

Research Article

Fourth-Order Hankel Determinants and Toeplitz Determinants for Convex Functions Connected with Sine Functions

Farah Zulfiqar ¹, Sarfraz Nawaz Malik ¹, Mohsan Raza ² and Md. Shajib Ali ³

¹Department of Mathematics, COMSATS University Islamabad, Wah Campus, Wah Cantt, Pakistan

²Department of Mathematics, Government College University, Faisalabad, Pakistan

³Department of Mathematics, Islamic University, Kushtia 7003, Bangladesh

Correspondence should be addressed to Md. Shajib Ali; shajib_301@yahoo.co.in

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This article deals with the upper bound of fourth-order Hankel and Toeplitz determinants for the convex functions which are defined by using the sine function. The main tools in this study are the coefficient inequalities for the class P of functions with positive real parts. Also, the investigation of the upper bound of the fourth-order Hankel determinant for 3-fold symmetric convex functions associated with the sine function is included.

1. Introduction

Let the family of all functions f be denoted by A which are analytic in an open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ with Taylor series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in D), \quad (1)$$

and S represent a family of functions $f \in A$ which are univalent in D . Let S^* , C , and K_g denote the families of starlike, convex, and close-to-convex functions, respectively, and they are defined as

$$S^* = \left\{ f \in S : \Re \left(\frac{z f'(z)}{f(z)} \right) > 0, (z \in D) \right\},$$

$$C = \left\{ f \in S : \Re \left(1 + \frac{z f''(z)}{f'(z)} \right) > 0, (z \in D) \right\}, \quad (2)$$

$$K_g = \left\{ f \in S : \Re \left(\frac{z f'(z)}{g(z)} \right) > 0, \text{ for } g \in S^*, (z \in D) \right\}.$$

Let P denote the family of all analytic functions p of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in D, \quad (3)$$

with the positive real parts in D . As the n^{th} coefficient for the functions belonging to the family is bounded by n , this bound helps in the study of geometric properties of functions $f \in S$. Specifically, the second coefficient a_2 helps in finding the distortion and growth properties of a normalized univalent function. Likewise, the problems involving power series with integral coefficients and investigating the singularities are successfully handled by using Hankel determinants. Pommerenke [1, 2] introduced the idea of Hankel determinants, and he defined those for univalent functions $f \in S$ of form (7) as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \quad (4)$$

In the theory of analytic functions, finding the upper bound of $|H_{q,n}(f)|$ is one of the most studied problems. Several researchers found the above-mentioned bound for different subfamilies of univalent functions for fixed values of q and n . A few remarkable contributions in this regard are included here for reference. For the subfamilies S^* , C , and $K_z = R$ (the class of functions with bounded turnings) of the set S , the sharp bounds of $|H_{(2,2)}(f)|$ were investigated by Janteng et al. [3, 4]. They proved the bounds as follows:

$$|H_{(2,2)}(f)| \leq \begin{cases} 1 & \text{for } f \in S^*, \\ \frac{1}{8} & \text{for } f \in C, \\ \frac{4}{9} & \text{for } f \in R. \end{cases} \quad (5)$$

The accurate estimate of $|H_{(2,2)}(f)|$ was obtained by Krishna et al. [5] for the family of Bazilevic functions. For subfamilies of S , more studies regarding $H_{(2,2)}(f)$ can be seen in [6–12]. According to Thomas’ conjecture [13], if $f \in S$, then $|H_{(q,2)}(f)| \leq 1$, but it was shown by Li and Srivastava in [14] that this conjecture is not true for $n \geq 4$. Also, Raducanu and Zaprawa [15] showed that it is false for $n = 2$. Rather, they showed that $\max\{|H_{(2,2)}(f)| : f \in S\} \geq 1.175$. As compared to $|H_{(2,2)}(f)|$, estimation of $|H_{(3,1)}(f)|$ is much more difficult. Babalola [16] published the first paper on $H_{(3,1)}(f)$ in 2010 in which he obtained the upper bound of $|H_{(3,1)}(f)|$ for subfamilies of S^* , C , and R . After that, for different subfamilies of analytic and univalent functions, few other authors [17–25] also published their work regarding $|H_{(3,1)}(f)|$. Zaprawa [26] improved the results of Babalola [16] recently in 2017, by showing

$$|H_{(3,1)}(f)| \leq \begin{cases} 1 & \text{for } f \in S^*, \\ \frac{49}{540} & \text{for } f \in C, \\ \frac{41}{60} & \text{for } f \in R. \end{cases} \quad (6)$$

He claimed that these bounds are not sharp. Furthermore, he considered the subfamilies of S^* , C , and R for sharpness, having functions with m -fold symmetry, and obtained the sharp bounds. Arif et al. [27–30] made a remarkable contribution in studying the fourth- and fifth-order Hankel determinants $H_{(4,1)}(f)$ and $H_{(5,1)}(f)$ for certain subfamilies of univalent functions. Mashwani et al. [31] have studied the fourth-order Hankel determinant for starlike functions related to sigmoid functions, whereas Kaur et al. [32] studied the same problem for a subclass of bounded turning functions. Wang et al. [33] studied the problem for bounded turning functions related to the lemniscate of Bernoulli. Recently, Zhang and Tang [34] have studied the fourth-order Hankel determinant for the class of starlike functions related to sine functions. Motivated by the

above-mentioned work, we intend to add some contributions to the fourth-order Hankel determinant for the class of convex functions associated with sine functions. Recently, the following class C_s of convex functions was introduced, which is associated with the sine function:

$$C_s = \left\{ f \in A : \frac{(zf'(z))'}{f'(z)} \prec 1 + \sin z \ (z \in D) \right\}, \quad (7)$$

where \prec is a subordination symbol and it also implies that the region defined by $(zf'(D))'/f'(D)$ lies in the eight-shaped region in the right-half plane. For different subfamilies of univalent functions, growth of $H_{q,n}(f)$ has been studied for fixed values of q and n . Particularly, we have

$$H_{4,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{vmatrix} \quad (n = 1, q = 4). \quad (8)$$

Also, Thomas and Halim defined the symmetric Toeplitz determinant $T_q(n)$ as follows:

$$T_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_n \end{vmatrix} \quad (n \geq 1, q \geq 1). \quad (9)$$

The Toeplitz determinants are closely related to Hankel determinants. As Hankel matrices consist of constant entries along the reverse diagonal, the Toeplitz matrices consist of constant entries along the diagonal.

As a special case, when $n = 1$ and $q = 4$, we have

$$T_{4,2}(f) = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{vmatrix}. \quad (10)$$

In this paper, we intend to find the upper bound of $|H_{4,1}(f)|$ and $|T_{4,2}(f)|$ for the class of functions defined by (6). The following sharp results would be useful for investigating our main results.

Lemma 1. *If $p \in P$ and p is of form (2), then for each $n, k, m, l \in \mathbb{N} = \{1, 2, \dots\}$, the following sharp inequalities hold:*

$$|c_n| \leq 2, \quad (11)$$

$$|c_{n+k} - \mu c_n c_k| \leq \begin{cases} 2, & \text{for } 0 \leq \mu \leq 1 \\ 2|2\mu - 1|, & \text{elsewhere} \end{cases}$$

$$c_{n+2k} - c_n c_k^2 \leq 6, \quad (12)$$

$$|c_n c_k - c_m c_l| \leq 4, \quad \text{for } n + k = m + l. \quad (13)$$

Inequalities (10)–(12) are proved in [26, 35, 36], respectively. Inequality (13) is obvious.

Libera and Zlotkiewicz proved the following result [37].

Lemma 2. *Let $p \in P$ be of form (2). Then, the modulus of the expressions*

$$\begin{aligned} A_3 &= c_1^3 - 2c_1c_2 + c_3, \\ A_4 &= c_1^4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 - c_4, \\ A_5 &= c_1^5 + 3c_1c_2^2 + 3c_1^2c_3 - 4c_1^3c_2 - 2c_1c_4 - 2c_2c_3 + c_5, \\ A_6 &= c_1^6 + 6c_1^2c_2^2 + 4c_1^3c_3 + 2c_1c_5 + 2c_2c_4 + c_3^2 \\ &\quad - c_2^3 - 5c_1^4c_2 - 3c_1^2c_4 - 6c_1c_2c_3 - c_6 \end{aligned} \quad (14)$$

are all bounded by 2.

2. Main Results

2.1. Bounds of $|H_{4,1}(f)|$ and $|T_{4,2}(f)|$ for the Set C_s Connected with the Sine Function. Following (7), we can write $H_{4,1}(f)$, where $f \in S$ and $a_1 = 1$, as

$$H_{4,1}(f) = a_7H_{3,1}(f) - a_6R_1 + a_5R_2 - a_4R_3, \quad (15)$$

where

$$H_{3,1}(f) = (a_3a_5 - a_4^2) - a_2(a_2a_5 - a_3a_4) + a_3(a_2a_4 - a_3^2) \quad (16)$$

and R_1, R_2 , and R_3 are determinants of order 3, given by

$$R_1 = (a_3a_6 - a_4a_5) - a_2(a_2a_6 - a_3a_5) + a_4(a_2a_4 - a_3^2), \quad (17)$$

$$R_2 = (a_4a_6 - a_5^2) - a_2(a_3a_6 - a_4a_5) + a_3(a_3a_5 - a_4^2), \quad (18)$$

$$R_3 = a_2(a_4a_6 - a_5^2) - a_3(a_3a_6 - a_4a_5) + a_4(a_3a_5 - a_4^2). \quad (19)$$

Also,

$$T_{4,1}(f) = a_1C_1 - a_2C_2 + a_3C_3 - a_4C_4, \quad (20)$$

where

$$C_1 = a_1(a_1^2 - a_2^2) - a_4(a_1a_4 - a_2a_3) + a_3(a_4a_2 - a_1a_3), \quad (21)$$

$$\begin{aligned} C_2 &= a_2(a_1^2 - a_2^2) - a_4(a_1a_3 - a_2a_4) + a_3(a_2a_3 - a_1a_4), \\ C_3 &= a_2(a_1a_4 - a_2a_3) - a_3(a_1^2 - a_3^2) + a_4(a_1a_2 - a_3a_4), \\ C_4 &= a_2(a_2a_4 - a_1a_3) - a_3(a_1a_2 - a_3a_4) + a_4(a_1^2 - a_4^2). \end{aligned} \quad (22)$$

As, from (7), $H_{4,1}(f)$ is a polynomial of six coefficients of function f of the given class, these coefficients are taken as

a_2, a_3, a_4, a_5, a_6 , and a_7 . However, there is a connection between these coefficients and the coefficients of function p in the class P in many problems. Consider that $f \in C_s$ has form (1); then, there is a Schwartz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$, such that

$$\frac{(zf'(z))'}{f'(z)} = 1 + \sin(w(z)). \quad (23)$$

Now,

$$\begin{aligned} \frac{(zf'(z))'}{f'(z)} &= \frac{1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}} \\ &= 1 + 2a_2z + (6a_3 - 4a_2^2)z^2 \\ &\quad + (-18a_2a_3 + 12a_4 + 8a_2^3)z^3 \\ &\quad + (-32a_2a_4 + 20a_5 - 18a_3^2 + 48a_3a_2^2 - 16a_4^2)z^4 \\ &\quad + (-50a_2a_5 + 30a_6 - 60a_4a_3 \\ &\quad + 80a_4a_2^2 + 90a_2a_3^2 - 120a_3a_2^3 + 32a_2^5)z^5 + \dots \end{aligned} \quad (24)$$

Consider

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + c_3z^3 + \dots \quad (25)$$

Since we have $p \in P$,

$$w(z) = \frac{p(z) - 1}{1 + p(z)} = \frac{c_1z + c_2z^2 + c_3z^3 + \dots}{2 + c_1z + c_2z^2 + c_3z^3 + \dots}. \quad (26)$$

Also,

$$\begin{aligned} 1 + \sin(w(z)) &= 1 + \frac{1}{2}c_1z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{1}{2}c_3 - \frac{1}{2}c_1c_2 + \frac{5}{48}c_1^3\right)z^3 \\ &\quad + \left(\frac{5}{16}c_2c_1^2 - \frac{1}{32}c_1^4 + \frac{1}{2}c_4 - \frac{1}{2}c_1c_3 - \frac{1}{4}c_2^2\right)z^4 \\ &\quad + \left(\frac{1}{3840}c_1^5 + \frac{5}{16}c_1c_2^2 - \frac{1}{8}c_2c_1^3 - \frac{1}{2}c_2c_3 + \frac{5}{16}c_3c_1^2 \right. \\ &\quad \left. - \frac{1}{2}c_1c_4 + \frac{1}{2}c_5\right)z^5 + \dots \end{aligned} \quad (27)$$

On comparing coefficients between (25) and (28), we get

$$\begin{aligned} a_2 &= \frac{1}{4}c_1, \\ a_3 &= \frac{1}{12}c_2, \\ a_4 &= \frac{1}{96}c_1c_2 - \frac{1}{576}c_1^3 + \frac{1}{24}c_3, \\ a_5 &= -\frac{1}{960}c_2c_1^2 + \frac{1}{1152}c_1^4 - \frac{1}{120}c_1c_3 - \frac{1}{160}c_2^2 + \frac{1}{40}c_4, \\ a_6 &= -\frac{11}{28800}c_1^5 + \frac{71}{34560}c_2c_1^3 + \left(\frac{1}{1152}c_2^2 - \frac{1}{160}c_4\right)c_1 - \frac{7}{720}c_2c_3 + \frac{1}{60}c_5, \end{aligned} \quad (28)$$

and

$$a_7 = \frac{2399}{14515200}c_1^6 - \frac{347}{241920}c_2c_1^4 + \frac{1}{756}c_3c_1^3 + \left(\frac{1}{3360}c_4 + \frac{29}{16128}c_2^2\right)c_1^2 + \left(\frac{1}{210}c_5 + \frac{23}{10080}c_2c_3\right)c_1 + \frac{1}{84}c_6 - \frac{1}{252}c_3^2 + \frac{5}{8064}c_2^3 - \frac{5}{672}c_4c_2. \tag{29}$$

By using these coefficients, we can write (16)–(19) in the following way:

$$H_{3,1}(f) = -\frac{19}{331776}c_1^6 + \frac{1}{34560}c_2c_1^4 + \frac{23}{34560}c_3c_1^3 + \left(-\frac{11}{46080}c_2^2 - \frac{1}{640}c_4\right)c_1^2 + \frac{11}{5760}c_3c_1c_2 + \frac{1}{12}c_2\left(\frac{1}{40}c_4 - \frac{1}{160}c_2^2\right) - \frac{1}{1728}c_3^2 - \frac{1}{576}c_3^2, \tag{30}$$

$$R_1 = \frac{289}{11059200}c_1^7 - \frac{29}{230400}c_2c_1^5 - \frac{1}{11520}c_3c_1^4 + \left(\frac{187}{1658880}c_2^2 + \frac{1}{2304}c_4\right)c_1^3 + \left(\frac{1}{5760}c_2c_3 - \frac{1}{960}c_5\right)c_1^2 + \frac{1}{12}c_2\left(\frac{1}{1152}c_2^2 - \frac{1}{160}c_4\right) + \frac{1}{1280}c_3^2 + \frac{1}{32}c_2\left(\frac{1}{40}c_4 - \frac{1}{160}c_2^2\right) + \frac{1}{13824}c_2^3c_1 + \frac{1}{12}c_2\left(-\frac{7}{720}c_2c_3 + \frac{1}{60}c_5\right) - \frac{1}{3456}c_3c_2^2 - \frac{1}{24}c_3\left(\frac{1}{40}c_4 - \frac{1}{160}c_2^2\right), \tag{31}$$

$$R_2 = -\frac{31}{66355200}c_1^8 + \frac{101}{12441600}c_2c_1^6 + \frac{31}{2764800}c_3c_1^5 + \left(-\frac{197}{4147200}c_2^2 - \frac{1}{23040}c_4\right)c_1^4 + \left(\frac{7}{64800}c_2c_3 - \frac{1}{34560}c_5\right)c_1^3 + \left(-\frac{1}{6400}c_3^2 + \frac{7}{38400}c_2c_4 - \frac{37}{921600}c_2^3\right)c_1^2 + \left(\frac{1}{5400}c_2^2c_3 - \frac{1}{1920}c_2c_5 + \frac{1}{2400}c_3c_4\right)c_1 + \frac{1}{1440}c_3c_5 + \frac{7}{14400}c_2^2c_4 - \frac{1}{1600}c_4^2 - \frac{19}{230400}c_2^4 - \frac{19}{34560}c_2c_3^2, \tag{32}$$

and

$$R_3 = -\frac{83}{4777574400}c_1^9 + \frac{317}{796262400}c_2c_1^7 - \frac{49}{66355200}c_3c_1^6 + \left(-\frac{1}{122880}c_4 - \frac{169}{132710400}c_2^2\right)c_1^5 + \left(-\frac{1}{138240}c_5 + \frac{209}{8294400}c_2c_3\right)c_1^4 + \frac{61}{2764800}c_2c_4 - \frac{23}{2764800}c_3^2 - \frac{2993}{199065600}c_2^3\right)c_1^3 + \left(-\frac{1}{23040}c_2c_5 + \frac{11}{5529600}c_2^2c_3\right) + \frac{1}{25600}c_3c_4\right)c_1^2 + \left(\frac{1}{12800}c_2^2c_4 - \frac{41}{8294400}c_4^2 + \frac{1}{5760}c_3c_5 - \frac{1}{6400}c_4^2 - \frac{29}{276480}c_2c_3^2\right)c_1 - \frac{1}{8640}c_2^2c_5 + \frac{1}{41472}c_2^3c_3 - \frac{1}{13824}c_3^3 + \frac{1}{5760}c_2c_3c_4. \tag{33}$$

$$C_1 = -\frac{1}{331776}c_1^6 - \frac{1}{9216}c_1^4c_2 + \frac{1}{6912}c_1^3c_3 + \left(-\frac{1}{16} - \frac{5}{9216}c_2^2\right)c_1^2 + \frac{1}{384}c_1c_2c_3 + 1 - \frac{1}{576}c_3^2 - \frac{1}{144}c_2^2, \tag{34}$$

$$C_2 = \frac{1}{1327104}c_1^7 + \frac{1}{110592}c_1^5c_2 - \frac{1}{27648}c_1^4c_3 + \left(\frac{1}{36864}c_2^2 - \frac{1}{64} + \frac{1}{3456}c_2\right)c_1^3 - \frac{1}{4608}c_1^2c_2c_3 + \left(\frac{1}{4} + \frac{1}{288}c_2^2 + \frac{1}{2304}c_3^2\right)c_1 - \frac{1}{144}c_3c_2, \tag{35}$$

$$C_3 = -\frac{1}{3981312}c_1^6c_2 + \left(-\frac{1}{1152} - \frac{1}{331776}c_2^2\right)c_1^4 + \frac{1}{82944}c_1^3c_3c_2 + \left(-\frac{1}{110592}c_2^3 - \frac{1}{96}c_2\right)c_1^2 + \left(\frac{1}{48}c_3 + \frac{1}{13824}c_2^2c_3\right)c_1 + \frac{1}{1728}c_2^3 - \frac{1}{6912}c_3^2c_2 - \frac{1}{12}c_2^2, \tag{36}$$

Similarly, in case of Toeplitz determinants,

$$\begin{aligned}
 C_4 = & \frac{1}{191102976}c_1^9 + \frac{1}{10616832}c_1^7c_2 - \frac{1}{2654208}c_1^6c_3 \\
 & + \left(\frac{1}{1769472}c_2^2 - \frac{1}{9216}\right)c_1^5 - \frac{1}{221184}c_1^4c_2c_3 \\
 & + \left(\frac{1}{884736}c_2^3 - \frac{1}{1536}c_2\right) \\
 & - \frac{1}{82944}c_2^2 + \frac{1}{110592}c_3^2 - \frac{1}{576}\Big)c_1^3 + \left(\frac{1}{384}c_3 - \frac{1}{73728}c_2^2c_3\right)c_1^2 \\
 & + \left(-\frac{15}{96}c_2 + \frac{1}{18432}c_3^2c_2 - \frac{1}{13824}c_3^3\right)c_1 + \frac{1}{24}c_3 - \frac{1}{13824}c_3^3 \\
 & + \frac{1}{3456}c_2^2c_3.
 \end{aligned}
 \tag{37}$$

By using the previous computations, we prove the following.

Theorem 1. *If the function $f \in C_s$ and is of form (1), then*

$$|H_{4,1}(f)| \leq \frac{112267159597}{15049359360000} \approx 0.0074599 \dots \tag{38}$$

Proof. As $f \in C_s$, then by using (30)–(33) in (15), we get

$$\begin{aligned}
 H_{4,1}(f) = & \frac{38723}{481579499520000}c_1^{12} + \frac{23}{2322432}c_3^4 - \frac{1159}{6967296000}c_2^6 \\
 & - \frac{4169}{6967296000}c_1^7c_5 - \frac{19}{27869184}c_6c_1^6 \\
 & - \frac{19}{1451520}c_6c_2^3 - \frac{1}{48384}c_6c_3^2 - \frac{17}{6967296}c_3^2c_2^3 + \frac{2539}{580608000}c_2^4c_4 + \frac{1}{1792000}c_2^2c_4^2 \\
 & + \frac{1}{57600}c_1^2c_5^2 - \frac{1}{43200}c_2c_5^2 - \frac{1}{64000}c_4^3 + \frac{1619}{580608000}c_1^3c_3c_2c_4 - \frac{1}{67200}c_1^2c_5c_2c_3 - \frac{1}{53760}c_1c_5c_2c_4 \\
 & - \frac{713}{48384000}c_2^2c_1c_3c_4 + \frac{11}{483840}c_6c_2c_1c_3 + \frac{247}{58060800}c_2c_1c_3^3 + \frac{589}{145152000}c_1^5c_5c_2 \\
 & + \frac{53}{36864000}c_2c_1^4c_3^2 + \frac{23}{2903040}c_6c_1^3c_3 + \frac{563}{32256000}c_1^2c_4^2c_2 + \frac{1}{64000}c_1c_4^2c_3 + \frac{29}{1036800}c_2^2c_3c_5 \\
 & + \frac{1}{28800}c_5c_3c_4 - \frac{913}{348364800}c_1^3c_5c_2^2 - \frac{43}{2419200}c_1c_5c_3^2 - \frac{3019}{1032192000}c_2^2c_1^5c_3 \\
 & - \frac{11}{3870720}c_6c_1^2c_2^2 + \frac{67}{9676800}c_1c_5c_2^3 + \frac{1}{2903040}c_6c_2c_1^4 + \frac{151}{96768000}c_1^2c_4c_2^2 + \frac{251}{139345920}c_1^6c_2c_4 \\
 & + \frac{1073}{580608000}c_1^2c_4c_2^3 - \frac{11077}{55738368000}c_1^7c_2c_3 + \frac{1}{40320}c_6c_2c_4 - \frac{1}{3628800}c_1^4c_5c_3 \\
 & - \frac{1919}{1741824000}c_2^4c_1c_3 - \frac{29}{20736000}c_1^5c_3c_4 - \frac{1}{53760}c_6c_1^2c_4 - \frac{30703}{4644864000}c_2^2c_1^4c_4 \\
 & + \frac{195313}{41803776000}c_1^3c_3c_2^3 + \frac{4699}{2322432000}c_1^2c_2^2c_3^2 - \frac{17}{2419200}c_1^3c_5c_4 - \frac{1}{37800}c_3^2c_2c_4 \\
 & - \frac{136291}{20065812480000}c_1^{10}c_2 + \frac{14201}{1003290624000}c_1^9c_3 + \frac{67139}{1337720832000}c_1^8c_2^2 - \frac{77}{7962624000}c_1^8c_4 \\
 & - \frac{40073}{1003290624000}c_1^6c_2^3 + \frac{8419}{9289728000}c_1^6c_3^2 + \frac{852083}{668860416000}c_2^4c_1^4 \\
 & - \frac{5947}{1741824000}c_1^3c_3^3 + \frac{1}{2867200}c_1^4c_4^2 - \frac{391}{217728000}c_1^2c_2^5.
 \end{aligned}
 \tag{39}$$

$$\begin{aligned}
H_{4,1}(f) = & \frac{1}{481579499520000} \{ 38723c_1^{12} - 3270984c_1^{10}c_2 + 6816480c_1^9c_3 \\
& + (24170040c_2^2 \\
& - 4656960c_4)c_1^8 + (-288161280c_5 - 95705280c_2c_3)c_1^7 + (-328320000c_6 \\
& + 436440960c_2^2 - 19235040c_2^3 + 867456000c_4c_2)c_1^6 + (-673505280c_3c_4 \\
& - 1408544640c_3c_2^2 + 1954160640c_2c_5)c_1^5 + (613499760c_2^4 + 165888000c_6c_2 \\
& - 3183287040c_2^2c_4 + 167961600c_2^2 - 132710400c_3c_5 + 692375040c_2c_3^2)c_1^4 \\
& + (-1644226560c_3^3 + 1342863360c_2c_3c_4 + 2250005760c_2^3c_3 \\
& + 3815424000c_6c_3 - 3384115200c_5c_4 - 1262131200c_2^2c_5)c_1^3 \\
& + (8360755200c_5^2 - 8957952000c_6c_4 - 7166361600c_5c_2c_3 - 1368576000c_6c_2^2 \\
& + 8405544960c_4^2c_2 + 751472640c_4c_3^2 - 864829440c_2^5 + 974384640c_2^2c_3^2 \\
& + 889989120c_4c_2^3)c_1^2 + (3334348800c_5c_2^3 + 7524679680c_4^2c_3 \\
& + 10948608000c_6c_2c_3 + 2048716800c_2c_3^3 - 7096688640c_2^2c_3c_4 \\
& - 8957952000c_5c_2c_4 - 530565120c_2^4c_3 - 8559820800c_5c_3^2)c_1 \\
& - 1175040000c_2^2c_3^3 + 2105948160c_2^4c_4 - 12740198400c_2^2c_3c_4 \\
& + 11943936000c_6c_2c_4 + 16721510400c_5c_3c_4 - 7524679680c_4^3 + 4769280000c_4^4 \\
& - 80110080c_2^6 - 9953280000c_6c_2^2 - 11147673600c_2c_5^2 + 13470105600c_2^2c_3c_5 \\
& - 6303744000c_6c_2^3 + 268738560c_2^2c_4^2 \}.
\end{aligned} \tag{40}$$

After rearranging the terms, we get

$$\begin{aligned}
H_{4,1}(f) = & \frac{1}{481579499520000} \\
& \cdot \left\{ -38723A_6(c_2 - c_1^2)^3 - 2961200c_2A_5^2 + 6661588c_3A_3A_6 \right. \\
& - 448912A_6(c_3 - c_1c_2)^2 - 45407915c_4A_3A_5 - 288238726c_1 \\
& A_6\left(c_5 - \frac{1}{288238726}c_2c_3\right) - 328281275A_6\left(c_6 - \frac{403543209}{328281275}c_3^2\right) - A_6(c_6 - 827940501c_2c_4) - A_6 \\
& \cdot (c_6 - 45022105c_2^3) + 519121748c_2A_5\left(c_5 - \frac{1592920565}{519121748}c_2c_3\right) + 635357352 \\
& \cdot c_3A_4(c_5 - c_1c_4) - 1475634554c_2A_4\left(c_6 - \frac{2857592366}{1475634554}c_3^2\right) - 371563976c_5A_3 \\
& \cdot (c_4 - c_1c_3) + 158880018c_4^2A_3c_1 + 769228159c_2^4A_3c_1 + 1052977475c_2^2c_4A_3c_1 \\
& + 5135210696(c_2 - c_1^2)(c_4 - c_1c_3)\left(c_6 - \frac{3289911688}{5135210696}c_3^2\right) - 3872726611(c_2 - c_1^2) \\
& \cdot \left(c_4 - \frac{2520764034}{3872726611}c_2^2\right)(c_6 - c_1c_5) - 4015593821A_3c_2c_4c_3 - 4121711992A_3c_3c_2^3 \\
& \left. - 8680311746c_4c_1^2\left(c_6 - \frac{11237134463}{8680311746}c_2c_4\right) - 24834316c_3^2c_1^2 \right\}
\end{aligned}$$

$$\begin{aligned}
 & \cdot \left(c_4 - \frac{11757634360}{24834316} c_2^2 \right) - 8937232652 c_5 (c_2 - c_1^2) \left(c_5 - \frac{7761341290}{8937232652} c_2 c_3 \right) \\
 & - 1898862450 c_2^3 c_1^2 \left(c_4 - \frac{430044643}{1898862450} c_2^2 \right) - 1305360709 c_2^2 c_1^2 c_6 + 17052974791 \\
 & \cdot c_3 c_2 c_1 \left(c_6 - \frac{13683368690}{17052974791} c_2 c_4 \right) - 13635892999 c_5 c_2 c_1 \left(c_4 - \frac{3937309300}{13635892999} c_2^2 \right) \\
 & + (-4485418959 c_2 c_3^3 - 4542515027 c_2^4 c_3 + 6721360728 c_4^2 c_3) c_1 + 368323828 c_5 c_1 \\
 & \cdot \left(c_6 - \frac{10733372524}{368323828} c_3^2 \right) - 2726601496 c_2 c_5 \left(c_5 - \frac{7692726219}{2726601496} c_2 c_3 \right) + 10690320416 \\
 & \cdot c_2 c_4 \left(c_6 - \frac{4233634378}{10690320416} c_3^2 \right) + 2843921897 c_2^4 \left(c_4 - \frac{35126698}{2843921897} c_2^2 \right) - 9215242838 \\
 & \cdot c_3^2 \left(c_6 - \frac{4359524115}{9215242838} c_3^2 \right) - 7632171375 c_6 c_2^3 - 2720144201 c_3^2 c_2^3 + 17732972519 \\
 & \cdot c_5 c_3 c_4 - 328281277 c_6^2 - 7524679680 c_4^3 - 31507775 c_2 c_1^7 c_3 - 1387142442 c_2^2 c_4^2.
 \end{aligned} \tag{41}$$

After using triangular inequalities and lemmas, we get the following expression:

$$\begin{aligned}
 |H_{4,1}(f)| &= \frac{1}{481579499520000} \\
 & \{ (38723 \times 2) \times 2^3 + 4(2961200 \times 2) + 4 \times (6661588 \times 2) + (448912 \times 2) \times \\
 & 2^2 + 2 \times (45407915 \times 2) \times 2 + 2(288238726 \times 2) \times 2 + \frac{957610286}{328281275} \times (328281275 \times 2) + 2 \times \\
 & 3311762002 + 2 \times 3311762002 + 2 \times (519121748 \times 2) \times \frac{1333359691}{129780437} + 2 \times (635357352 \times 2) \\
 & \times 2 + 2 \times (1475634554 \times 2) \times \frac{4239550178}{737817277} + 371563976 \times (2) \times (2) \times (2) + 158880018 \times (4) \\
 & \times (2) \times (2) + 769228159 \times (16) \times (2) \times (2) + 1052977475 \times (4) \times (2) \times (2) \times (2) \\
 & + 5135210696 \times (2) \times (2) \times \frac{51593310}{91700191} + 872726611 \times (2) \times (2) \times (2) + 4015593821 \times (16) \\
 & + 4121711992 \times (32) + 8680311746 \times (8) \times \frac{13793957180}{4340155873} + 24834316 \times (16) \times \frac{11745217202}{6208579} \\
 & + 8937232652 \times (2) \times (2) \times (2) + 1898862450 \times (32) \times (2) + 1305360709 \times (32) \\
 & + 17052974791 \times (8) \times (2) + 13635892999 \times (8) \times (2) + (4485418959 \\
 & + 6721360728 \times (8) \times (2) + 368323828 \times (4) \times \frac{10549210610}{92080957} + 2726601496 \times (4) \times \frac{6329425471}{681650374} \\
 & + 10690320416 \times (4) \times (2) + 2843921897 \times (16) \times (2) + 9215242838 \times (4) \times (2) \\
 & + 7632171375 \times (16) + 2720144201 \times (32) + 17732972519 \times (8) + 328281277 \times (4) \\
 & + 7524679680 \times (8) + 31507775 \times (512) + 1387142442 \times (16) \}.
 \end{aligned} \tag{42}$$

Hence,

$$|H_{4,1}(f)| \leq \frac{3592549107104}{481579499520000} \approx 0.0074599, \quad (43) \quad |T_{4,1}(f)| \leq \frac{501434459}{286654464} \approx 1.7493. \quad (44)$$

which completes the proof. \square

Theorem 2. *If the function $f \in C_s$ and is of form (1), then*

Proof. As $f \in C_s$, using (34)–(37) in (20), we get

$$\begin{aligned} T_{4,1}(f) = & 1 + \frac{1}{256}c_1^4 - \frac{1}{288}c_3^2 - \frac{1}{165888}c_1^6 - \frac{1}{8}c_1^2 - \frac{1}{72}c_2^2 + \frac{5}{576}c_1c_2c_3 + \frac{1}{9216}c_1^3c_3c_2 \\ & + \frac{1}{3456}c_1^3c_3 - \frac{1}{221184}c_1^6c_2 + \frac{1}{497664}c_1^3c_3c_2^2 - \frac{1}{5308416}c_1^3c_2^3c_3 \\ & + \frac{1}{82944}c_1c_2^3c_3 - \frac{1}{331776}c_1c_2c_3^3 + \frac{1}{884736}c_1^2c_2^2c_3^2 - \frac{1}{10616832}c_1^5c_2^2c_3 \\ & - \frac{1}{63700992}c_1^7c_2c_3 + \frac{1}{2654208}c_1^4c_2c_3^2 - \frac{13}{4608}c_1^2c_2^2 + \frac{1}{55296}c_1^5c_3 \\ & - \frac{1}{73728}c_1^4c_2^2 - \frac{1}{4608}c_1^2c_2^2 - \frac{1}{23887872}c_1^6c_2^2 - \frac{1}{1990656}c_1^4c_2^3 \\ & - \frac{1}{663552}c_1^2c_2^4 - \frac{1}{41472}c_2^2c_3^2 + \frac{1}{84934656}c_1^4c_2^4 + \frac{1}{127401984}c_1^6c_2^3 \\ & + \frac{1}{509607936}c_1^8c_2^2 + \frac{1}{4586471424}c_1^{10}c_2 - \frac{1}{1146617856}c_1^9c_3 + \frac{1}{31850496}c_1^6c_3^2 \\ & - \frac{1}{1990656}c_1^3c_3^3 - \frac{1}{2654208}c_1^8 + \frac{1}{20736}c_2^4 + \frac{1}{110075314176}c_1^{12}. \end{aligned} \quad (45)$$

Rearranging the terms, we may write

$$\begin{aligned} T_{4,1}(f) = & \frac{1}{110075314176} \{ c_1^{12} + 24c_1^{10}c_2 - 96c_1^9c_3 + (-41472 + 216c_2^2)c_1^8 - 1728c_1^7c_2c_3 \\ & + (-663552 + 3456c_3^2 - 497664c_2 + 864c_2^3 - 4608c_2^2)c_1^6 \\ & + (-10368c_2^2c_3 + 1990656c_3)c_1^5 \\ & + (-1492992c_2^2 + 41472c_3^2c_2 - 55296c_2^3 - 39813120c_2 + 1296c_2^4 + 429981696)c_1^4 \\ & + (11943936c_2c_3 + 31850496c_3 - 55296c_3^3 - 20736c_2^3c_3 + 221184c_3c_2^2)c_1^3 \\ & + (-310542336c_2^2 - 13759414272 + 124416c_2^2c_3^2 - 165888c_2^4 - 23887872c_3^2)c_1^2 \\ & + (1327104c_2^3c_3 + 955514880c_2c_3 - 331776c_2c_3^3)c_1 \\ & - 1528823808c_2^2 - 382205952c_3^2 + 331776c_3^4 + 5308416c_2^4 \\ & + 110075314176 - 2654208c_2^2c_3^2 \}. \end{aligned} \quad (46)$$

After rearranging the terms, we get

$$\begin{aligned}
T_{4,1}(f) = & \frac{1}{110075314176} \left\{ -A_6(c_2 - c_1^2)^3 - 100A_3A_6\left(c_3 - \frac{32}{100}c_1c_2\right) - 2c_1A_6\left(c_5 - \frac{3}{2}c_1c_4\right) \right. \\
& - 2570c_1c_2A_6\left(c_3 - \frac{416}{2570}c_1c_2\right) - 41472A_3A_5 + 2156544A_5\left(c_3 - \frac{746496}{2156544}c_1c_2\right) \\
& - 22704c_2^2A_5\left(c_3 - \frac{2467}{22704}c_1c_2\right) - 312c_4A_5\left(c_3 - \frac{100}{312}c_1c_2\right) + 29c_2A_4\left(c_6 - \frac{68}{29}c_1c_5\right) \\
& + (-663552 + c_6 - 4608c_2^2 + 3955c_3^2)A_6 + 22311936c_2A_3\left(c_3 - \frac{4022784}{22311936}c_1c_2\right) \\
& - 902c_2^2A_3\left(c_5 - \frac{1389}{902}c_1c_4\right) - 12A_3c_4\left(c_5 - \frac{9}{12}c_1c_4\right) - 104c_3A_3\left(c_6 - \frac{208}{104}c_1c_5\right) \\
& - 70616c_3^2A_3\left(c_3 - \frac{69470}{70616}c_1c_2\right) - 102891c_2^3A_3\left(c_3 - \frac{9058}{102891}c_1c_2\right) + 239616c_2^2A_3 \\
& \left(c_3 - \frac{78336}{239616}c_1c_2\right) + 41472A_3\left(c_5 - \frac{82944}{41472}c_1c_4\right) + 12498c_3c_4c_1^2\left(c_3 - \frac{8344}{12498}c_1c_2\right) \\
& + (5c_6c_2 - 4313088c_2 + 429981696)A_4 + 34504704c_3A_3 - 633c_4^2c_1\left(c_3 - \frac{212}{633}c_1c_2\right) \\
& + 4313088c_3c_1\left(c_4 - \frac{30233088}{4313088}c_1c_3\right) - 188c_5c_2c_1\left(c_4 - \frac{5216}{188}c_1c_3\right) + 1857024 \\
& c_2^3c_1\left(c_3 - \frac{294912}{1857024}c_1c_2\right) - 144138c_2^4c_1\left(c_3 - \frac{11076}{144138}c_1c_2\right) - 859963392c_1. \\
& \left(c_3 - \frac{1289945088}{859963392}c_1c_2\right) - 4c_6c_1\left(c_5 - \frac{6}{4}c_1c_4\right) - 517010c_2c_2^2c_1\left(c_3 - \frac{293955}{517010}c_1c_2\right) \\
& + 746496c_2c_1\left(c_5 - \frac{1492992}{746496}c_1c_4\right) + 9216c_2^2c_1\left(c_5 - \frac{13824}{9216}c_1c_4\right) - 2608c_6c_2c_1 \\
& \left(c_3 - \frac{451}{2608}c_1c_2\right) + 40974336c_2^2c_1\left(c_3 - \frac{6054912}{40974336}c_1c_2\right) - 40985c_2^2c_4c_1 \\
& \left(c_3 - \frac{6702}{40985}c_1c_2\right) + 1106804736c_2c_1\left(c_3 - \frac{435953664}{1106804736}c_1c_2\right) \\
& + (4c_5^2 - 1990656c_4 - 13759414272)c_1^2 + 23606c_2^2c_5\left(c_3 - \frac{4203}{23606}c_1c_2\right) + 3958c_3^2 \\
& \left(c_6 - \frac{7916}{3958}c_1c_5\right) - 41803776c_2\left(c_4 - \frac{42467328}{41803776}c_2^2\right) + 429981696 \\
& \left(c_4 - \frac{1958805504}{429981696}c_2^2\right) + 9216c_2^3\left(c_4 - \frac{4608}{9216}c_2^2\right) + 32c_2c_6\left(c_4 - \frac{34}{32}c_2^2\right) \\
& + 2c_2^4\left(c_4 - \frac{1}{2}c_2^2\right) - 8334c_3^2c_2\left(c_4 - \frac{61339}{8334}c_2^2\right) - 663552\left(c_6 - \frac{1327104}{663552}c_1c_5\right) \\
& - 4608c_6c_2^2 + 398537c_3^4 + 5308416c_2^4 + 110075314176 - 416047104c_2^2 \\
& \left. + 324c_4c_5c_3 + c_6^2 - 18081792c_2c_3^2 - 2889216c_2^2c_3^2 - 2156544c_5c_3\right\}.
\end{aligned} \tag{47}$$

$$\begin{aligned}
& \tag{48}
\end{aligned}$$

After using the triangular inequality and above-stated lemmas, we get

$$\begin{aligned}
 |T_{4,1}(f)| &= \frac{1}{110075314176} \\
 &\{2 \times (2)^3 + 100 \times (2) \times (2) \times (2) + 2 \times (2) \times (2) \times (4) + 2570 \times (2) \times (2) \times \\
 &(2) \times (2) + 41472 \times (2) \times (2) + 2156544 \times (2) \times (2) + 22704 \times (4) \times (2) \times (2) + 312 \times (2) \times \\
 &(2) \times (2) + 29 \times (2) \times (2) \times \frac{214}{29} + (663552 + 2 + 4608 \times (4) + 3955 \times 4) \times (2) + 22311936 \times \\
 &(2) \times (2) \times (2) + 902 \times (4) \times (2) \times \frac{1876}{451} + 12 \times (2) \times (2) \times (2) + 104 \times (2) \times (2) \times (6) + \\
 &70616 \times (4) \times (2) \times (2) + 102891 \times (8) \times (2) \times (2) + 2 \times (239616 \times 4) \times 2 + 6 \times (41472 \times 2) \\
 &+ 2 \times (2 \times (12498 \times 2) \times 4) + (5 \times 4 + 43130880 \times 2 + 429981696) \times 2 + 34504704 \times (2) \times \\
 &(2) + 2 \times (633 \times 4) \times 2 + 2 \times (4313088 \times 2) \times \frac{677}{26} + \frac{5122}{47} \times (2 \times (188 \times 2) \times 2) + 2 \times \\
 &(1857024 \times 8) \times 2 + 2 \times (144138 \times 16) \times 2 + 4 \times (859963392 \times 2) + 2 \times (4 \times 2) \times 4 + 2 \times \\
 &(4 \times (517010 \times 2) \times 2) + 2 \times (746496 \times 2) \times 6 + 2 \times (9216 \times 4) \times 4 + 2 \times (2 \times (2608 \times 2)) \\
 &\times (2) + 2 \times (40974336 \times 4) \times 2 + 2 \times (2 \times (40985 \times 4) \times 2) + 2 \times ((1106804736 \times 2) \times 2) + \\
 &(4 \times 4 + 1990656 \times 2 + 13759414272) \times 4 + 23606 \times (4) \times (2) \times (2) + 3958 \times (4) \times (6) + \\
 &41803776 \times (2) \times \frac{130}{63} + 429981696 \times \frac{146}{9} + 9216 \times (8) \times (2) + 32 \times (2) \times (2) \times \frac{9}{4} + 2 \\
 &\times (16) \times (2) + 8334 \times (4) \times (2) \times \frac{114344}{4167} + 663552 \times (6) + 4608 \times (2) \times (4) + 398537 \\
 &\times (16) + 5308416 \times (16) + 110075314176 + 416047104 \times (4) + 324 \times (2) \times (2) \times (2) + \\
 &(4) + 18081792 \times (2) \times (4) + 2889216 \times (4) \times (4) + 2156544 \times (2) \times (2)\}.
 \end{aligned} \tag{49}$$

This reduces to

$$|T_{4,1}(f)| \leq \frac{192550832256}{110075314176} = \frac{501434459}{286654464} \approx 1.7493\dots, \tag{50}$$

which completes the proof. \square

2.2. *Bounds of $|H_{4,1}|$ for the Set $C_s^{(3)}$.* Let $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. Rotation of a domain \mathcal{D} about the origin through an angle of $2\pi/n$ containing \mathcal{D} onto itself is said to be n -fold symmetric. An analytic function f is n -fold symmetric in D if

$$f(e^{2\pi i/n} z) = e^{2\pi i/n} f(z) \tag{51}$$

holds for any $z \in D$. Denote $S^{(n)}$ as the set of n -fold univalent functions which have the following Taylor series form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{nk+1} z^{nk+1} \quad (z \in D). \tag{52}$$

Denote $C^{(n)}$ as the subfamily of $S^{(n)}$ of n -fold symmetric convex functions. We can see that an analytic function f of form (52) belongs to the family $C^{(n)}$, if and only if

$$\frac{(zf'(z))'}{f'(z)} = p(z), \tag{53}$$

where $p \in P^{(n)}$. The family $P^{(n)}$ is defined as

$$P^{(n)} = \left\{ p \in P: p(z) = 1 + \sum_{k=1}^{\infty} c_{nk} z^{nk}, (z \in D) \right\}. \tag{54}$$

Now, consider the following.

Theorem 3. *Let $f \in C_s^{(3)}$ be of form (52). Then,*

$$|H_{4,1}(f)| \leq \frac{1}{6048} \approx 0.00016534. \tag{55}$$

Proof. Let $f \in C_s^{(3)}$ of form (52). Consider the function $p \in P^{(3)}$ as

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_3z^3 + c_6z^6 + c_9z^9 + \dots \quad (56)$$

Now,

$$w(z) = \frac{p(z) - 1}{p(z) + 1} \quad (57)$$

The class $C_s^{(3)}$ which is associated with the sine functions can be written in the following form:

$$\frac{(zf'(z))}{f'(z)} = 1 + \sin(w(z)), \quad (z \in D). \quad (58)$$

By expanding and equating them, we get the following expression:

$$1 + 12a_4z^3 + (42a_7 - 48a_4^2)z^6 + \dots = 1 + \frac{1}{2}c_3z^3 + \left(\frac{1}{2}c_6 - \frac{1}{4}c_3^2\right)z^6 + \dots \quad (59)$$

This implies

$$a_4 = \frac{1}{24}c_3, \quad (60)$$

$$a_7 = -\frac{1}{252}c_3^2 + \frac{1}{84}c_6.$$

By using these coefficients, we can get $H_{3,1}(f)$, R_1, R_2 , and R_3 as

$$H_{3,1} = -\frac{1}{576}c_3^2, \quad (61)$$

$$R_1 = 0, \quad R_2 = 0,$$

$$R_3 = -\frac{1}{13824}c_3^3.$$

By using these values in $H_{4,1}(f)$, we get

$$H_{4,1}(f) = a_7H_{3,1}(f) - a_6R_1 + a_5R_2 - a_4R_3$$

$$= \left(-\frac{1}{252}c_3^2 + \frac{1}{84}c_6\right)\left(-\frac{1}{576}c_3^2\right) - \left(\frac{1}{24}c_3\right)\left(-\frac{1}{13824}c_3^3\right)$$

$$= \frac{23}{2322432}c_3^4 - \frac{1}{48384}c_3^2c_6$$

$$= \frac{1}{48384}c_3^2\left(c_6 - \frac{23}{48}c_3^2\right). \quad (62)$$

The triangle inequality and the application of Lemma 1 lead us to

$$|H_{4,1}(f)| = \frac{1}{6048} \approx 0.00016534, \quad (63)$$

which completes the proof. □

3. Conclusion

In this paper, we have found the upper bounds of fourth-order Hankel and Toeplitz determinants, followed by a review of such findings obtained so far for certain analytic functions. We have studied them for the convex functions associated with the function $1 + \sin z$. A similar bound of the fourth-order Hankel determinant for 3-fold symmetric convex functions associated with $1 + \sin z$ has also been investigated.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors' Contributions

All authors contributed equally to this study and approved the final manuscript.

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