Research Article

Inverse Eigenvalue Problem and Least-Squares Problem for Skew-Hermitian \{P, k + 1\}-Reflexive Matrices

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This paper involves related inverse eigenvalue problem and least-squares problem of skew-Hermitian \{P, k + 1\}-reflexive(antireflexive) matrices and their optimal approximation problems. The above problems are studied by converting them into two simpler cases: \(k = 1\) and \(k = 2\). Firstly, with some special properties of skew-Hermitian \{P, k + 1\}-reflexive(antireflexive) matrices, the necessary and sufficient conditions for the solvability and the general solution are presented, and the solution of corresponding optimal approximation problems also given, respectively. Then, we give the least-squares solution of \(AX = B\) satisfying the special condition by the singular value decomposition. Finally, we give an algorithm and an example to illustrate our results.

1. Introduction

Throughout this paper, \(C^{n \times m}_{\text{scm}}\) and \(SHC^{n \times m}_{\text{scm}}\) stand for the sets of all \(n \times m\) complex matrices and \(n \times n\) skew-Hermitian matrices, respectively. The symbols \(A^H\), \(\|A\|\), \(r(A)\), and \(A^*\) stand for the conjugate transpose, the Frobenius norm, the rank, and the Moore-Penrose inverse of a matrix \(A \in C^{n \times m}_{\text{scm}}\), respectively. Let \(I_k\) be identity matrix of size \(k\), and \(J_k\) cross-identity matrix of size \(k\) having the elements 1 along the southwest-northeast diagonal and the remaining elements being zeros. Symbol \(O_{m \times n}\) is the \(m \times n\) matrix of all zeros entries (if no confusion occurs, we will omit the subscript).

For two matrices \(A, B, \ldots\), the inner product is defined by \(\langle A, B \rangle = tr(B^H A)\). Then, it is apparent that \(C^{n \times m}_{\text{scm}}\) is a Hilbert inner product space and the norm that is generated by this inner product is the Frobenius norm.

A matrix \(P \in C^{n \times m}_{\text{scm}}\) is called Hermitian and \(\{k + 1\}\)-potent matrix if \(P^{k+1} = P = P^H\).

**Definition 1.** Let \(P \in C^{n \times n}_{\text{scm}} \in C^{n \times m}_{\text{scm}}\) be two Hermitian and \(\{k + 1\}\)-potent matrices, we say that matrix \(A \in C^{n \times m}_{\text{scm}}\) is \(\{P, Q, k + 1\}\)-reflexive(antireflexive) if \(A = PAQ (A = -PAQ)\).

**Definition 2.** Let \(P \in C^{n \times n}_{\text{scm}}\) be a Hermitian and \(\{k + 1\}\)-potent matrix, we say that matrix \(A \in C^{n \times n}_{\text{scm}}\) is skew-Hermitian \(\{P, k + 1\}\)-reflexive(antireflexive) if \(A \in SHC^{n \times n}_{\text{scm}}, A = PAP (A = -PAP)\).

The (skew-)Hermitian \(\{P, k + 1\}\)-reflexive(antireflexive) matrices, \(\{P, k + 1\}\)-reflexive(antireflexive) matrices can be seen as a particular case of \(\{P, Q, k + 1\}\)-reflexive(antireflexive) matrices. These kind of matrices have many special properties and have been discussed widely. In [1], Li and Wang obtained the \(\{P, Q, k + 1\}\)-reflexive solutions to a system of matrix equations. In [2], Liang et al. discussed the \(\{P, Q, k + 1\}\)-reflexive solution of matrix equation \(AXB = C\), the general solution was given by using Moore–Penrose inverse and the associated optimal approximation problem was considered. In [3, 4], Cao et al. considered the \(\{P, Q, k + 1\}\)-reflexive solutions and \(\{P, k + 1\}\)-reflexive solutions of \(AX = C, XB = D\), respectively. In [5, 6], Shen and Yu et al. studied Hermitian \(\{P, k + 1\}\) (anti)-reflexive solutions of a linear matrix equation of \(AX = B\).

There are still some other properties worth exploring about skew-Hermitian \(\{P, k + 1\}\)-reflexive(antireflexive)
matrices, in this paper, we will discuss the following three problems:

**Problem 1** (inverse eigenvalue problem). Given \( X = [ x_1, x_2, \ldots, x_n ] \in \mathbb{C}^{m \times n} \) and \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{C}^{m \times m} \), find skew-Hermitian \( \{ P, k+1 \} \)-reflexive(antireflexive) matrix \( A \) such that
\[
AX = X\Lambda.
\] (1)

**Problem 2** (optimal approximation problem). Given \( A^* \in \mathbb{C}^{m \times n} \), find \( X \in \mathbb{C}^{m \times n} \) such that
\[
\| A^* - A^* X \| = \min_{X \in \mathbb{C}^{m \times n}} \| A - A^* \|.
\] (2)

where \( \mathbb{C}^{m \times n} \) is the solution set of Problem 1.

In the first problem, it is clear that \( \Lambda^H = -\Lambda \). The inverse problem and the approximation problem have been discussed extensively. For reflexive and antireflexive matrices, bisymmetric matrices, symmetric or ortho-symmetric matrices, skew-Hermitian reflexive and antireflexive matrices, and the inverse eigenvalue problem were discussed in [7–13], respectively. As for the left and right inverse eigenvalues of reflexive and antireflexive matrices, it was also considered in [14]. But the kinds of skew-Hermitian \( \{ P, k+1 \} \)-reflexive(antireflexive) matrices have not been discussed, which is the motivation of this work.

**Problem 3** (least-squares problem). Given \( A, B \in \mathbb{C}^{m \times n} \), find skew-Hermitian \( \{ P, k+1 \} \)-reflexive(antireflexive) matrix \( X \) such that
\[
\| AX - B \| = \min X \in \mathbb{C}^{m \times n}.
\] (3)

For the matrix equation \( AX = B \), the Hermitian reflexive, antireflexive, generalized reflexive, and antireflexive solutions have been discussed in [15, 16]. And different solution sets for other equations have also been discussed in [17–20]. Besides, the optimization problems of different matrices have also been studied in [21–24]. To our knowledge, so far, there are relatively few research on the least-squares skew-Hermitian \( \{ P, k+1 \} \)-reflexive and anti-reflexive solutions of \( AX = B \), in this paper, we will explore this problem.

### 2. Properties

In this section, we will give some properties of skew-Hermitian \( \{ P, k+1 \} \)-reflexive(antireflexive) matrices.

**Lemma 1** (Lemma 2.1 in [2]). Let \( P \in \mathbb{C}^{m \times n} \) be the Hermitian, then \( P \) is \( k+1 \)-potent matrix if and only if \( P \) is idempotent (i.e., \( P^2 = P \)) when \( k \) is odd, or tripotent (i.e., \( P^3 = P \)) when \( k \) is even. Moreover, there exist \( U \in \mathbb{U}^{m \times n} \) such that
\[
P = U \begin{bmatrix} I_p & 0 \\ 0 & O \end{bmatrix} U^H,
\] (4)
or if \( k \) is even,
\[
P = U \begin{bmatrix} I_p & -I_{p-r} \\ O & O \end{bmatrix} U^H,
\] (5)
where \( p = r(P) \).

Throughout this paper, we always assume the Hermitian and \( \{ k+1 \} \)-potent matrix \( P \) is fixed, which is given by (4) and (5).

From Lemma 1, we can see that the general \( \{ k+1 \} \)-potent matrix \( P \) can be reduced to only two simpler cases: \( P^2 = P \) (i.e., \( k = 1 \)) and \( P^3 = P \) (i.e., \( k = 2 \)). Consequently, we will discuss our problems through \( \{ P, 2 \} \)-reflexive(antireflexive) and \( \{ P, 3 \} \)-reflexive(antireflexive) constraints.

**Lemma 2.** \( X \in \mathbb{C}^{m \times n} \) is skew-Hermitian \( \{ P, 2 \} \)-reflexive if and only if
\[
X = U \begin{bmatrix} X_{11} & 0 \\ 0 & X_{12} \end{bmatrix} U^H, \]
(6)
where \( X_{11} \in \mathbb{SHC}^{p \times p} \).

**Proof.** If \( X \) is skew-Hermitian \( \{ P, 2 \} \)-reflexive, by Definition 1 and (4), it follows that
\[
X = PXP = U \begin{bmatrix} I_p & 0 \\ 0 & O \end{bmatrix} U^H X U \begin{bmatrix} I_p & 0 \\ 0 & O \end{bmatrix} U^H.
\] (7)

Let
\[
U^H X U = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix},
\] (8)
then (7) becomes
\[
U^H X U = U \begin{bmatrix} I_p & 0 \\ 0 & O \end{bmatrix} U^H X U \begin{bmatrix} I_p & 0 \\ 0 & O \end{bmatrix} U^H U,
\] (9)
i.e.,
\[
\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & O \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} I_p & 0 \\ O & O \end{bmatrix},
\] (10)
yielding \( X_{12} = 0, X_{21} = 0, \) and \( X_{22} = 0 \). So, (7) becomes (6).

Conversely, if \( X \) can be expressed as the form (6), then \( X^H = -X \) and \( PXP = X \), so \( X \) is skew-Hermitian \( \{ P, 2 \} \)-reflexive matrix.

**Lemma 3.** \( X \in \mathbb{C}^{m \times n} \) is skew-Hermitian \( \{ P, 3 \} \)-reflexive if and only if
\[
X = U \begin{bmatrix} X_{11} & 0 \\ X_{22} & 0 \end{bmatrix} U^H,
\] (11)
where \( X_{11} \in \mathbb{SHC}^{r \times r} \) and \( X_{22} \in \mathbb{SHC}^{(p-r) \times (p-r)} \).

The proof of it is similar to Lemma 2.
In this case, the matrix equation $AX$ can be expressed as

$$X = U \begin{bmatrix} O & X_{12} & O \\ -X_{12}^H & O & O \\ O & O & O \end{bmatrix} U^H,$$  
(12)

where $X_{12} \in C^{r 	imes (p-r)}$.

**Proof.** If $X$ is skew-Hermitian $\{P,3\}$-antireflexive, by Definition 1 and (5), it follows that

$$-X = PXP = U \begin{bmatrix} I_r \\ -I_{p-r} \\ O \end{bmatrix} U^H X U \begin{bmatrix} I_r \\ -I_{p-r} \\ O \end{bmatrix} U^H,$$  
(13)

Let

$$U^H X U = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix},$$  
(14)

then (13) becomes

$$-U^H X U = U^H \begin{bmatrix} I_r \\ -I_{p-r} \\ O \end{bmatrix} U^H X U \begin{bmatrix} I_r \\ -I_{p-r} \\ O \end{bmatrix} U^H,$$  
(15)

i.e.,

$$-U^H X U = \begin{bmatrix} I_r \\ -I_{p-r} \\ O \end{bmatrix} U^H X U \begin{bmatrix} I_r \\ -I_{p-r} \\ O \end{bmatrix},$$  
(16)

yielding $X_{11} = X_{13} = X_{22} = X_{23} = X_{13} = X_{32} = X_{33} = 0$, and

$$X_{21} = U^H U_1 X_1 = -(U_1^H U_1)^H = -X_{12}^H,$$  
so (13) becomes (12).

Conversely, if $X$ can be expressed as the form (12), then $X^H = -X$ and $PXP = -X$, so $X$ is skew-Hermitian $\{P,3\}$-antireflexive matrix. $\square$

**Lemma 5** (Lemma 2.5 in [12]). Given $X, B \in C^{p\times m}$, then $AX = B$ has a solution $A \in SHC^{p\times m}$ if and only if

$$BX^+ X = B, X^+ B = -B^H X,$$  
(17)

and its general solution can be expressed as

$$A = BX^+ - \left(BX^+ \right)^H \left(I_n - XX^+ \right) + \left(I_n - XX^+ \right) G \left(I_n - XX^+ \right),$$  
(18)

where $G$ is arbitrary.

**Lemma 6** (Lemma 12 in [17]). If $A \in C^{m\times n}$, $B \in C^{m\times k}$, $C \in C^{k\times l}$, and $D \in C^{l\times m}$ be known and $X \in C^{m\times k}$ is variable. Then, the following equation is as follows:

$$\begin{bmatrix} AX = B \\ XC = D \end{bmatrix},$$  
(19)

has a common solution if and only if

$$\begin{cases} \text{AA}^T B = B, \\ \text{DC} C = D, \\ \text{BC} = AD. \end{cases}$$  
(20)

Also, the general solution

$$X = A^T B + \left(I_n - A^T A\right) \text{DC}^+ + \left(I_n - A^T A\right) Z \left(I_n - C C^+ \right),$$  
(21)

where $Z$ is arbitrary.

### 3. The Solution of Problem 1

Based on the analysis in previous section, this section solves the inverse eigenvalue problems of skew-Hermitian $\{P,2\}$-reflexive, skew-Hermitian $\{P,3\}$-(anti-)reflexive matrices, respectively. And give the general solutions by the Moore-Penrose inverse.

**Theorem 1.** Given $X = [x_1, x_2, \ldots, x_n] \in C^{m\times n}$, $\Lambda = \text{diag} \ (\lambda_1, \lambda_2, \ldots, \lambda_m)$, and $P$ is $[k + 1]$-potent matrix, let

$$U^H X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$  
(22)

where $X_1 \in C^{p\times m}$ and $X_2 \in C^{(n-p)\times m}$. Then, $AX = XA$ is consistent for $\{P,2\}$-reflexive matrix $A$ if and only if $X_1 \Lambda X_2', X_2 = X_1' \Lambda$ and $X_1'^H X_2 = \Lambda X_1'^H X_1$, and the general solution can be expressed as

$$A = U \begin{bmatrix} X_1 \Lambda X_2' - (X_1 \Lambda X_2')^H (I_p - X_1 X_2') \\ + (I_p - X_1 X_2') G (I_p - X_1 X_2') \end{bmatrix} U^H,$$  
(23)

where $G \in SHC^{p\times p}$ is arbitrary.

**Proof.** Assume $A$ is skew-Hermitian $\{P,2\}$-reflexive matrix is the solution of $AX = XA$. By Lemma 2, there exist $A_{11} \in SHC^{p\times p}$ such that

$$A = U \begin{bmatrix} A_{11} & O \\ O & O \end{bmatrix} U^H,$$  
(24)

and $AX = XA$. Thus, combining with (22), the above matrix equation $AX = XA$ can be expressed as

$$U \begin{bmatrix} A_{11} & O \\ O & O \end{bmatrix} U^H X = XA,$$  
(25)

i.e.,
Problem 1. 

\[ A_{11} O \begin{bmatrix} X_1 \\ O \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} . \] 
(26)

So, it is equivalent to \( A_{11} X_1 = X_1 \Lambda \). By Lemma 5, this equation is consistent if and only if

\[ X_1 \Lambda X_1^H X_1 = X_1 \Lambda. \] 
(27)

Moreover, the general solution is

\[ A_{11} = X_1 \Lambda X_1^H (I_p - X_1 X_1^H) + (I_p - X_1 X_1^H) G (I_p - X_1 X_1^H) \] \( G \in \text{SHC}^{p,p} \) is arbitrary, which implies that (23) holds.

Assume (23) holds. For any \( p \times p \in \text{P} \), is arbitrary, which implies that (23) holds.

(23) hold. Thus, combining with (29), the above matrix equation is equivalent to

\[ \begin{bmatrix} A_{12} X_2 = X_1 \Lambda, \\ -A_{12}^H X_1 = X_2 \Lambda, \end{bmatrix} \] 
(33)

and \( AX = X \Lambda \). Therefore, \( A \) is a solution of Problem 1.

Remark 1. The skew-Hermitian \( \{P,3\}\)-reflexive solution of Problem 1 can be reduced similar to Theorem 1, the conclusion is omitted here.

Theorem 2. Given \( X = [x_1, x_2, \ldots, x_n] \in \mathbb{C}^{m \times n} \), \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \), and \( P \) is positive definite, let

\[ U^H X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} . \] 
(29)

Then, \( AX = X \Lambda \) is consistent for \( \{P,3\}\)-antireflexive matrix \( A \) if and only if

\[ X_1^H (X_1^H)^+ \Lambda X_2 = \Lambda X_2, \] \( X_1 \Lambda X_1^H X_2 = X_1 \Lambda, \) \( X_1^H X_1 \Lambda = \Lambda X_2^H X_2, \) 
(30)

and the general solution can be expressed as

\[ A = U \begin{bmatrix} O & A_{12} & O \\ -A_{12}^H & O & O \end{bmatrix} U^H, \] 
(31)

where \( A_{12} = (X_1^H)^+ \Lambda X_2 + (I - (X_1^H)^+ X_1) X_1 \Lambda X_1 + (I - (X^H)^+ X^H) U (I - X_2 X_2^H) \) and \( U \in \mathbb{C}^{r \times (p-r)} \) is arbitrary.

Proof. Assume \( A \) is skew-Hermitian \( \{P,3\}\)-antireflexive matrix is the solution of \( AX = X \Lambda \). By Lemma 4, there exist \( A_{12} \in \mathbb{C}^{r \times (p-r)} \) such that

\[ A = U \begin{bmatrix} O & A_{12} & O \\ -A_{12}^H & O & O \end{bmatrix} U^H, \] 
(32)

and \( AX = X \Lambda \). Thus, combining with (29), the above matrix equation is equivalent to

\[ \begin{bmatrix} A_{12} X_2 = X_1 \Lambda, \\ -A_{12}^H X_1 = X_2 \Lambda, \end{bmatrix} \] 
(33)

i.e.,

\[ X_1^H X_2 = \Lambda X_2, \] \( A_{12} X_2 = X_1 \Lambda, \) 
(34)

and \( U \in \mathbb{C}^{r \times (p-r)} \) is arbitrary, which implies that (31) holds.

Assume (31) holds. We can obtain \( A^H = -A \) and \( \text{PAP} = -A \). So \( A \) is skew-Hermitian \( \{P,3\}\)-antireflexive matrix. From Lemma 6, there exist unique \( A_{12} \) such that \( X_1^H A_{12} = -(X_1^H) X_1 \Lambda \) and \( A_{12} X_2 = X_1 \Lambda \), which is equivalent to

\[ \begin{bmatrix} O & A_{12} & O \\ -A_{12}^H & O & O \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} X_1 \Lambda \\ X_2 \Lambda \\ X_3 \Lambda \end{bmatrix}, \] 
(37)

i.e., \( AX = X \Lambda \). Therefore, \( A \) is a solution of Problem 1.
4. The Solution of Problem 2

In this section, we will obtain the unique optimal approximation solution for Problem 2.

If the solution set \( \Psi (A) \) of Problem 1 is nonempty, it is easy to verify that \( \Psi (A) \) is a closed convex set. Thus, there exist unique solution of Problem 2.

**Lemma 7** (Lemma 4 in [3]). If \( A \in C^{m \times m} \), \( B \in C^{p \times q} \), \( C \in C^{m \times n} \), \( B^2 = B = B^H \), and \( C^2 = C = C^H \). Then, 
\[
\| A - BAC \| = \min_{G \in C^{m \times n}} \| A - BGC \| \quad \text{if and only if} \quad \| B(A - G)C \| = 0.
\] (38)

In that case,
\[
\min_{G \in C^{m \times n}} \| A - BGC \| = \| BAC - A \|.
\] (39)

**Theorem 3.** If \( X \in C^{m \times m} \), \( A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \in C^{m \times m} \). Let \( \Psi (A) \) be the set of all skew-Hermitian \([P, 2]\)-reflexive solutions to Problem 1 and \( A^* \in C^{m \times n} \) is a given matrix. Then, 
\[
\| A^* - A^* \| = \min_{A \in \Psi (A)} \| A - A^* \| \quad \text{has a unique solution} \quad A^*,
\]
which can be expressed as
\[
A^* = U \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} U^H,
\] (40)
where 
\[
G = X_1 AX_1^H + (X_1 AX_1^H) (I_p - X_1 X_1^*) - A_{11}^\dagger
\]
and 
\[
A_{11}^\dagger = X_1 AX_1^* - (X_1 AX_1^H) (I_r - X_1 X_1^*)
\]
\[
+ (I_r - X_1 X_1^*) G (I_p - X_1 X_1^*),
\]
\[
A_{22}^\dagger = X_2 AX_2^H + (X_2 AX_2^H) (I_p^2 - X_1 X_1^*) - A_{22}^\dagger
\]
\[
+ (I_p^2 - X_1 X_1^*) G (I_p - X_1 X_1^*),
\]
\[
A_{11}^* = U_1^H A_{11}^\dagger U_1,
\]
\[
A_{22}^* = U_2^H A_{22}^\dagger U_2.
\]

Proof. It is easy to verify that the solution set \( \Psi (A) \) is a closed and convex set in matrix space \( C^{m \times n} \) under the Frobenius norm, so Problem 2 has a unique solution. Let 
\[
U^H A^* U = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
\] (41)

From Theorem 1 and the unitary invariance of the Frobenius norm, we have
\[
\| A - A^* \|^2 = \left\| U \begin{bmatrix} A_{11} & O \\ O & O \end{bmatrix} U^H - A^* \right\|^2
\]
\[
= \left\| \begin{bmatrix} A_{11} & O \\ O & O \end{bmatrix} - \left( A^* \right)^H \right\|^2
\] (42)
\[
= \| A_{11} - A_{11}^\dagger \|^2 + \| A_{12} - A_{12}^\dagger \|^2 + \| A_{21} - A_{21}^\dagger \|^2 + \| A_{22} - A_{22}^\dagger \|^2,
\]
where 
\[
A_{11} = X_1 AX_1^H - (X_1 AX_1^H) (I_p - X_1 X_1^*) + (I_p - X_1 X_1^*) G (I_p - X_1 X_1^*).
\]
\[
A_{12} = X_2 AX_2^H - (X_2 AX_2^H) (I_p - X_1 X_1^*) + (I_p - X_1 X_1^*) G (I_p - X_1 X_1^*).
\]
Hence, \( \min_{\Psi (A)} \| A - A^* \|^2 \) is equivalent to the problem
\[
\min \left\| X_1 AX_1^* - (X_1 AX_1^H) (I_p - X_1 X_1^*) + (I_p - X_1 X_1^*) G (I_p - X_1 X_1^*) - A_{11}^\dagger \right\|^2,
\] (43)
\[
\left( X_1 AX_1^* + (X_1 AX_1^H) (I_p - X_1 X_1^*) - A_{11}^\dagger \right)^2.
\] (44)

From Lemma 7, it is equivalent to the problem
\[
\left( I_n - X_1 X_1^* \right) \left[ G - \left( X_1 AX_1^* + (X_1 AX_1^H) \right) \left( I_p - X_1 X_1^* \right) - A_{11}^\dagger \right] \left( I_n - X_1 X_1^* \right) = 0.
\] (45)

Thus, \( G = X_1 AX_1^* + (X_1 AX_1^H) (I_p - X_1 X_1^*) - A_{11}^\dagger \). \( \square \)

The following two theorems obtain the skew-Hermitian \([P, 3]\)-reflexive constraint and skew-Hermitian \([P, 3]\)-anti-reflexive constraint of Problem 1 and the proof of them is similar to that of Theorem 2.

**Theorem 4.** If \( X \in C^{m \times m} \) and \( A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \in C^{m \times m} \). Let \( \Psi (A) \) be the set of all skew-Hermitian \([P, 3]\)-reflexive solutions to Problem 1 and \( A^* \in C^{m \times n} \) is a given matrix. Then, \( \| A^* - A^* \| = \min_{A \in \Psi (A)} \| A - A^* \| \quad \text{has a unique solution} \quad A^* \), which can be expressed as
\[
A^* = U \begin{bmatrix} A_{11}^* & O \\ O & A_{22}^* \end{bmatrix} U^H,
\] (46)

where
\[
A_{11}^* = X_1 AX_1^* - (X_1 AX_1^H) (I_r - X_1 X_1^*)
\]
\[
+ (I_r - X_1 X_1^*) G (I_p - X_1 X_1^*),
\]
\[
A_{22}^* = X_2 AX_2^H + (X_2 AX_2^H) (I_p^2 - X_1 X_1^*) - A_{22}^*,
\]
\[
A_{11}^* = U_1^H A_{11}^\dagger U_1,
\]
\[
A_{22}^* = U_2^H A_{22}^\dagger U_2.
\]

**Theorem 5.** If \( X \in C^{m \times m} \) and \( A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \in C^{m \times m} \). Let \( \Psi (A) \) be the set of all skew-Hermitian \([P, 3]\)-anti-reflexive solutions to Problem 1 and \( A^* \in C^{m \times n} \) is a given matrix. Then, \( \| A^* - A^* \| = \min_{A \in \Psi (A)} \| A - A^* \| \quad \text{has a unique solution} \quad A^* \), which can be expressed as
\[
A^* = U \begin{bmatrix} O & A_{12}^* \\ -A_{12}^H & O \end{bmatrix} U^H,
\] (48)

where
\[
A_{11}^* = X_1 AX_1^* - (X_1 AX_1^H) (I_r - X_1 X_1^*)
\]
\[
+ (I_r - X_1 X_1^*) G (I_p - X_1 X_1^*),
\]
\[
A_{22}^* = X_2 AX_2^H + (X_2 AX_2^H) (I_p^2 - X_1 X_1^*) - A_{22}^\dagger,
\]
\[
A_{11}^* = U_1^H A_{11}^\dagger U_1,
\]
\[
A_{22}^* = U_2^H A_{22}^\dagger U_2.
\]
\[ A_{12} = -\left( X_{11}^T \right)^H (X_2 A)^H + X_1 \Lambda X_2^T \]
\[ + \left( I - (X_{11}^T)^T X_{11}^T \right) X_1 \Lambda X_2 \]
\[ + \left( I - (X_{11}^T)^T X_{11}^T \right) U (I - X_2 X_2^T), \]
\[ U = -\left( X_{11}^T \right)^T (X_2 A)^H + X_1 \Lambda X_2^T \]
\[ + \left( I - (X_{11}^T)^T X_{11}^T \right) X_1 \Lambda X_2 - A_{12}^*, \]
\[ A_{12}^* = U^H A^* U_2. \]

5. The Solution of Problem 3

In this section, by the singular value decomposition, we will consider the least-squares skew-Hermitian \{P,2\}-reflexive and skew-Hermitian \{P,3\}-(anti-)reflexive solution of matrix equation \( AX = B \) and obtain the general expression.

Lemma 8 (Lemma 11 in [4]). Given \( E, F \in \mathbb{C}^{m \times n} \), \( \Sigma_1 = \text{diag}(\sigma_1, \ldots, \sigma_s) \), \( \sigma_i > 0 \), and \( i = 1, 2, \ldots, s \). Then, there exist unique matrix \( S \in \mathbb{C}^{m \times n} \) such that
\[ \| \Omega_1 S - E \|_F^2 + \| \Sigma_2 - F \|_F^2 = \min, \]
and \( S^* \) can be expressed as
\[ S^* = \Phi \ast (\Omega_1 E + F \Omega_2), \]
where \( \Phi = 1/\alpha_i^2 + b_j^2 \).

Lemma 9 (Lemma 2.7 in [6]). Suppose \( G \in \mathbb{C}^{m \times n} \) and \( \Sigma_0 = \text{diag}(\sigma_1, \ldots, \sigma_s) \), \( \sigma_i > 0 \), and \( i = 1, 2, \ldots, s \). Then, there exists a unique \( S^* \in \mathbb{S}^{[m \times n]} \) such that
\[ \min \| \Sigma_0 S - G \|_F = \| \Sigma_0 S^* - G \|_F, \]
and \( S^* = \Phi \ast (G^H \Sigma_0 - \Sigma_0 G) \), where
\[ \Phi = (\varphi_{ij}) \in \mathbb{H}^{m \times n}, \]
\[ \varphi = \frac{1}{\sigma_i + \sigma_j}, 1 \leq i, j \leq s. \]

Theorem 6. Let \( A, B \in \mathbb{C}^{m \times n} \), and \( A U = (A_1, A_2), B U = (B_1, B_2) \). Assume that the singular value decomposition of \( A_1 \) is as follows:
\[ A_1 = W \begin{bmatrix} \Sigma_1 & O \\ O & O \end{bmatrix} V_1^H, \]
where \( W = (W_1, W_2) \), and \( V = (V_1, V_2) \) are unitary matrices and \( \Sigma_1 = \text{diag}(\sigma_1, \ldots, \sigma_s) \), \( \sigma_i > 0 \), \( i = 1, 2, \ldots, r_1 \), and \( r_1 = \text{rank}(A_1) \). Then, the least-square skew-Hermitian \{P,2\}-reflexive solution can be expressed as
\[ X = U \begin{bmatrix} V_1 \left( \Phi \ast \left( [W_1^H B_1 V_1] \Sigma_1 - [W_1^H B_1 V_1] \Sigma_1 [W_1^H B_1 V_1]^H \right) \right) \end{bmatrix} V_1^H, \]
where \( Y_4 \) is arbitrary.

Proof. If \( X \) is skew-Hermitian \{P,2\}-reflexive matrix, from Lemma 2 and (54), we have
\[ \| AX - B \|_F^2 = \| AU \begin{bmatrix} X_{11} & O \\ O & O \end{bmatrix} U^H - B \|_F^2 \]
\[ = \| AU \begin{bmatrix} X_{11} & O \\ O & O \end{bmatrix} - BU \|_F^2 \]
\[ = \left\| \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} X_{11} \\ O \end{bmatrix} \right\|_F^2 - \left\| \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} O \\ B \end{bmatrix} \right\|_F^2 \]
\[ = \left\| A_1 X_{11} - B \right\|_F^2 + \left\| B \right\|_F^2. \]

Thus, \( \| AX - B \| = \min \) is equivalent to \( \| A_1 X_{11} - B_1 \|_2 = \min \). It yielding from (54) that
\[ \| A_1 X_{11} - B_1 \|_2 = \| W \left( \begin{bmatrix} \Sigma_1 & O \\ O & O \end{bmatrix} V_1^H \right) X_{11} - B_1 \|_2 \]
\[ = \left\| \Sigma_1 \begin{bmatrix} O \\ O \end{bmatrix} V_1^H X_{11} - B_1 \right\|_F^2. \]

Let
\[ V_1^H X_{11} V = \begin{bmatrix} Y_1 & Y_2 \\ -Y_2^H & Y_4 \end{bmatrix}, \]
\[ W_1^H B_1 V = \begin{bmatrix} W_1^H B_1 V_1 & W_1^H B_1 V_2 \\ W_2^H B_1 V_1 & W_2^H B_1 V_2 \end{bmatrix}, \]
then (57) becomes
\[ \left\| \begin{bmatrix} W_1^H B_1 V_1 & W_1^H B_1 V_2 \\ W_2^H B_1 V_1 & W_2^H B_1 V_2 \end{bmatrix} \right\|_F^2. \]
Let 
\[ \sigma_i > N \]

Then from Lemma 4 and (62), we have

By Lemma 9,
\[ Y_1 = \Phi * \left( (W_1^H B_1 V_1) \Sigma_1 - \Sigma_1 W_1^H B_1 V_1 \right), \]

\[ \Phi = \frac{1}{\sigma_i + \sigma_j}, \]

\[ Y_2 = \Sigma_1 W_1^H B_1 V_2. \]

Substituting \( Y_1, Y_2, \) and \( \Phi \) into (58) and \( X \) can be expressed as (55).

**Remark 2.** The skew-Hermitian \([P,3]-\)antireflective constraint least squares problem can be reduced similar to Theorem 5.1, the conclusion is omitted here.

**Theorem 7.** Let \( A, B \in \mathbb{C}^{m \times n}, \) \( AU = (A_1, A_2, A_3), \) and \( BU = (B_1, B_2, B_3). \) Assume that the singular value decomposition of \( A_1, A_2 \) are as follows:

\[ A_1 = W \left[ \begin{array}{c} \Sigma_1 O \\ O O \end{array} \right] V_1^H, A_2 = M \left[ \begin{array}{c} \Sigma_2 O \\ O O \end{array} \right] N_1^H, \]

where \( W = (W_1, W_2), \) \( V = (V_1, V_2), \) \( M = (M_1, M_2), \) and \( N = (N_1, N_2) \) are unitary matrices; \( \Sigma_1 = \text{diag}(\sigma_1, \ldots, \sigma_r), \) \( \Sigma_2 = \text{diag}(\delta_1, \ldots, \delta_r), \) \( \sigma_i > 0 (i = 1, 2, \ldots, r_1), \) \( r_1 = \text{rank}(A_1), \) \( \delta_i > 0 (i = 1, 2, \ldots, r_2), \) and \( r_2 = \text{rank}(A_2). \) Then, the least-square skew-Hermitian \([P,3]-\)antireflective solution can be expressed as

\[ X = U \left[ \begin{array}{c} O X_{12} O \\ -X_{12}^H O O \end{array} \right] U^H, \]

where

\[ X_{12} = V \left[ \Phi * \left( \Sigma_1 W_1^H B_2 N_1 - V_1^H B_1^H M_1 \Sigma_2 \right) \Sigma_1 W_1^H B_2 N_2 \right] N_1^H. \]

**Proof.** If \( X \) is skew-Hermitian \([P,3]-\)antireflective matrix, then from Lemma 4 and (62), we have

\[ \| AX - B \|^2 = \| AU \left[ \begin{array}{c} O X_{12} O \\ O O O \end{array} \right] U^H - B \| \]

Thus, \( \| AX - B \| \) is min equivalent to \( \| AX - B \|^2 + \| -X_{12} A_2^H - B_1^H \|^2 = \| A_1 X_{12} - B_1 \|^2 + \| -X_{12} A_2^H - B_1^H \|^2 = \| A_1 X_{12} - B_1 \|^2 + \| -X_{12} A_2^H - B_1^H \|^2 + \| B_3 \|^2. \]

(65)

Thus, \( \| AX - B \| \) is min equivalent to \( \| A_1 X_{12} - B_2 \|^2 + \| -X_{12} A_2^H - B_1^H \|^2 = \| A_1 X_{12} - B_2 \|^2 + \| -X_{12} A_2^H - B_1^H \|^2 \) min. It yielding from (62) that

\[ A_1 X_{12} - B_2 \|^2 + \| -X_{12} A_2^H - B_1^H \|^2 = \]

\[ \| \Sigma_1 O \| V_1^H X_{12} N - W_1^H B_2 N \|^2 \]

(66)

\[ + \| -V_1^H X_{12} N \| - V_1^H B_1^H M \|^2 \]

Let

\[ V_1^H X_{12} N = \left[ \begin{array}{c} Z_1 Z_2 \\ Z_3 Z_4 \end{array} \right], \]

\[ W_1^H B_2 N = \left[ \begin{array}{c} W_1^H B_2 N_1 \\ W_1^H B_2 N_2 \end{array} \right], \]

\[ V_1^H B_1^H M = \left[ \begin{array}{c} V_1^H B_1^H M_1 \\ V_1^H B_1^H M_2 \end{array} \right], \]

then (66) becomes

\[ \| \Sigma_1 Z_1 - W_1^H B_2 N_1 \|^2 + \| -Z_1 \Sigma_2 - V_1^H B_2^H M_2 \|^2 \]

\[ + \| Z_2 \Sigma_2 - V_1^H B_2^H M_1 \|^2 \]

(69)

\[ + \| W_2^H B_2 N_1 \|^2 + \| W_2^H B_2 N_2 \|^2 \]

(69)

\[ + \| V_1^H B_1^H M_2 \|^2 + \| V_1^H B_1^H M_2 \|^2 \]

Thus, \( \| AX - B \| \) is min solvable if and only if there exist \( Z_1, Z_2, Z_3 \) such that

\[ X = U \left[ \begin{array}{c} O X_{12} O \\ -X_{12}^H O O \end{array} \right] U^H, \]

(63)

(64)
By Lemma 8,
\[
Z_1 = \Phi \ast \left( \Sigma_1 W_1^H B_2 N_1 - V_1^H B_1^H M_1 \Sigma_2 \right), \\
\Phi = \frac{1}{\sigma_i^2+\sigma_j^2}, \\
Z_2 = \Sigma_1 W_1^H B_2 N_2, \\
Z_3 = -\Sigma_2 V_2^H B_1^H M_1.
\]
Substituting \( Z_1, Z_2, Z_3 \Phi \) into (67), \( X \) can be expressed as (63).

\[ \tag{70} \]

\section*{6. An Algorithm and Numerical Example}

We propose the following algorithm to compute the optimal approximate solution of Problem 2 over skew-Hermitian \{P,2\}-reflexive matrix.

\begin{example}
Suppose \( X \in C^{4 \times 3}, P \in C^{4 \times 4}, A^* \in C^{4 \times 4}, \Lambda \in C^{3 \times 3} \), and \( \Lambda = \text{diag}(-4.3i, -4.3i, -4.3i) \),
\[
X = \begin{bmatrix}
2.1 + i & -3.7i & 0.62 \\
3.4i & 2.3 - 0.5i & -2.7i \\
4.2 - i & 2 - 1.8i & 2.3 \\
\end{bmatrix},
\]
\[
P = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix},
\]
\[
A^* = \begin{bmatrix}
1.7 + 2i & 3 - 4i & 0.25 - 2i & 2 + 0.5i \\
-3.1i & 0.28i & 2.7 & 0.6i \\
4.1 & 2.8 + 3i & 3.04 & 1.5 \\
\end{bmatrix}.
\]

From Algorithm 1, we can obtain
\[
U = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix},
\]
\[
X_1 = \begin{bmatrix}
4.2 - i & -2i & -6.8i \\
1 & 2 - 1.8i & 2.3 \\
3.4i & 2.3 - 0.5i & -2.7i \\
\end{bmatrix},
\]
\[
A_{11}^* = \begin{bmatrix}
1.5 & 3.04 & 2.8 + 3i \\
0.6i & 2.7 & 0.28i \\
2 + 0.5i & 0.25 - 2i & 3 - 4i \\
\end{bmatrix}.
\]

So, the conditions \( X_1 A X_1^H = X_1 \Lambda, X_1^H X_1 = \Lambda X_1^H X_1 \) is hold. By Theorem 1 and problem (40), we can obtain
\[
A^* = U \begin{bmatrix}
-4.3i & 0 & 0 \\
0 & -4.3i & 0 \\
0 & 0 & -4.3i \\
\end{bmatrix} U^H. \tag{74}
\]

\section*{Data Availability}
There is no data available to support this article.

\section*{Conflicts of Interest}
The authors declare that they have no conflicts of interest.

\begin{references}


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