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Research Article

Residual Division Graph of Lattice Modules

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Let L be a multiplicative lattice and M be a lattice module over L. In this paper, we assign a graph to M called *residual division graph RG(M)* in which the element $N \in M$ is a vertex if there exists $0_M \neq P \in M$ such that $NP = 0_M$ and two vertices N_1, N_2 are adjacent if $N_1N_2 = 0_M$ (where $N_1N_2 = (N_1\colon I_M)(N_2\colon I_M)I_M$). It is proved that such a graph with the greatest element I_M which does not belong to the vertex set is nonempty if and only if M is a prime lattice module. Also, we provide conditions such that $RG(M)$ is isomorphic to a subgraph of Zariski topology graph $\mathcal{G}_X(M)$ with respect to X.

1. Introduction

In our everyday life, we found that numerous issues are dealt with the assistance of graphs. Extraordinarily, the idea of the coloring of graphs assumes a significant function in computer sciences. To examine the coloring of rings, I. Beck first presented the zero-divisor graphs of a commutative ring with unity (see [1]). This examination of the coloring of a commutative ring was further studied by Anderson and Naseer (see [2]).

The ring structure is firmly associated with ideals more than elements and so it has the right to present a graph with vertices as ideals instead of elements. In this direction, M. Behboodi et al. studied the annihilating-ideal graph $AG(R)$ with vertex set $V(AG(R))$ contain all those ideals of ring R whose annihilators are nonzero(see [3, 4]). Thereafter, Ansari-Toroghy et al. expanded this work for R-module M, where R is a commutative ring. They studied the algebraic as well as topological properties of M with the help of annihilating-submodule graph and the *Zariski topology* graph (see [5]). Recently, the idea of the zero divisor has likewise been applied in Boolean algebra, poset, and lattices (see [6, 7, 78]).

A complete lattice $L = (L, \vee, \wedge, 0_L, 1_L)$ is a *multiplicative* if there is a defined binary operation called multiplication, denoted by "$\cdot$", which is commutative and associative such that the greatest element 1_L works as the multiplicative identity and for an arbitrary index set I $a.(\vee_{\alpha\in I} b_\alpha) = \vee_{\alpha\in I} a.b_\alpha$, where $a, b_{\alpha\in L}$ and 0_L is the least element of L. It is interesting to note that the purpose behind the development of multiplicative lattice is to generalize lattices of ring ideals (see [9, 10]). Also, it is observed that the annihilating-ideal graph of a commutative ring R with unity has close ties with a multiplicative lattice of ideals of R. As far as the study of Johnson [11] is concerned, a lattice module is just an extension of a multiplicative lattice. It becomes worthy to study the graph $AG(R)$ over a commutative ring R with unity with the help of a lattice module M over L.

Definition 1 (see [12]). A *lattice module M* over the multiplicative lattice L is a complete lattice with least element 0_M and greatest element I_M if a multiplication between elements of L and M, represented by $xN \in M$, where $x \in L$ and $N \in M$, which satisfies the following properties:

(1) $(xy)N = x(yN)$
(2) $(\vee_\alpha x_\alpha)(\vee_\beta N_\beta) = (\vee_{\alpha\beta} x_\alpha N_\beta)$
(3) $1_L N = N$
(4) $0_L N = 0_M$, for all $x, y, x_\alpha \in L$ and for all $N, N_\beta \in M$

Note that, for $P, Q \in M$ and $x \in L$, we define $(P: Q) = \vee\{a \in L \mid aQ \leq P\}$ and $(P: x) = \vee\{K \in M | xK \leq P\}$. Here, the operation "$:$" is called *residual division* (see [12]). Also, note that $A, C \in M$ with $A \leq C$; the interval $\{B \in M | A \leq B \leq C\}$ which is denoted by C/A is a lattice module over a multiplicative lattice L with the multiplication $a \cdot B = aB \vee A$, where $a \in L$ (see [12]).

Furthermore, for more definitions and concept of lattice modules and multiplicative lattice, see [9–19].

The semicomplement graph $\Gamma(M)$ of lattice module M introduced and investigated by Phadatare et al. (see [20]). In the recent paper [5], Ansari-Toroghy and Habibi have highlighted the closed sets in Zariski topology on prime spectrum Spec (M) of R-module M and defined new graph called *Zariski topology graph* $G(\tau_T)$, where $T \subseteq Spec(M)$ which is nonempty. They studied the relationship between $G(\tau_T)$ and annihilating-submodule graph $AG(M/\cap(T))$ (see [5]). Thereafter, this graph $G(\tau_T)$ generalized to lattice modules M over a C-lattice L and the *Zariski topology graph* $Ģ_X(M)$ was studied (see [21]).

Throughout the paper, M denotes a lattice module over a multiplicative lattice L, and for $N, K \in M$, we define $NK = (N: I_M)(K: I_M)I_M$.

The aim of this paper is to generalize the annihilating-submodule graph of a module to the lattice module M over a multiplicative lattice L and introduce the *residual division graph* $RG(M)$ whose vertex set is $\{N \in M |$there exists $0_M \neq K \in M$ such that $NK = 0_M\}$ and in $RG(M)$; two vertices A, B are adjacent if and only if $AB = 0_M$. Apart from this, we will investigate interrelationship between $Ģ_X(M)$ and $RG(M/\wedge(X))$, where $\wedge(X)$ is the meet of all elements in X.

Throughout the paper, M denotes a lattice module over a multiplicative lattice L.

2. Some Graph Theoretic Notions

We consider only undirected graphs. Thus, we adopt the notation $G = (V, E)$, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges of G. A graph G is an empty if $|V(G)| = \emptyset$. The number of edges incident on a vertex v is called a degree of vertex v and it is denoted by $d(v)$. A graph G is said to be *k-regular* if the degree of each vertex in G is k. Distance between the vertices a and b is the length of the shortest path between them, which is denoted by $d(a, b)$. Consider $d(a, b) = \infty$ if there is no path between a and b. diam $(G) = sup\{d(a, c) | a, c \in V(G)\}$ is the diameter of a graph G. Length of shortest cycle in G is called the *girth* of G, denoted by $gr(G)$. A *clique* of graph G is its maximal complete subgraph, and the minimum number of cliques required to cover all the vertices of graph G is called the *partition number*, denoted by $\theta(G)$. In a graph G, a subset $S \subseteq V(G)$ is supposed to be independent if no two vertices in S are adjacent. The size of maximum independent set in a graph G called as independence number, denoted by $\alpha(G)$. For a vertex $x \in V(G)$, $\Gamma_G(x) = \{y | xy \in E(G)\}$ denotes the set of all neighbors of x' in G. A graph G is said to be *perfect* if $\theta(H) = \alpha(H)$, for every induced subgraph H of G. Strongly perfect and very strongly perfect graphs are the classes of the perfect graph.

For further information, the reader may refer [22, 23].

3. Residual Division Graph $RG(M)$

Definition 2. The *residual division* graph $RG(M)$ of M is a graph with vertices $V(RG(M)) = \{N \in M |$ there exists $0_M \neq K \in M$ such that $NK = 0_M\}$, where distinct vertices P and Q are adjacent if and only if $PQ = 0_M$.

Example 1. Figure 1 represents the residual division graph $RG(M)$ of M with the vertex set $V(RG(M)) = \{0_M, A, B, C, Q\}$, where M represents lattice module over L (see Figure 2 and 3).

We essentially need the following two Lemmas throughout this article.

Lemma 1 (see [12]). *For $x \in L$ and $P, Q, R \in M$, the following holds:*

(1) If $P \leq Q$, then $(P: R) \leq (Q: R)$

(2) $(Q \wedge R: P) = (Q: P) \wedge (R: P)$

(3) $x \leq (xP: P)$

(4) $x(P: x) \leq P$

(5) $(P: Q)Q \leq P$

(6) If $P \leq Q$, then $(R: Q) \leq (R: P)$

Lemma 2 (see [12]). *For $N \in M$, $((N: I_M)I_M: I_M) = (N: I_M)$.*

The following lemma gives a condition under which a nonzero proper element of lattice module M is a vertex of $RG(M)$.

Lemma 3. *Let $0_M \neq N$ be a proper element of M. Then, $N \in V(RG(M))$ if $(0_M: N) \neq (0_M: I_M)$ or $(0_M: (N: I_M)) \neq 0_M$.*

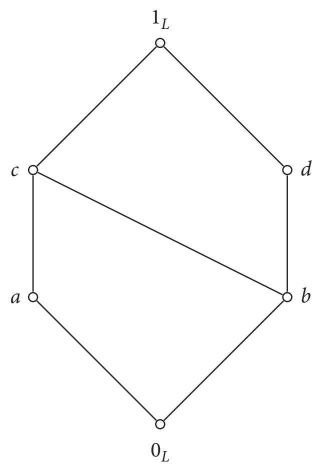
Proof. Suppose that $0_M \neq N$ is a proper element of M and $(0_M:N) \neq (0_M:I_M)$. Then, $(0_M:N)I_M \neq 0_M$. Now, let $K = (0_M: N)I_M$. Therefore, $NK = (N:I_M)((0_M:N)I_M:I_M)I_M \leq (N:I_M)(0_M:N)I_M = (0_M:N)(N:I_M)I_M \leq (0_M:N)\ N = 0_M$. This implies that $NK = 0_M$; consequently, $N \in V(RG(M))$. □

Callialp and Tekir [14] introduced the notion of multiplication lattice modules.

Definition 3 (see [14]). A lattice module M is said to be multiplication lattice module if for each $K \in M$ there exists an element $x \in L$ such that $K = xI_M$.

Lemma 4 (see [14]). *A lattice module M is a multiplication lattice module if and only if $K = (K: I_M)I_M$, for all $K \in M$.*

Lemma 5. *Let $0_M \neq N$ be a proper element of multiplication lattice module M. Then, $N \in V(RG(M))$ if and only if $(0_M: (N: I_M)) \neq 0_M$.*



.	0_L	a	b	c	d	1_L
0_L	0_L	0_L	0_L	0_L	0_L	0_L
a	0_L	a	0_L	a	0_L	a
b	0_L	0_L	0_L	0_L	b	b
c	0_L	a	0_L	a	b	c
d	0_L	0_L	b	b	d	d
1_L	0_L	a	b	c	d	1_L

Figure 1: Multiplicative lattice L.

Proof. Suppose that $0_M \neq N \in M$ is a vertex of $RG(M)$. Then, by definition, there exists $0_M \neq K \in M$ such that $NK = 0_M$. Therefore, $(N\colon I_M)(K\colon I_M)I_M = 0_M$. Since M is multiplication, by Lemma 4, $(K\colon I_M)I_M = K$; therefore, $(N\colon I_M)(K\colon I_M)I_M = (N\colon I_M)K = 0_M$. This implies $(0_M\colon (N\colon I_M)) = K$. However, $K \neq 0_M$; therefore, $(0_M\colon (N\colon I_M)) \neq 0_M$. Converse part follows from Lemma 3. □

Co-multiplication lattice module is introduced and characterized by F. Callialp et al. (see [16]).

Definition 4 (see [16]). Lattice module M is called a co-multiplication lattice module if for each $P \in M$, there exists an element $x \in L$ such that $P = (0_M\colon x)$.

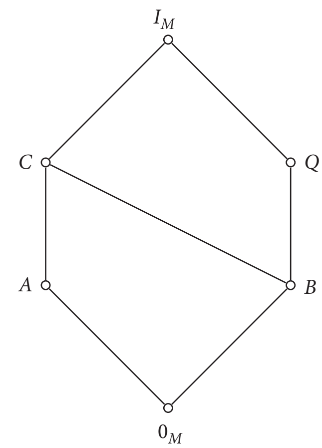
The following characterization plays an important role in the study of residual division graph $RG(M)$.

Lemma 6 (see [16]). *Lattice module M is a co-multiplication if and only if $K = (0_M\colon (0_M\colon K))$ for every element $K \in M$.*

The following theorem is the immediate consequence of Lemma 6.

Theorem 1. *Every proper element $0_M \neq N$ of a co-multiplication lattice module M is a vertex of $RG(M)$.*

Proof. Let $0_M \neq N$ is a proper element of co-multiplication lattice module M. By Lemma 3, to prove $N \in V(RG(M))$, we have to show that $(0_M\colon N) \neq (0_M\colon I_M)$ or $(0_M\colon (N\colon I_M)) \neq 0_M$. Suppose that $(0_M\colon N) = (0_M\colon I_M)$. Then, by Lemma 1(6), $(0_M\colon (0_M\colon N)) = (0_M\colon (0_M\colon I_M))$. Since M is a co-multiplication lattice module over a C-lattice L, by Lemma 6, $N = (0_M\colon (0_M\colon N)) = (0_M\colon (0_M\colon I_M)) = I_M$ which is contradiction to $N < I_M$; consequently, $(0_M\colon N) \neq (0_M\colon I_M)$. □



.	0_M	A	B	C	Q	1_M
0	0_M	0_M	0_M	0_M	0_M	0_M
a	0_M	A	0_M	A	0_M	A
b	0_M	0_M	0_M	0_M	B	B
c	0_M	A	0_M	A	B	C
d	0_M	0_M	B	B	Q	Q
1_L	0_M	A	B	C	Q	1_M

:	0_M	A	B	C	Q	1_M
0_M	1_L	d	c	b	a	0_L
A	1_L	1_L	c	c	a	a
B	1_L	d	1_L	d	c	b
C	1_L	1_L	1_L	c	c	c
Q	1_L	d	1_L	d	1_L	d
1_M	1_L	1_L	1_L	1_L	1_L	1_L

Figure 2: Lattice module M over L.

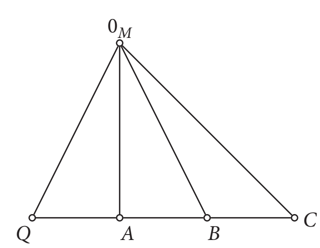


Figure 3: $RG(M)$.

In the above three results, we studied various conditions on lattice module M under which proper elements $0_M \neq N \in M$ becomes a vertex. However, then the natural question arises: can the greatest element I_M be a vertex in residual division graph $RG(M)$?

The following lemma answers the above question.

Lemma 7. *Greatest element I_M of M is a vertex if and only if there exists a proper element $0_M \neq N \in M$ such that $(N\colon I_M) = (0_M\colon I_M)$.*

Proof. Suppose that greatest element I_M of a lattice module M is a vertex. Then, there exists a $0_M \neq P \in M$ such that $PI_M = 0_M$. Therefore, $(P\colon I_M)(I_M\colon I_M)I_M = 0_M$. However, $(I_M\colon I_M)I_M = I_M$; therefore, $(P\colon I_M)I_M = 0_M$, and hence, $(P\colon I_M) = (0_M\colon I_M)$. Conversely, suppose that $(N\colon I_M) = (0_M\colon I_M)$, where N is a proper element of M with $0_M \neq N$. Then, $NI_M = (N\colon I_M)(I_M\colon I_M)I_M = (N\colon I_M)I_M$ because of $(I_M\colon I_M)I_M = I_M$. Since $(N\colon I_M) = (0_M\colon I_M)$, we have $NI_M = (N\colon I_M)I_M = (0_M\colon I_M)I_M \leq 0_M$. By Lemma 1(5), $(0_M\colon I_M)I_M \leq 0_M$; therefore, $NI_M = 0_M$; hence, the greatest element I_M is a vertex. □

Theorem 2. *Let the greatest element I_M of M is not a vertex of $RG(M)$. Then, $0_M \neq N \in M$ is a vertex if and only if $(0_M\colon (N\colon I_M)) \neq 0_M$.*

Proof. Suppose that $0_M \neq N \in M$ is in $V(RG(M))$. Then, there exists $0_M \neq P \in M$ such that $NP = 0_M$. Therefore, $NP = (N\colon I_M)(P\colon I_M)I_M = 0_M$ and so $(P\colon I_M)I_M \leq (0_M\colon (N\colon I_M))$. Since I_M is not a vertex, by Lemma 7, $(P\colon I_M)I_M \neq 0_M$, and hence, $(0_M\colon (N\colon I_M)) \neq 0_M$. Converse follows from Lemma 3. □

In [13], Al-Khouja studied the relationship between the maximal (prime) elements of lattice module M and the maximal (prime) elements of multiplicative lattice L. "If N is a prime element of a lattice module M over a multiplicative lattice L, then $(N\colon I_M)$ is a prime element of multiplicative lattice L (see [13])."

According to Callialp et al. (see [15]), a lattice module M over a multiplicative lattice L is prime if the least element 0_M is prime element of M.

The following characterization is done by Callialp et al. (see [15]).

Lemma 8 (see [15]). *Least element 0_M of M is a prime if and only if $(0_M\colon I_M) = (0_M\colon N)$, for all $0_M \neq N \in M$.*

Lemma 8 helps us to characterize the prime lattice module.

Theorem 3. *Let the greatest element I_M of M is not a vertex of $RG(M)$. Then, $RG(M) = \varnothing$ if and only if M is a prime lattice module.*

Proof. Suppose that $RG(M) = \varnothing$ and M is not a prime lattice module over L. Then, least element 0_M is not a prime element of M. Therefore, by Lemma 8, there exists proper element $0_M \neq N \in M$ such that $(0_M\colon I_M) \neq (0_M\colon N)$, and hence, by Lemma 3, N is in $V(RG(M))$, contradiction to $RG(M) = \varnothing$; consequently, M is prime. Conversely, M is a prime, and there exists $N \in M$ such that $N \in V(RG(M))$. Then, by definition, there exists $0_M \neq K$ in M such that $NK = 0_M$. This implies that $(N\colon I_M)(K\colon I_M)I_M = 0_M$, so $(N\colon I_M)(K\colon I_M) = (0_M\colon I_M)$. Since 0_M is a prime element of M, $(0_M\colon I_M)$ prime element of L. Therefore, $(N\colon I_M) = (0_M\colon I_M)$ or $(K\colon I_M) = (0_M\colon I_M)$. This follows by Lemma 7 that I_M is a vertex, a contradiction. Consequently, $RG(M) = \varnothing$. □

Theorem 4. *For given M, $RG(M)$ is connected and* $\mathrm{diam}\,(RG(M)) \leq 3$.

Proof. Suppose that $P, Q \in V(RG(M))$ such that $P \neq Q$. If $PQ = 0_M$, by definition, we have a path $P - Q$ of length one. Now, suppose that $PQ \neq 0_M$.

Case (1): if $P^2 = PP = 0_M$ and $Q^2 = QQ = 0_M$, then $PPQ = (P\colon I_M)(P\colon I_M)(Q\colon I_M)I_M = 0_M$ and $PQQ = (P\colon I_M)(Q\colon I_M)(Q\colon I_M)I_M = 0_M$. This implies that P is adjacent to PQ and PQ is adjacent to Q, i.e., $P - PQ - Q$ is a path of length equal to 2.

Case (2): if $P^2 = 0_M$ and $Q^2 \neq 0_M$, since $Q \in V(RG(M))$, there exists $0_M \neq Q_1 \in M$ such that $QQ_1 = 0_M$. If $PQ_1 = 0_M$, then we have a path $P - Q_1 - Q$ with $d(P, Q) = 2$. Suppose that $PQ_1 \neq 0_M$. Then, $PPQ_1 = P^2Q_1 = 0_M$, and therefore, $P - PQ_1 - Q$ is a path of length 2 because $QQ_1 = 0_M$. Similarly, if $Q^2 = 0_M$ and $P^2 \neq 0_M$, then we have a path with length 2.

Case (3): if $PQ \neq 0_M$, $P^2 \neq 0_M$, and $Q^2 \neq 0_M$, by definition, there exist $0_M \neq N, K$ such that $PN = 0_M = QK$. If $N = K$, then $P - N = K - Q$ is a path of length 2. Now, suppose that $N \neq K$ and $NK = 0_M$. Since $PN = 0_M = QK$, we have a path $P - N - K - Q$ of length 3, i.e., $d(P, Q) = 3$. Above cases implies that $d(P, Q) \leq 3$; consequently, $\mathit{diam}\,(RG(M)) \leq 3$. □

Corollary 1. *If $RG(M)$ contain a cycle, then $gr\,(RG(M)) \leq 4$.*

In [21], Phadatare et al. introduced the quasi-prime element of M.

Definition 5 (see [21]). A proper element $N \in M$ is said to be *quasi-prime* if $(N\colon I_M)$ is a quasi-prime element of L.

The following lemma follows from Theorem 3.

Lemma 9. *Let the greatest element I_M of M is not a vertex of $RG(M)$. If $RG(M) = \varnothing$, then the least element 0_M is a quasi-prime element of M.*

$Spec^q(M)$ is a collection of all quasi-prime elements of M and for $N \in M$, set $D(N) = \{K \in Spec^q(M) \mid (N\colon I_M) \leq (K\colon I_M)\}$ (see [21]).

We basically need the following lemma.

Lemma 10 (see [21]). *Let M be a lattice module and $Y \subseteq Spec^q(M)$. Then, $Y \subseteq D(N)$ if and only if $(N\colon I_M) \leq (\wedge(Y)\colon I_M)$.*

Phadatare et al. [21] employed closed set $D(N)$ to introduce the Zariski topology graph $\mathcal{G}_X(M)$ with respect to $X \subseteq Spec^q(M)$.

Definition 6. For $X \subseteq Spec^q(M)$, we define an undirected graph $\mathcal{G}_X(M)$ associated with X, called *Zasiski topology graph* with respect to X with vertex set $V(\mathcal{G}_X(M)) = \{N \in M \mid$ there exists $0_M \neq K \in M$ such that $D(N) \cup D\ (K) = X$ and D

$(N) \neq X, D(K) \neq X\}$ and distinct vertices N and K are adjacent if and only if $D(N) \cup D(K) = X$.

Remark 1. By Proposition 3.1 of [21] and Lemma 2, for N, $K \in M$, $D(NK) = D((N: I_M)(K: I_M)I_M) = D(N) \cup D(K)$.

Note that, for $Y \subseteq Spec^q(M)$, an interval $\{K \in M: \wedge(Y) \leq K \leq M\}$ denoted by $M/\wedge(Y) = \overline{M}$ is a lattice module over a multiplicative lattice L with the multiplication $a \cdot K = aK \vee \wedge(Y)$, where $a \in L$.

Theorem 5. *Let M be a lattice module, and $M/\wedge(Y) = \overline{I_M}$ is not a vertex of $RG(M/\wedge(Y))$. Then, $RG(M/\wedge(Y))$ is isomorphic to subgraph of $Ç_Y(M)$.*

Proof. Suppose that $\overline{N} = N/\wedge(Y) \in V(RG(M/\wedge(Y)))$. By definition, there exists $\wedge(Y) \neq K/\wedge(Y) \in M/\wedge(Y)$ such that $N/\wedge(Y)$ is adjacent to $K/\wedge(Y)$. Therefore, $\overline{NK} = N/\wedge(Y)K/\wedge(Y) = (N/\wedge(Y): I_M/\wedge(Y))\ (K/\wedge(Y): I_M/\wedge(Y))I_M/\wedge(Y) = \wedge(Y)$. Note that $(N/\wedge(Y): I_M/\wedge(Y))\ (K/\wedge(Y): I_M/\wedge(Y)) = ((N: I_M)\ (K: I_M)I_M \vee \wedge(Y))/\wedge(Y)$. Therefore, we have $\overline{NK} = N/\wedge(Y)K/\wedge(Y) = ((N: I_M)(K: I_M)I_M \vee \wedge(Y))/\wedge(Y) = \wedge(Y)$. This impies that $NK = (N: I_M)\ (K: I_M)\ I_M \leq \wedge(Y))$; therefore, by Lemma 1(1), $(NK: I_M) \leq (\wedge(Y): I_M)$, and hence, by Lemma 10, $D(NK) = Y$. Since $D(NK) = D(N) \cup D(K)$, we have $Y = D(N) \cup D(K)$. If $D(N) = Y$, then $(N: I_M) = (\wedge(X): I_M)$; therefore, by Lemma 7, $\overline{I_M}$ is a vertex of $RG(M/\wedge(Y))$ which is a contradiction; consequently, $D(N) \neq Y$. In the same line, we have $D(K) \neq Y$. Thus, $N \in V(Ç_Y(M))$ such that N is adjacent to K. □

Corollary 2 (see [21]). *For $X \subseteq Spec^q(M)$, $Ç_X(M)$ is nonempty if and only $X = D(\wedge(X))$ and $\wedge(X) \notin Spec^q(M)$.*

The following is the characterization of nonempty residual division graph $RG(M/\wedge(Y))$.

Theorem 6. *If $M/\wedge(Y) = \overline{I_M}$ is not a vertex of $RG(M/\wedge(Y))$, then $RG(M/\wedge(Y)) = \varnothing$ if and only if $Ç_Y(M) = \varnothing$.*

Proof. Suppose that $\overline{I_M}$ is not a vertex of $RG(M/\wedge(Y))$ and $RG(M/\wedge(Y)) = \varnothing$. Then, by Lemma 9, $\wedge(Y)$ is a quasi-prime element of M; therefore, by Corollary 2, $Ç_Y(M) = \varnothing$. Converse follows from Theorem 6. □

Theorem 7. *The greatest element $M/\wedge(Y) = \overline{I_M}$ is a vertex in $RG(M/\wedge(Y))$ if there exists $N < I_M$ such that $\wedge(Y) < N$ and $D(N) = Y$.*

Proof. Suppose there exists $N < I_M$ such that $\wedge(Y) < N$ and $D(N) = Y$. To prove that $\overline{I_M} \in V(RG(M/\wedge(Y)))$, it suffices to prove that there exists $\wedge(Y) \neq \overline{K}$ such that $\overline{KI_M} = \wedge(Y)$. Note that $\wedge(Y) < N < I_M$. By definition, $\overline{NI_M} = N/\wedge(Y)I_M/\wedge(Y) = (N/\wedge(Y): I_M/\wedge(Y))(I_M/\wedge(Y): I_M/\wedge(Y))\ I_M/\wedge(Y) = (N: I_M)I_M \vee \wedge(Y)/\wedge(Y)$. Since $D(N) = Y$, we have, for all $P \in Y$, $(N: I_M) \leq (P: I_M)$; therefore, $(N: I_M) \leq \wedge_{P\in Y}(P: I_M)$, and hence, by Lemma 1(2), $(N: I_M) \leq (\wedge_{P\in Y}P: I_M)$. This implies that $(N: I_M) \leq (\wedge(Y): I_M)$, i.e., $(N: I_M)I_M \leq \wedge(Y)$; therefore, $(N: I_M)I_M \vee \wedge(Y)/\wedge(Y) = \wedge(Y)$; consequently, $\overline{I_M}$ is a vertex in $RG(M/\wedge(Y))$.

For $N < I_M$, $\sqrt{N}$ represents *q-radical* of N, and it is defined as $\sqrt{N} = \wedge D(N)$. If $\sqrt{N} = N$, then N is said to be *q*-radical element of M.

"If the natural map $\psi: Spec^q(M) \longrightarrow Spec^q(L/(0_M: I_M))$ defined by $\psi(N) = \overline{(N: I_M)}$ surjective, then $\sqrt{(N: I_M)} = (\sqrt{N}: I_M)$ (see [24])." □

Theorem 8. *If the natural map ψ is surjective and A, B are adjacent vertices in $Ç_Y(M)$. Then, $\sqrt{A}/\wedge(Y)$ and $\sqrt{B}/\wedge(Y)$ are adjacent in $RG(M/\wedge(Y))$.*

Proof. To show that $\sqrt{A}/\wedge(Y)$ and $\sqrt{B}/\wedge(Y)$ are adjacent in $RG(M/\wedge(Y))$, we have to prove that $\sqrt{(A: I_M)I_M}/\wedge(Y) \neq \wedge(Y)$ and $\sqrt{(B: I_M)I_M}/\wedge(Y) \neq \wedge(Y)$ with $\sqrt{(A: I_M)I_M}/\wedge(Y)\ \sqrt{(B: I_M)I_M}/\wedge(Y) = \wedge(Y)$. Suppose that A, B are adjacent vertices in $Ç_Y(M)$. Then, we have $D(A) \cup D(B) = Y$ with $D(A), D(B) \neq Y$. However, $D(A) \cup D(B) = D((A: I_M)\ (B: I_M)I_M)$; therefore, $Y = D((A: I_M)\ (B: I_M)I_M)$, and hence, $\wedge(Y) = \wedge D((A:I_M)(B:I_M)I_M) = \sqrt{(A:I_M)(B:I_M)I_M} \leq \sqrt{(A:I_M)I_M} \cap \sqrt{(B:I_M)I_M}$. Thus, $\wedge(Y) \leq \sqrt{(A:I_M)I_M} \cap \sqrt{(B:I_M)I_M}$. Also, note that $\sqrt{(A: I_M)I_M}/\wedge(Y)\ \sqrt{(B: I_M)}\ I_M/\wedge(Y) = (\sqrt{(A: I_M)I_M}/\wedge(Y): I_M/\wedge(Y))\ (\sqrt{(B: I_M)I_M}/\wedge(Y): I_M/\wedge(Y))I_M/\wedge(Y) = ((\sqrt{(A: I_M)I_M}: I_M)\ (\sqrt{(B: I_M)I_M}: I_M))I_M \vee \wedge(Y)/\wedge(Y)$. Since ψ is surjective and $D(N) = D((N: I_M)I_M)$, we have $(\sqrt{(A: I_M)I_M}: I_M) = (\sqrt{A}: I_M) = \sqrt{(A: I_M)}$; therefore, $\sqrt{(A: I_M)I_M}/\wedge(Y)\sqrt{(B: I_M)I_M}/\wedge(Y) = ((\sqrt{(A: I_M)}\ I_M: I_M)\ (\sqrt{(B: I_M)I_M}: I_M))I_M \vee \wedge(Y)/\wedge(Y) = \sqrt{(A: I_M)}\ \sqrt{(B: I_M)}I_M \vee I_M/\wedge(Y)$. However, $\sqrt{(A: I_M)}\ \sqrt{(B: I_M)}I_M \leq \sqrt{(A: I_M)(B: I_M)}I_M \leq \sqrt{(A: I_M)(B: I_M)I_M} = \sqrt{AB} = \wedge Y)$; therefore, $\sqrt{(A: I_M)I_M}/\wedge(Y)\ \sqrt{(B: I_M)I_M}/\wedge(Y) = \sqrt{(A: I_M)}\ \sqrt{(B: I_M)}I_M \vee I_M/\wedge(Y) = \wedge(Y)$. Now, it remains to prove that $\sqrt{(A: I_M)I_M}/\wedge(Y) \neq \wedge(Y)$ and $\sqrt{(A: I_M)I_M}/\wedge(Y) \neq \wedge(Y)$. If $\sqrt{(A: I_M)I_M}/\wedge(Y) = \wedge(Y)$, then $\sqrt{(A: I_M)I_M} = \wedge(Y)$. Therefore, $(A: I_M)I_M \leq \sqrt{(A: I_M)I_M} = \wedge(Y)$, and hence, $(A: I_M) \leq (\wedge(Y): I_M)$. By Lemma 10, $D(N) = Y$, a contradiction. Consequently, $\sqrt{(A: I_M)I_M}/\wedge(Y) \neq \wedge(Y)$. Similarly, we have $\sqrt{(B: I_M)I_M}/\wedge(Y) \neq \wedge(Y)$. □

4. Conclusion

In this paper, we introduced the residual division graph of the lattice module and characterized the co-multiplication lattice module and prime lattice module. Also, we characterized vertex set of residual division graph of the multiplication lattice module. We found that the residual division graph of an interval lattice module is isomorphic with the Zariski topology graph.

Data Availability

The data from previous studies were used to support this study. They are cited at relevant places within the article as references.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] I. Beck, "Coloring of commutative rings," *Journal of Algebra*, vol. 116, no. 1, pp. 208–226, 1988.
[2] D. D. Anderson and M. Naseer, "Beck′s coloring of a commutative ring," *Journal of Algebra*, vol. 159, no. 2, pp. 500–514, 1993.
[3] M. Behboodi and Z. Rakeei, "The annihilating-ideal graph of commutative rings-I," *Journal of Algebra and Its Applications*, vol. 10, no. 4, pp. 727–739, 2011.
[4] M. Behboodi and Z. Rakeei Z, "The annihilating-ideal graph of commutative rings-II," *Journal of Algebra and Its Applications*, vol. 10, no. 4, pp. 741–753, 2011.
[5] H. Ansari-Toroghy and S. Habibi, "The zariski topology-graph of modules over commutative rings," *Communications in Algebra*, vol. 42, no. 8, pp. 3283–3296, 2014.
[6] D. F. Anderson, R. Levy, and J. Shapiro, "Zero-divisor graphs, von Neumann regular rings, and Boolean algebras," *Journal of Pure and Applied Algebra*, vol. 180, no. 3, pp. 221–241, 2003.
[7] E. Estaji and K. Khashyarmanesh, "The zero-divisor graph of a lattice," *Results in Mathematics*, vol. 61, no. 1-2, pp. 1–11, 2012.
[8] R. Halaš and M. Jukl, "On Beck's coloring of posets," *Discrete Mathematics*, vol. 309, pp. 4584–4589, 2009.
[9] R. P. Dilworth, "Abstract commutative ideal theory," *Pacific Journal of Mathematics*, vol. 12, no. 2, pp. 481–498, 1962.
[10] M. Ward and R. P. Dilworth, "Residuated lattices," *Transactions of the American Mathematical Society*, vol. 45, no. 3, pp. 335–354, 1939.
[11] E. W. Johnson, *A-Transforms of Noether Lattices*, Ph.D. Dissertation, University of California, Los Angeles, CA, USA, 1966.
[12] J. A. Johnson, "a-adic completions of Noetherian lattice modules," *Fundamenta Mathematicae*, vol. 66, pp. 341–371, 1970.
[13] E. A. AL-Khouja, "Maximal elements and prime elements in lattice modules," *Damascus University Journal of Basic Sciences*, vol. 19, pp. 9–20, 2003.
[14] F. Callialp and U. Tekir, "*Multiplication lattice modules,* Iran," *Iranian Journal of Science and Technology, Transaction A, Science*, vol. 4, pp. 309–313, 2011.
[15] F. Callialp, U. Tekir, and E. AslanKarayigit, "On multiplication lattice modules," *Hacettepe Journal of Mathematics and Statistics*, vol. 43, no. 4, pp. 571–579, 2014.
[16] F. Callialp, U. Tekir, and G. Ulucak, "*Comultiplication lattice modules,* Iran," *Iranian Journal of Science and Technology, Transaction*, vol. A2, no. 39, pp. 213–220, 2015.
[17] R. H. Lee, "Covers and associated primes in noetherian lattice modules," *Houston Journal of Mathematics*, vol. 5, no. 1, pp. 219–239, 1979.
[18] N. K. Thakare and C. S. Manjarekar, "Abstract spectral theory: multiplicative lattices in which every character is contained in a unique maximal character," in *Algebra and its Applications*, pp. 265–276, Marcel Dekker, New York, USA, 1984.
[19] N. K. Thakare, C. S. Manjarekar, and S. Maeda, "Abstract spectral theory II: minimal characters and minimal spectrums of multiplicative lattices," *Acta Scientiarum Mathematicarum*, vol. 52, pp. 53–67, 1988.
[20] N. Phadatare, V. Kharat, and S. Ballal, "Semi-complement graph of lattice modules," *Soft Computing*, vol. 23, no. 12, pp. 3973–3978, 2019.
[21] N. Phadatare, V. Kharat, and S. Ballal, *Zariski Topology Graph of Quasi-Prime Spectrum of Lattice Modules*.
[22] F. Harary, *Graph Theory*, Narosa, New Delhi, India, 1988.
[23] D. B. West, *Introduction to Graph Theory*, Prentice-Hall of India, New Delhi, India, 2nd edition, 2002.
[24] N. Phadatare, S. Ballal, and V. Kharat, "On the maximal spectrum of lattice modules," *Southeast Asian Bulletin of Mathematics*, vol. 44, pp. 105–117, 2020.