \begin{cases} y_1, & \text{if } p^* = \theta p \text{ and } 0 \leq a < S^{-\theta}, \\ y_2, & \text{if } p^* > \theta p \text{ and } a \geq 0. \end{cases} \tag{70}$$

Then, by (69) and the definition of Ψ, we obtain

$$-\left[S^{-1}y^\sigma - aS^{\theta-1}y - b\right] = -\Psi(y) = 0, \tag{71}$$

which implies from the proof of Lemma 1 that $\Psi(y_*) = 0$. Therefore, $h'(t_*) = 0$, where $t_* = S^{1/p}\|z_\varepsilon\|^{-1}(y_*)^{1/(\theta-1)p}$. As $\Psi(y)$ is concave, then $h'(t)$ is convex, so

$$\max_{t\geq 0} h(t) = h(t_*) = -\frac{1}{p^*}S^{-(p^*/p)}\|z_\varepsilon\|^{p^*}t_*^{p^*} + \frac{a}{\theta p}\|z_\varepsilon\|^{\theta p}t_*^{\theta p} + \frac{b}{p}\|z_\varepsilon\|^p t_*^p. \tag{72}$$

By $h'(t_*) = 0$, we have

$$S^{-(p^*/p)}\|z_\varepsilon\|^{p^*}t_*^{p^*} = a\|z_\varepsilon\|^{\theta p}t_*^{\theta p} + b\|z_\varepsilon\|^p t_*^p. \tag{73}$$

So, from (73), we deduce that

$$\begin{aligned}\max_{t\geq 0} h(t) &= -\frac{1}{p^*}\left(a\|z_\varepsilon\|^{\theta p}t_*^{\theta p} + b\|z_\varepsilon\|^p t_*^p\right) + \frac{a}{\theta p}\|z_\varepsilon\|^{\theta p}t_*^{\theta p} + \frac{b}{p}\|z_\varepsilon\|^p t_*^p \\ &= a\left(\frac{1}{\theta p} - \frac{1}{p^*}\right)t_*^{\theta p}\|z_\varepsilon\|^{\theta p} + b\left(\frac{1}{p} - \frac{1}{p^*}\right)t_*^p\|z_\varepsilon\|^p \\ &= a\left(\frac{1}{\theta p} - \frac{1}{p^*}\right)S^\theta y_*^{\theta/(\theta-1)} + b\left(\frac{1}{p} - \frac{1}{p^*}\right)S y_*^{1/(\theta-1)} \\ &= C^*.\end{aligned} \tag{74}$$

Consequently, by (9),

$$\begin{aligned}\sup_{t\geq 0} I_\lambda(tz_\varepsilon) &\leq \sup_{t\geq 0} h(t) + \frac{1}{p^*}\left(S^{-(p^*/p)}\|z_\varepsilon\|^{p^*} - \int_\Omega |z_\varepsilon|^{p^*}\mathrm{d}x\right)t_1^{p^*} - \frac{\lambda}{p}t_0^p\int_\Omega |z_\varepsilon|^p \mathrm{d}x \\ &\leq C^* + C_1\left(S^{-(p^*/p)}S^{(p^*/p)} + O\left(\varepsilon^{(N-p)/p}\right) - 1\right) + \\ &\quad - C_2\begin{cases} C\varepsilon^{(N-p)/p}|\ln\varepsilon|, & \text{if } N = p^2, \\ C\varepsilon^{(N-p)/p}, & \text{if } N < p^2, \end{cases} \\ &\leq C^* + O\left(\varepsilon^{(N-p)/p}\right) - \begin{cases} C\varepsilon^{(N-p)/p}|\ln\varepsilon|, & \text{if } N = p^2, \\ C\varepsilon^{(N-p)/p}, & \text{if } N < p^2, \end{cases} \\ &< C^*.\end{aligned} \tag{75}$$

which is the desired result.

Now, we can proof the existence of a Mountain Pass-type solution. □

Proof of Theorem 1. Note that $I_\lambda(0) = 0$, so from Lemma 2, I_λ satisfies the geometry conditions of the Mountain Pass Theorem [15]. Then, there exists a Palais–Smale sequence (u_n) at level c, such that

$$I_\lambda(u_n) \longrightarrow c, I'_\lambda(u_n) \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty, \tag{76}$$

with

$$0 < c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) \leq \sup_{t\geq 0} I_\lambda(te) < C^*, \tag{77}$$

where

$$\Gamma = \left\{\gamma \in C\left([0,1], W_0^{1,p}(\Omega)\right), \gamma(0) = 0,\ \gamma(1) = e\right\}. \tag{78}$$

Using Lemma 3, we have that (u_n) has a subsequence, still denoted by (u_n), such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ as $n \longrightarrow +\infty$. Hence, from Lemma 4 and 5, we have $u_n \longrightarrow u$ in $W_0^{1,p}(\Omega)$ as $n \longrightarrow +\infty$. Hence, $I'_\lambda(u) = 0$ and $I_\lambda(u) = c > 0$. So, as $c > 0 = I_\lambda(0)$, we can conclude that u is a nonzero solution of (Υ_λ) with positive energy. Now, we show that $u > 0$ because

$$\begin{aligned}
0 &= \langle I'_\lambda(u), u^- \rangle \\
&= \left(a\|u\|^{(\theta-1)p} + b\right)\left(\int_\Omega |\nabla u|^{p-2}\nabla u \nabla u^- \,\mathrm{d}x\right) + \\
&\quad - \int_\Omega |u|^{p^*-2}uu^- \,\mathrm{d}x - \lambda\int_\Omega |u|^{p-2}uu^- \,\mathrm{d}x \\
&\geq \left(a\|u\|^{(\theta-1)p} + b\right)\left(\int_\Omega |\nabla u^-|^p \mathrm{d}x\right) + \int_\Omega |u|^{p^*} \mathrm{d}x + \lambda\int_\Omega |u|^p \mathrm{d}x \\
&\geq b\|u^-\|^p,
\end{aligned} \tag{79}$$

which implies that $u^- = 0$. By the strong maximum principle [16], one has $u > 0$. This completes the Proof of Theorem 1. □

Data Availability

The functions, functionals, and parameters used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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