Research Article

Q-Differential Equations of Higher Order and Structural Properties of the Approximate Roots for q-Modified Derangements’ Polynomials

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The objective of this study is to find q-differential equations of higher order related to q-modified derangements’ polynomials and confirm the structure of approximation roots. Furthermore, it states several symmetric properties of q-differential equations of higher order and the special properties of the approximation roots of q-modified derangements’ polynomials.

1. Introduction

1.1. Bernoulli Differential Equation

\[
\frac{dy}{dx} + p(x)y - g(x)y^m = 0, \tag{1}
\]

is an equation, where \( m \) is any real number, and \( p(x) \) and \( g(x) \) are continuous functions on the interval. Amongst all of the differential equations, the Bernoulli differential equation converts nonlinear equations into linear equations.

If \( m = 0 \) or \( m = 1 \), the above equation is linear, and if not, the equation is nonlinear. By substituting \( u = y^{1-m} \), the Bernoulli differential equation can be reduced to a linear differential equation. Then, it concludes to a linear equation \((du/dx) + (1-m)p(x)u = (1-m)g(x)\) for \( u \). This equation can be applied to problems related to nonlinear differential equations, equations about the population expressed in logistic equations, or Verhulst equations, physics, etc.

If \( m = 0 \) in (1), the Bernoulli differential equation,

\[
\frac{d}{dx} D_n(x) - D_n(x) + \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^k = 0, \tag{2}
\]

has a solution which is the generating function of modified derangements’ polynomials, see [1, 2].

For \( t \neq 1 \), the modified derangements’ numbers and polynomials can be expressed as

\[
\sum_{n=0}^{\infty} D_n t^n = \frac{e^{-t}}{1-t}, \tag{3}
\]

\[
\sum_{n=0}^{\infty} D_n(x) t^n = \frac{e^{-t}}{1-t} e^{tx},
\]

respectively.

Table 1 is the first few examples of the modified derangements’ numbers \( D_n \) and polynomials \( D_n(x) \).

Based on the above concept and q-numbers, we consider the q-Bernoulli differential equation of the first order \( D_q y + p(x)y - g(x)y^m = 0 \). In addition, it is possible to assume that the q-modified derangements’ polynomials are a solution of the following q-differential equation of the first order when \( m = 0 \) in (1).

The aim of this study is to find the q-Bernoulli differential equation with the solution of q-modified derangements’ polynomials. Also, we derive characteristic properties...
by visualizing the approximation roots of $q$-modified derangements’ polynomials.

Several mathematicians discovered $q$-differential equations by using special polynomials as a solution and studying their properties and identities, see [3–7]. The $q$-differential equation based on $q$-Hermit polynomials were studied by Hermoso, Huertas, and Lastra in [8]. In [9, 10], phenomena of roots for various kinds of polynomials related to differential equations were researched by Ryoo. Furthermore, various properties of polynomials using $q$-series, $q$-derivative, $q$-distribution, and so on were found, see [9, 11–13].

To lay out the foundation for achieving the goal of this study, the following summarizes the definitions and theorems and makes arrangements.

Jackson introduced the $q$-number, which plays an important role in $q$-calculus, see [4, 14]. Based on the discovery of $q$-number, useful results are studied in $q$-series, $q$-special functions, quantum algebras, $q$-discrete distribution, $q$-differential equations, $q$-calculus, etc., see [15, 16]. Here, we briefly review several concepts of $q$-calculus which we need for this study.

Let $n, q \in \mathbb{R}$ with $q \neq 1$. The number

$$[n]_q = \frac{1 - q^n}{1 - q},$$

is called $q$-number, see [6, 16]. We note that $\lim_{q \to 1} [n]_q = n$. In particular, for $k \in \mathbb{Z}$, $[k]_q$ is called $q$-integer.

The $q$-Gaussian binomial coefficients are defined by

$$\begin{bmatrix} m \cr r \end{bmatrix}_q = \frac{[m]_q!}{[m - r]_q![r]_q!},$$

where $m$ and $r$ are nonnegative integers, see [17]. For $r = 0$, the value is 1 since the numerator and the denominator are both empty products. One notes $[n]_1! = [n]_q[n - 1]_q \ldots [2]_q[1]_q$ and $[0]_1! = 1$.

**Definition 1.** Let $z$ be any complex number with $|z| < 1$. The $q$-exponential functions can be expressed as follows:

(i) $e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \sum_{n=0}^{\infty} \frac{(1 - q^n)(1 - q^{n+1}) \ldots (1 - q)}{\prod_{k=0}^{n-1} (1 - (1 - q)q^kz)}$

(ii) $E_q(z) = \sum_{n=0}^{\infty} \frac{n^2}{[n]_q!} (z^n/[n]_q!)$

where $(q; q)_n = (1 - q^n)(1 - q^{n+1}) \ldots (1 - q)$ is the $q$-Pochhammer symbol.

We note that $\lim_{q \to 1} e_q(z) = e^z$, see [12, 16].

**Theorem 1.** From Definition 1, we note that

(i) $e_q(x)e_q(y) = e_q(x + y)$, if $yx = qxy$,

(ii) $e_q(x)E_q(-x) = 1$,

(iii) $e_{q^{-1}}(x) = E_q(x)$.

**Definition 2.** The $q$-derivative of a function $f$ with respect to $x$ is defined by

$$D_{q,x}f(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \text{ for } x \neq 0,$$

and $D_{q,x}f(0) = f'(0)$.

We can prove that $f$ is differentiable at zero, and it is clear that $D_{q,x}^n = [n]_q x^{-n}$, see [6, 14–16]. From Definition 2, there are some formulae for $q$-derivative.

**Theorem 2.** From Definition 2, we note that

(i) $D_q(f(x)g(x)) = q(x)D_qf(x) + f(qx)D_qg(x) = f(x)D_qg(x) + g(qx)D_qf(x)$,

(ii) $D_q \left( \frac{f(x)}{g(x)} \right) = \frac{g(x)D_qf(x) - f(qx)D_qg(x)}{g(x)g(qx)} = \frac{g(x)D_qf(x) - f(x)D_qg(x)}{g(x)g(qx)}$,

(iii) for any constants $a$ and $b$, $D_q(a f(x) + b g(x)) = aD_qf(x) + bD_qg(x)$.

Our ultimate purpose is to find the solution of $q$-modified derangements’ polynomials by observing various $q$-differential equations of higher order. In Section 2, we define the $q$-modified derangements’ numbers and polynomials, mention several forms of $q$-differential equations of higher order, and check its associated symmetric properties. Lastly, in Section 3,
by observing the values of $q$-modified derangements’ numbers, the approximation roots of $q$-modified derangements’ polynomials will be shown, and several conjectures for those numbers and polynomials will be organized.

2. Various Types of $q$-Differential Equations of Higher Order Associated with the $q$-Modified Derangements’ Polynomials

In this section, $q$-modified derangements’ polynomials are defined, and various kinds of $q$-differential equation of higher order associated with these polynomials are introduced. Moreover, we find several symmetric properties of $q$-differential equation of higher order.

**Definition 3.** For $t \neq 1$ and $|q| < 1$, the $q$-modified derangements’ polynomials $D_{n,q}(x)$ is defined with the following generating function as follows:

$$\sum_{n=0}^{\infty} D_{n,q}(x) \frac{t^n}{[n]_q!} = e_q(t) - e_q(tx).$$

(9)

From Definition 3, we note that

$$\sum_{n=0}^{\infty} D_{n,q}(x) \frac{t^n}{[n]_q!} = e_q(t).$$

(10)

Here, we define $D_{n,q}$ as the $q$-modified derangements’ numbers. The $q$-modified derangements’ numbers $D_{n,q}$ has the relation $D_{n,q} = D_{n,q}(1)$ with the polynomials of $q$-derangements, see [1].

**Theorem 3.** The $q$-modified derangements’ polynomials $D_{n,q}(x)$ is a solution of the $q$-differential equation which can be given as

$$D^{(1)}_{q,x} D_{n,q}(x) - D_{n,q}(x) + \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^k = 0.$$  

(11)

**Proof.** When $t \neq 1$, the generating function of the $q$-modified derangements’ polynomials have

$$(1-t) \sum_{n=0}^{\infty} D_{n,q}(x) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^k \right) \frac{t^n}{[n]_q!}.$$

(12)

The left-hand side of (12) can be changed to

$$(1-t) \sum_{n=0}^{\infty} D_{n,q}(x) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \left( D_{n,q}(x) - [n]_q D_{n-1,q}(x) \right) \frac{t^n}{[n]_q!}.$$  

(13)

Using (12) and (13), applying the Cauchy product rule and comparing the coefficients of both sides of result equation, we have

$$D_{n,q}(x) - [n]_q D_{n-1,q}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^k.$$  

(14)

Applying $q$-derivative in $D_{n,q}(x)$, we find a relation between $D_{n,q}(x)$ and $D^{(1)}_{q,x} D_{n,q}(x)$ as

$$D_{n-1,q}(x) = \frac{1}{[n]_q} D^{(1)}_{q,x} D_{n,q}(x).$$  

(15)

Substituting $D^{(1)}_{q,x} D_{n,q}(x)$ of (15) to the left-hand side of (14), we achieve the shown result. $\square$

**Corollary 1.** When $q \rightarrow 1$ in Theorem 3, one holds that

$$\frac{d}{dx} D_n(x) - D_n(x) + \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^k = 0,$$

where $D_n(x)$ is the modified derangements’ polynomials.

**Theorem 4.** The solution of the following $q$-differential equation of higher order is the $q$-modified derangements’ polynomials $D_{n,q}(x)$ which can be given as follows:

$$D^{(n)}_{q,x} D_{n,q}(x) - D_{n,q}(x) + \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^k = 0.$$  

(16)

(17)
Proof. Using a property of $q$-exponential function $e_q(t)E_q(-t) = 1$ in the generating function of the $q$-modified derangements' polynomials, we find
\[
\sum_{n=0}^{\infty} \mathcal{D}_{n,q}(x) = \frac{e_q(t)}{1-t} e_q(tx),
\]
\[
= \frac{1}{(1-t)E_q(t)} e_q(tx).
\]

(18)

Suppose that \((1-t)E_q(t) \neq 0\) in (18). Then, we have
\[
(1-t)E_q(t) \sum_{n=0}^{\infty} \mathcal{D}_{n,q}(x) = \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!},
\]

(19)

From the power series of $q$-exponential functions, the left-hand side of (19) can be transformed as
\[
\sum_{n=0}^{\infty} \mathcal{D}_{n,q}(x) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \left[ n \right] \left( q \right) \frac{1}{1-q} \left( \frac{1}{2} \right) \left[ n \right] \left( q \right).
\]

(20)

Comparing (19) and (20), we have
\[
\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \left[ n \right] \left( q \right) \frac{1}{1-q} \left( \frac{1}{2} \right) \left[ n \right] \left( q \right) \mathcal{D}_{n-k,q}(x) = x^n.
\]

(21)

Applying $q$-derivative in $\mathcal{D}_{n,q}(x)$, we obtain a relation such as
\[
\mathcal{D}_{n-k,q}(x) = \frac{[n-k]_q!}{[n]_q!} D^{(k)}_{q,x} \mathcal{D}_{n,q}(x).
\]

(22)

Combining (21) and (22), we complete the desired result. □

Corollary 2. Let $q \rightarrow 1$ in Theorem 4. Then, we obtain
\[
1 - n \frac{d^n}{dx^n} \mathcal{D}_{n}(x) + 2 - \frac{n}{(n-1)!} \frac{d^{n-1}}{dx^{n-1}} \mathcal{D}_{n}(x) + \frac{3 - n}{(n-2)!} \frac{d^{n-2}}{dx^{n-2}} \mathcal{D}_{n}(x) + \ldots + \frac{2}{3!} \frac{d^{3}}{dx^{3}} \mathcal{D}_{n}(x) + \frac{1}{2!} \frac{d^{2}}{dx^{2}} \mathcal{D}_{n}(x) + \mathcal{D}_{n}(x) - x^n = 0,
\]

(24)
where $\mathcal{D}_n(x)$ is the modified derangements’ polynomials.

**Theorem 5.** The $q$-modified derangements’ polynomials $\mathcal{D}_{n,q}(x)$ is a solution of the following $q$-differential equation of higher order which is combined with the q-modified derangements’ numbers:

$$
\sum_{k=0}^{n-2} \binom{n-1}{k} \frac{(-1)^{k+1} + \mathcal{D}_{k,q}}{[n-1]_q^{k+1}} \mathcal{D}_{n-1,q}(x) + \sum_{k=0}^{n+2} \binom{n-2}{k} \frac{(-1)^{k+1} + \mathcal{D}_{k,q}}{[n-2]_q^{k+1}} \mathcal{D}_{n-2,q}(x) + \cdots
$$

$$
\sum_{k=0}^{n-3} \binom{1-k}{k} \frac{(-1)^{k+1} + \mathcal{D}_{k,q}}{[2]_q^{k+1}} \mathcal{D}_{n-1,q}(x) + \sum_{k=0}^{n+1} \binom{1}{k} \frac{(-1)^{k+1} + \mathcal{D}_{k,q}}{[1]_q^{k+1}} \mathcal{D}_{n-2,q}(x) + \cdots
$$

$$
-(1 - \mathcal{D}_{n,q}) q^{-n} \mathcal{D}_{n-1,q}(x) - \mathcal{D}_{n,q}(x) = 0.
$$

**Proof.** Initially, we apply the $q$-derivative after substituting $qx$ instead of $x$ in the generating function of the $q$-modified derangements’ polynomials $\mathcal{D}_{n,q}(x)$. Then, we obtain

$$
\sum_{n=0}^{\infty} q^n \mathcal{D}_{n,q}(x) \frac{t^n}{[n]_q!} = e_q(qtx) \left( \frac{e_q(-t) - (1-t)e_q(-t)}{(1-t)(1-qt)} \right) + \frac{qxe_q(-qt)}{1-t} - e_q(qtx),
$$

$$
\sum_{n=0}^{\infty} q^n \mathcal{D}_{n,q}(x) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} q^n \mathcal{D}_{n,q}(x) \frac{t^n}{[n]_q!} \left( \frac{e_q(-t)}{(1-t)e_q(-qt)} + \frac{e_q(-qt)}{e_q(-qt)} + qx \right),
$$

$$
\sum_{n=0}^{\infty} q^n \mathcal{D}_{n,q}(x) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} q^n \mathcal{D}_{n,q}(x) \frac{t^n}{[n]_q!} \left( \frac{(-1)^{k+1} + \mathcal{D}_{k,q}}{[2]_q^{k+1}} \mathcal{D}_{n-1,q}(x) + \frac{(-1)^{k+1} + \mathcal{D}_{k,q}}{[1]_q^{k+1}} \mathcal{D}_{n-2,q}(x) + \cdots \right),
$$

From (26), we have

$$
tD_{q,t} \sum_{n=0}^{\infty} q^n \mathcal{D}_{n,q}(x) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} q^n \mathcal{D}_{n,q}(x) \frac{t^n}{[n+1]_q!} + \sum_{n=0}^{\infty} n q^n x \mathcal{D}_{n-1,q}(x) \frac{t^n}{[n]_q!},
$$

From the generating function of the $q$-modified derangements’ polynomials $\mathcal{D}_{n,q}(x)$, we also have

$$
tD_{q,t} \sum_{n=0}^{\infty} q^n \mathcal{D}_{n,q}(x) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} q^n \mathcal{D}_{n,q}(x) \frac{t^n}{[n+1]_q!} + \sum_{n=0}^{\infty} n q^n x \mathcal{D}_{n-1,q}(x) \frac{t^n}{[n]_q!}.
Comparing (27) and (28), we obtain

\[
D_{n,q}(x) - q^n D_{n-1,q}(x) = \sum_{k=0}^{n-1} \sum_{l=0}^{k} \binom{n-1}{k} \binom{l}{k} \left( (-1)^{k-1} + D_{k,q} \right) q^{l-k+1} D_{n-1,q}^{l-k+1}(x).
\]

Applying relation (22) in (29), we derive

\[
D_{n,q}(x) - q^n x D_{n-1,q}(x) = \sum_{l=0}^{n-1} \sum_{k=0}^{l} \binom{l}{k} \left( (-1)^{k-1} + D_{k,q} \right) q^{l-k+1} D_{n-1,q}^{l-k+1}(x),
\]

and this equation gives us the required result.

\[\square\]

**Corollary 3.** Setting \( q \to 1 \) in Theorem 5, we obtain

\[
\sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{k-1} + D_{k,q} \\frac{d^{n-1}}{dx^{n-1}} D_{n-1,q}(x) + \sum_{k=0}^{n-2} \binom{n-2}{k} (-1)^{k-1} + D_{k,q} \\frac{d^{n-2}}{dx^{n-2}} D_{n-1,q}(x)
\]

\[+ \cdots + \sum_{k=0}^{2} \binom{2}{k} (-1)^{k-1} + D_{k,q} \frac{d^2}{dx^2} D_{n-1,q}(x) + \left( \frac{1}{k} \right) (-1)^{k-1} + D_{k,q} \frac{d}{dx} D_{n-1,q}(x) - (1 - D_0 - x) D_{n-1,q}(x) - D_n(x) = 0,
\]

\[
\text{Table 2: Approximate values of } D_{n,q}.
\]

<table>
<thead>
<tr>
<th>n</th>
<th>0.1</th>
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<th>0.99</th>
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<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>1</td>
<td>1</td>
<td>1</td>
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</tr>
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<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
</tr>
</tbody>
</table>

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where $D_n$ is the modified derangements' number and $D_n(x)$ is the modified derangements' polynomials.

**Theorem 6.** The $q$-modified derangements' polynomials $D_{n,q}(x)$ is a solution of the $q$-differential equation of higher order which is

$$
\sum_{k=0}^{n-1} \binom{n-1}{k} q^k \frac{(n-k-1)^{n-k-1} + q^k D_{n-k-1,q}}{[n-1]_q!} x^{n-k} + \cdots + \binom{2}{n-2} q^k \frac{2^{n-k} + q^k D_{2-k,q}}{[2]_q!} x^{n-2} + \binom{1}{n-1} q^{n-1} D_{n-1,q}/[n-1]_q! x^{n-1} = 0,
$$

where $D_{n,q}(x)$ is the modified derangements' polynomials for $0 \leq n \leq 8$ and $q = 0.6, 0.4, 0.1$.

![Figure 1: Locations of $D_{n,q}$ for $0 \leq n \leq 8$ and $q = 0.6, 0.4, 0.1$.](image)
where $D_{n,q}$ is the $q$-modified derangements’ numbers.

Proof. apply the $q$-derivative after substituting $qt$ instead of $t$ in $D_{n,q}(x)$. Then, we have

$$D_q \sum_{n=0}^{\infty} q^n D_{n,q}(x) \frac{t^n}{[n]_q!} = e_q(q t) \left( \frac{q e_q(-qt) - q (1 - qt) e_q(-qt)}{1 - qt} \right) + \frac{q x e_q(-qt)}{1 - qt} e_q(q x),$$

$$= \sum_{n=0}^{\infty} q^{2n+1} D_{n,q}(q^{-1} x) \frac{t^n}{[n]_q!} \left( E_q(q t) \sum_{n=0}^{\infty} q^n D_{n,l}(q^{-1} x) \frac{t^n}{[n]_l!} - E_q(q t) e_q(-qt) + x \right).$$

$$= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ l \\ k \end{array} \right] \left[ \begin{array}{c} l \\ k \end{array} \right] (-1)^{k+1} q \left( \begin{array}{c} l-k \\ 2 \end{array} \right) + q \left( \begin{array}{c} k \\ 2 \end{array} \right) D_{l-k,q} \right) q^{2n+1} D_{n,l}(q^{-1} x) \frac{t^n}{[n]_l!}$$

$$+ \sum_{n=0}^{\infty} q^{2n+1} D_{n,q}(q^{-1} x) \frac{t^n}{[n]_q!}.$$

### Figure 2: Locations of $D_{n,q}(x)$ for $0 \leq n \leq 50$: (a) $q = 0.01$, (b) $q = 0.001$, and (c) $q = 0.0001$.

### Figure 3: Approximation roots and circles of $D_{50,q}(x)$: (a) $q = 0.01$, (b) $q = 0.001$, and (c) $q = 0.0001$.

### Table 3: The circle of approximation roots of $D_{50,q}(x)$.

<table>
<thead>
<tr>
<th>$q$</th>
<th>The center $(x, y)$</th>
<th>The radius</th>
<th>The error range</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>$(0.0202672, 1.19233 \times 10^{-12})$</td>
<td>1.26217</td>
<td>0.0319478</td>
</tr>
<tr>
<td>0.001</td>
<td>$(0.0108304, -2.93695 \times 10^{-11})$</td>
<td>1.01593</td>
<td>0.000242633</td>
</tr>
<tr>
<td>0.0001</td>
<td>$(0.00120143, 4.84225 \times 10^{-14})$</td>
<td>1.00124</td>
<td>3.23829 \times 10^{-8}</td>
</tr>
</tbody>
</table>

The center $(x, y)$ The radius  The error range
In a similar way to find (27) and (28), we obtain

$$tD_{q,t} \sum_{n=0}^{\infty} q^n q_{n,q} (x) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} [n]_q q^n q_{n,q} (x) \frac{t^n}{[n]_q!}$$

(34)

$$tD_{q,t} \sum_{n=0}^{\infty} q^n q_{n,q} (x) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} [n]_q q^n q_{n,q} (x) \frac{t^n}{[n]_q!}$$

(35)

From (34) and (35), we have

$$\sum_{n=0}^{\infty} \sum_{l=0}^{n-1} \sum_{k=0}^{l} \left( \begin{array}{c} n-1 \\ l \\ k \end{array} \right)_q \left( \begin{array}{c} l-k \\ 2 \\ q \end{array} \right)_q q^{l-k} q^n q_{n,q} (x) \frac{t^n}{[n]_q!} = \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} [n]_q q^n q_{n,q} (q^{-1}x) \frac{t^n}{[n]_q!}$$

(36)

Here, we find a relation between $q_{n,q} (q^{-1}x)$ and $D_{n,q}(q^{-1}x)$ as follows:

$$q^{k[\frac{n-k}{q}]} q_{n,q} (q^{-1}x) = D_{n,q}^{(k)} (q^{-1}x).$$

(37)

Using (37) in the left-hand side of (36), we have

$$\sum_{n=0}^{\infty} \sum_{l=0}^{n-1} \sum_{k=0}^{l} \left( \begin{array}{c} n-1 \\ l \\ k \end{array} \right)_q \left( \begin{array}{c} l-k \\ 2 \\ q \end{array} \right)_q q^{l-k} q^n q_{n,q} (x) \frac{t^n}{[n]_q!} = \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} [n]_q q^n q_{n,q} (q^{-1}x) \frac{t^n}{[n]_q!}$$

(38)

$$\sum_{n=0}^{\infty} \sum_{l=0}^{n-1} \sum_{k=0}^{l} \left( \begin{array}{c} n-1 \\ l \\ k \end{array} \right)_q \left( \begin{array}{c} l-k \\ 2 \\ q \end{array} \right)_q q^{l-k} q^n q_{n,q} (x) \frac{t^n}{[n]_q!} = \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} [n]_q q^n q_{n,q} (q^{-1}x) \frac{t^n}{[n]_q!}$$

$$D_{n,q}^{(l)} (q^{-1}x) = D_{n,q}^{(l)} (q^{-1}x) - q^{n-1} x D_{n-1,q} (q^{-1}x).$$
By (38), it is possible to finish the proof of Theorem 6.

\[ \sum_{k=0}^{n-1} \binom{n-1}{k} \left( \frac{(-1)^{k+1}}{(n-1)!} \sum_{k=0}^{n-2} \binom{n-2}{k} \left( \frac{(-1)^{k+1}}{(n-2)!} \frac{d^{n-2}}{dx^{n-2}} D_{n-1}(x) + \right) \right) \]

\[ + \cdots + \sum_{k=0}^{2} \binom{2}{k} \left( \frac{(-1)^{k+1}}{2!} \frac{d^2}{dx^2} D_{n-1}(x) + \right) \]

\[ + \left( 1 + x + \mathcal{D}_0 \right) D_{n-1}(x) - \mathcal{D}_n(x) = 0, \]

where \( \mathcal{D}_n \) is the modified derangements’ numbers and \( \mathcal{D}_n(x) \) is the modified derangements’ polynomials.

From now on, we take several suitable forms to find symmetric properties of \( q \)-differential equation of higher order. To obtain various symmetric properties of \( q \)-
differential equation of higher order which combines other numbers and polynomials, the below forms \( A, B \) are the basic forms.

**Theorem 7.** Let \( a \neq 0, b \neq 0, \) and \( 0 < q < 1. \) Then, we obtain

**Corollary 4.** Putting \( q \rightarrow 1 \) in Theorem 6, one holds

\[ \frac{b^n a^n \mathcal{D}_{n,q}^{(n)} D_{q,x}^{(n)} D_{q,x}^{(n-1)} \mathcal{D}_{n,q}^{(n-1)} (b^{-1} x)}{[n]_q!} + \frac{b^n a^n \mathcal{D}_{n,q}^{(n)} D_{q,x}^{(n)} D_{q,x}^{(n-1)} \mathcal{D}_{n,q}^{(n-1)} (a^{-1} x)}{[n]_q!} \]

\[ + \frac{b^n a^n \mathcal{D}_{n,q}^{(n)} D_{q,x}^{(n)} D_{q,x}^{(n-1)} \mathcal{D}_{n,q}^{(n-1)} (b^{-1} x)}{[n]_q!} \]

\[ + \frac{b^n a^n \mathcal{D}_{n,q}^{(n)} D_{q,x}^{(n)} D_{q,x}^{(n-1)} \mathcal{D}_{n,q}^{(n-1)} (a^{-1} x)}{[n]_q!} \]

**Proof.** To obtain a symmetric property of \( q \)-differential equation of higher order for \( q \)-modified derangements’ polynomials \( \mathcal{D}_{n,q}(x) \), we consider a form \( A \) as

\[ \frac{e_q(-at) e_q(-bt) e_q(tx)}{1 - at} \frac{1}{1 - bt} e_q(tx) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \mathcal{D}_{k,q}^{(n-k)} \mathcal{D}_{n-k,q}^{(n-k)} (b^{-1} x) \]

\[ \frac{e_q(-bt) e_q(-at) e_q(tx)}{1 - at} \frac{1}{1 - bt} e_q(tx) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} \mathcal{D}_{k,q}^{(n-k)} \mathcal{D}_{n-k,q}^{(n-k)} (a^{-1} x) \]

From (42) and (43), we find

\[ \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \mathcal{D}_{k,q}^{(n-k)} \mathcal{D}_{n-k,q}^{(n-k)} (b^{-1} x) = \sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} \mathcal{D}_{k,q}^{(n-k)} \mathcal{D}_{n-k,q}^{(n-k)} (a^{-1} x). \]
Replacing (37) to (44), we obtain

\[ b^n \sum_{k=0}^{n} \frac{a^k}{[k]_q} D_{q,x} D^{(k)}_{q,x}(b^{-1} x) = a^n \sum_{k=0}^{n} \frac{b^k}{[k]_q} D_{q,x} D^{(k)}_{q,x}(a^{-1} x). \]  

(45)

From (45), we complete the proof of Theorem 7. □

**Corollary 5.** Setting \( a = 1 \) in Theorem 7, we have

\[ b^n[n]_q D_{q,x} D^{(n)}_{q,x}(b^{-1} x) + \frac{b^n}{[n-1]_q} D_{q,x} D^{(n-1)}_{q,x}(b^{-1} x) + \ldots \]

\[ + \frac{b^n}{[2]_q} D_{q,x} D^{(2)}_{q,x}(b^{-1} x) + b^n D_{q,x} D^{(1)}_{q,x}(b^{-1} x) + b^n D_{0,q} D_{q,x}(b^{-1} x) \]

\[ = b^n[n]_q D_{q,x} D^{(n)}_{q,x}(x) + \frac{b^n}{[n-1]_q} D_{q,x} D^{(n-1)}_{q,x}(x) + \ldots \]

\[ + \frac{b^n}{[2]_q} D_{q,x} D^{(2)}_{q,x}(x) + bD_{1,q} D^{(1)}_{q,x}(x) + D_{0,q} D_{q,x}(x). \]  

(46)

**Corollary 6.** consider \( q \rightarrow 1 \) in Theorem 7; then, we get

\[ b^n a^n \frac{d^n}{dx^n} D_{n}(b^{-1} x) + \frac{b^n}{(n-1)!} \frac{d^{n-1}}{dx^{n-1}} D_{n}(b^{-1} x) + \ldots \]

\[ + \frac{b^n a^2}{2!} \frac{d^2}{dx^2} D_{n}(b^{-1} x) + \frac{b^n a}{(n-1)!} \frac{d^{n-1}}{dx^{n-1}} D_{n}(b^{-1} x) + \frac{b^n}{(n-1)!} \frac{d^{n-1}}{dx^{n-1}} D_{n}(b^{-1} x) \]

\[ = a^n b^n \frac{d^n}{dx^n} D_{n}(a^{-1} x) + \frac{a^n b^{n-1}}{(n-1)!} \frac{d^{n-1}}{dx^{n-1}} D_{n}(a^{-1} x) + \ldots \]

\[ + \frac{a^n b^2}{2!} \frac{d^2}{dx^2} D_{n}(a^{-1} x) + a^n b \frac{d}{dx} D_{n}(a^{-1} x) + a^n D_{0} D_{n}(a^{-1} x). \]

(47)

**Theorem.** Let \( a \neq 0, b \neq 0, \) and \( 0 < q < 1. \) Then, we have

\[ \frac{b^n a^n}{[n]_q^2} D_{q,y} D^{(n)}_{q,y}(a^{-1} y) + \frac{b^n a^{n-1}}{[n-1]_q^2} D_{q,y} D^{(n-1)}_{q,y}(b^{-1} y) + \ldots \]

\[ + \frac{b^n a}{(n-1)!} D^{(1)}_{q,y} D_{q,y}(a^{-1} y) + \frac{b^n}{[n-1]_q} D^{(n-1)}_{q,y} D_{q,y}(a^{-1} y) \]

\[ = a^n b^n \frac{d^n}{dx^n} D_{n}(a^{-1} y) + \frac{a^n b^{n-1}}{(n-1)!} \frac{d^{n-1}}{dx^{n-1}} D_{n}(a^{-1} y) + \ldots \]

\[ + \frac{a^n b^2}{2!} \frac{d^2}{dx^2} D_{n}(a^{-1} y) + a^n b \frac{d}{dx} D_{n}(a^{-1} y) + a^n D_{0} D_{n}(a^{-1} y). \]

(48)
Proof. Suppose form $B$ as

$$B: = \frac{e_q(-at)e_q(-bt)e_q(tx)e_q(ty)}{(1-at)(1-bt)}.$$  

(49)

Using $B$, the desired result can be obtained in a similar way to the proof of Theorem 7. Therefore, we omit the proof process. □

**Corollary 7.** Setting $a = 1$ in Theorem 8, we have

$$\frac{b^n\mathcal{D}_{1,q}(x)}{\binom{n}{q}}\mathcal{D}_{n,q}(b^{-1}y) + \frac{b^n\mathcal{D}_{n-1,q}(x)}{\binom{n-1}{q}}\mathcal{D}_{n,q}(b^{-1}y) + \cdots
+ b^n\mathcal{D}_{1,q}(b^{-1}y)\mathcal{D}_{n,q}(b^{-1}y) + \mathcal{D}_{1,q}(b^{-1}y)\mathcal{D}_{n,q}(b^{-1}y).$$

(50)

**Corollary 8.** Consider $q \longrightarrow 1$ in Theorem 8; then, we get

$$\frac{b^n a^n\mathcal{D}_n(a^{-1}x)}{n!} \frac{d^n}{dy^n}\mathcal{D}_n(b^{-1}y) + \frac{b^n a^{-1}b_{n-1}(a^{-1}x)}{(n-1)!} \frac{d^{n-1}}{dy^{n-1}}\mathcal{D}_n(b^{-1}y) + \cdots
+ b^n a\mathcal{D}_1(a^{-1}x)\mathcal{D}_n(b^{-1}y) + \mathcal{D}_1(a^{-1}x)\mathcal{D}_n(b^{-1}y)$$

(51)

3. **Visualization of the Approximation Roots for $q$-Modified Derangements’ Polynomials**

This section handles the approximation roots of the $q$-modified derangements’ polynomial. To confirm several conjectures for $q$-modified derangements, we show the structure of the approximation roots of those polynomials. We use MATHEMATICA in order to obtain the pictures and the results of calculation.

Based on the generating function of $q$-modified derangements’ numbers, $\mathcal{D}_{n,q}$ are found as follows:

$$\mathcal{D}_{0,q} = 1,$$
$$\mathcal{D}_{1,q} = 0,$$
$$\mathcal{D}_{2,q} = 1,$$
$$\mathcal{D}_{3,q} = q(1 + q),$$
$$\mathcal{D}_{4,q} = (1 + q + q^3)(1 + q^2 + q^3),$$
$$\mathcal{D}_{5,q} = q(1 + q)(1 + q^2)(2 + 2q + 2q^2 + 2q^3 + 2q^4 + q^5),$$
$$\mathcal{D}_{6,q} = (1 + q + q^2 + q^3 + q^4) \times (1 + q + 4q^2 + 6q^3 + 8q^4 + 9q^5 + 8q^6 + 7q^7 + 5q^8 + 3q^9 + q^{10}), \ldots$$

(52)
From the above \( q \)-modified derangements’ numbers, Table 2 shows the approximation values of \( D_{n,q} \) which appear with changes in values of \( q \). In Table 1, as the value of \( q \) increases, we can observe that the approximation value of \( q \)-modified derangements’ numbers also increases.

By using Table 2, we can see the location of \( D_{n,q} \) shown by varying \( n \) and \( q \), as shown in Figure 1. The nonnegative integers of the \( x \)-axis represent the values of \( n \) in Figure 1. Here, the lines display variations of the approximation values for \( q \)-modified derangements’ numbers. The blue dots, the yellow squares, and the green rhombuses in Figure 1 are the approximation values of \( q \)-modified derangements’ polynomials for two purposes. The first is to check the structure of properties of approximations values that appear depending on the value of \( q \) of the \( q \)-modified derangements’ polynomials. Here, we set the red dots to indicate the approximation real root exists. Therefore, we can make some assumptions and think that the related research should be continued.

To obtain the first purpose, Figure 2 shows the structure of the approximate roots close to the origin regardless of the change in the \( q \)-number.

| \( D_{0,q}(x) \) | 1 |
| \( D_{1,q}(x) \) | \( x \) |
| \( D_{2,q}(x) \) | \( 1 + x^2 \) |
| \( D_{3,q}(x) \) | \( 2q + 2q^2 + q^3 + (1 + q + q^2)x + (q + q^2 + q^4)x^2 + x^3 \) |

Based on the above polynomial, we find special properties of approximations values that appear depending on the value of \( q \) of the \( q \)-modified derangements’ polynomials. Here, we set the values of \( q \) to 0.01, 0.001, and 0.0001, respectively, because the extremely small value of \( q \) represents the properties of the \( q \)-modified derangements’ polynomials. We experiment with the \( q \)-modified derangements’ polynomials for two purposes. The first is to check the structure of the approximation roots and find the polygon that is most similar to it. The second is to find the approximation values of the real number among the approximation roots appearing in the \( q \)-modified derangements’ polynomials.

To obtain the first purpose, Figure 2 shows the structure of the approximate roots of \( q \)-modified derangements’ polynomials. Here, the range of \( n \) is from 0 to 50, and the value of \( q \) changes. In Figure 2, the value of \( q \) in (a) is 0.01, the value of \( q \) in (b) is 0.001, and the value of \( q \) in (c) is 0.0001. A characteristic that can be confirmed in the structure of the approximation roots in Figure 2(c) is that, as \( n \) increases, one approximation root close to the origin continues to accumulate. The structures of the approximation roots shown in Figure 2 seem to become more circular as \( n \) increases.

Based on the results of Figure 2, we find Figure 3. Figure 3 shows the structure of approximation roots when the value of \( n \) is 50. In Figure 3, the value of \( q \) in (a) is 0.01, the value of \( q \) in (b) is 0.001, and the value of \( q \) in (c) is 0.0001. Here, we set the red dots to indicate the approximation roots, the blue dots to indicate the center of the circle, and the blue lines to connect the red dots. Also, to obtain the blue lines closest to the approximation roots, the real roots are excluded.

Table 3 is the result of calculating the exact values in Figure 3. From Table 3, we can see that, as the value of \( q \) gets smaller, the structure of the approximation roots has a shape closer to a circle, and the radius of the circle gets closer to 1.

From Figures 2 and 3 and Table 3, we make the following conjecture:

**Conjecture 1.** As \( n \) increases and \( q \) approaches 1, the approximation value of \( D_{n,q} \) increases.

Using the generating function of \( q \)-modified derangements’ polynomials, \( D_{n,q}(x) \) are found as follows:

\[
\begin{align*}
D_{0,q}(x) &= 1, \\
D_{1,q}(x) &= x, \\
D_{2,q}(x) &= 1 + x^2, \\
D_{3,q}(x) &= 2q + 2q^2 + q^3 + (1 + q + q^2)x + (q + q^2 + q^4)x^2 + x^3, \ldots.
\end{align*}
\]

4. **Conclusion**

We organized the \( q \)-differential equation with the solution of \( q \)-modified derangements’ polynomials and find the corresponding symmetric properties. We also looked at the distribution of approximation roots of \( q \)-modified derangements’ polynomials and their phenomena. As a result, we can make some assumptions and think that the related research should be continued.
Data Availability

The data used to support the findings of this study can be obtained from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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