Local RBF-FD-Based Mesh-free Scheme for Singularly Perturbed Convection-Diffusion-Reaction Models with Variable Coefficients

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This work analyzes singularly perturbed convection-diffusion-reaction (CDR) models with two parameters and variable coefficients by developing a mesh-free scheme based on local radial basis function-finite difference (LRBF-FD) approximation. In the evolution of the scheme, time derivative is discretized by forward finite difference. After that, LRBF-FD approximation is used for spatial discretization, and we obtained a system of linear equations. Then, the obtained linear system is solved by LU decomposition method in MATLAB. For numerical simulation, four singularly perturbed models are considered to check the efficiency and chastity of the proposed scheme.

1. Introduction

Singularly perturbed models (SPMs) can be seen in different areas of science, medicine, and engineering. Herein, we consider the following class of parabolic SPMs with two small parameters and variable coefficients:

\[
L_{ε,μ}w(x, t) ≡ εw_{xx}(x, t) + μa(x, t)w_x(x, t) - b(x, t)w_t(x, t) - c(x, t)w = f(x, t), \quad (x, t) ∈ Ω = (0, 1) × (0, T],
\]

with initial and boundary conditions (BCs):

\[
\begin{align*}
  w(x, 0) &= w_0(x), \quad x ∈ (0, 1), \\
  w(0, t) &= g_1(t), \\
  w(1, t) &= g_2(t), \quad t ∈ (0, T],
\end{align*}
\]

where \(ε\) and \(μ\) are small positive parameters and the variable coefficients \(a(x, t), b(x, t), c(x, t)\), and \(f(x, t)\) are sufficiently smooth functions on \(\overline{Ω}\) and satisfies the following conditions:

\[
\begin{align*}
a(x, t) &≥ α > 0, \\
b(x, t) &≥ β > 0, \\
c(x, t) &≥ γ > 0, \quad ∀ (x, t) ∈ \overline{Ω}.
\end{align*}
\]

We also suppose that the initial and BCs have enough smoothness on \(\partial Ω\) and sufficient compatibility conditions on the edges of the domain \(Ω\). These conditions on the initial and BCs ensure that there exists a unique classical solution \(w(x, t) ∈ C^{4,2}(\overline{Ω})\) [1].
In the SP case, the solutions of the abovementioned models may possess a multiscale character, presenting rapid variations in some narrow regions adjacent to both the left and right lateral boundary depending on the magnitude of the parameters $\varepsilon$ and $\mu$. For $\mu = 1$, $\varepsilon \rightarrow 0$, model (1) becomes convection dominated model (CDM) and for $\mu = 0$, $\varepsilon \rightarrow 0$, model (1) was converted into the reaction-diffusion model (RDM). There are various numerical techniques available in literature for the simulation of CDM and RDM, e.g., adaptive B-spline collocation method [2], $\varepsilon$-uniform schemes [3], parameter uniform difference scheme [4], uniformly convergent B-spline collocation method [5], finite difference fitted schemes [6], uniformly convergent difference schemes [7, 8], SDFEM [9], parameter-uniform hybrid finite difference scheme [10], finite difference domain decomposition algorithms [11], high order methods [12], Layer-adapted meshes and FEM [13], parameter uniform approximations [14], uniformly convergent scheme [15], and finite difference scheme [16]. Except these, there are many schemes for these types of problems [17–22].

The singularly perturbed CDR models occurred in mathematical modelling of physical problems, such as fluid dynamics, elasticity, quantum mechanics, heat and mass transfer in mechanical engineering, theory of plates and shells, oil and gas reservoir simulation, and magneto-hydrodynamic flow. Two parameter problems arise in chemical flow reactor theory [23] as well as in the case of boundary layers controlled by suction (or blowing) of some fluids [17, 18, 24]. It is important to study these CDR models with two parameters and variable coefficients so that other information can be clarify by the interaction between convection and diffusion.

Recently, singularly perturbed convention diffusion problems are solved by Ahmad and Kalarestaghi [25] using LRBFs methods. But, these problems are steady state and with constant coefficients. Secondly, upwind LRBFs and central LRBFs methods are used for solving these problems. This motivates the authors to develop LRBF-FD scheme for unsteady-state singularly perturbed CDR models. To achieve this aim, we present a mesh-free scheme based on LRBF-FD approximation for the unsteady-state singularly perturbed CDR models with two parameters and variable coefficients. The novelty of the work is that we solved unsteady-state singularly perturbed problems with two parameters and variable coefficients. Second, we used LRBF-FD approximation in place of upwind LRBFs and central LRBFs methods. In the evolvement of the scheme, time derivative is discretized by forward finite difference. After that, RBF-FD approximation is used for spatial discretization, and we obtained a system of linear equations. Finally, the obtained linear system is solved by the LU decomposition method in MATLAB. For numerical simulation, four problems are pondered to check the efficiency and chastity of the proposed scheme.

Rest of the paper is organized as follows: in Section 2, the details of RBFs and Local RBF-FD approximation has been described. Mesh-free numerical scheme based on LRBF-FD approximation will be introduced in Section 3. In Section 4, MATLAB implementation steps are given in form of algorithm of the developed scheme. Numerical simulation and discussion are carried out in Section 5. Concluding remarks of the work are given in the final Section 6.

2. RBFs and Local RBF-FD Approximation

Herein, we presented the brief introduction of RBFs and LRBF-FD approximation in $\mathbb{R}^n$ space as follows.

2.1. RBFs. For scattered points $x \in \Xi \subset \mathbb{R}^n$, the approximation of a function $\chi(x)$ can be written as a linear combination of RBFs [26, 27].

$$\chi(x) \equiv \sum_{j=1}^{M} \lambda_j \varphi \left( \| x - x_j \|_2 \right) + \psi(x),$$

(4)

where $M$ is the number of centers or sometimes called knots $x, \ X = \{x_1, x_2, \ldots, x_n\}, n$ is the dimension of the problem, $\lambda_j$’s are coefficients to be determined and $\varphi$ is a RBF. (4) can be written without the additional polynomial $\psi(x)$. If $\Psi_\alpha^n$ denotes the space of $n$-variate polynomials of order not exceeding $q$, and letting the polynomials $P_1, P_2, \ldots, P_m$ be the basis of $\Psi_\alpha^n$ in $\mathbb{R}^n$, then the polynomial $\psi(x)$, in (4), is usually written in the following form:

$$\psi(x) = \sum_{i=1}^{m} \zeta_i P_i(x),$$

(5)

where $m = (q - 1 + n)! / (q! (q - 1)!)$.

To determine the coefficients $(\lambda_1, \lambda_2, \ldots, \lambda_M)$ and $(\zeta_1, \zeta_2, \ldots, \zeta_m)$ extra $m$ equations are required in addition to the $M$ equations resulting from the collocating equation (4) at $M$ knots. This is insured by the $m$ conditions for (4), viz.,

$$\sum_{j=1}^{M} \lambda_j P_j(x_i) = 0, \quad i = 1, 2, \ldots, m.$$  

(6)

There are many RBFs (expression of $\varphi$) available in literature such as Multiquadric (MQ), Inverse Multiquadric (IMQ), Gaussian, Thin Plate spline [27–34] etc.

2.2. LRBF-FD Method Formulation. In this work, we used LRBF-FD approximation for the discretization of spatial derivatives. LRBF-FD approximation is more general form of classical finite difference method. To explain the concept of LRBF-FD approximation, assume that there are $M$ scattered points in the bounded domain of the problem i.e. $\Xi \subset \mathbb{R}^n$. In this approximation, the differential operator $\ell(\cdot)$ is approximated at a particular point $x_i \in \Xi$ as a weighted linear combination of function values at closest $n_s$ neighbor stencil of the point $x_i$ as follows:

$$\ell[w(x_i)] \equiv \sum_{j \in I(x_i)} \alpha_j w_j,$$

(7)

where $I(x_i) = \{ j \in \{1, 2, \ldots, M\} : \| x_j - x_i \| \leq \delta \}$ and $\delta$ is the radius of support domain.

Now, the main task is to calculate the unknown coefficients $\alpha_j$ occurred in (7). For this purpose, assume the
approximation of the unknown solution \( w(.) \) is as follows [35, 36].

\[
w(x) = \sum_{k \in I(x)} \beta_k \varphi \left( \| x - x_k \|_2 \right), \quad x \in \Xi, \tag{8}
\]

where \( \varphi(.) \) is MQ-RBF discussed in above subsection.

Apply the differential operator \( \ell [.] \) over (8) and take \( x = x_c \), we get

\[
\ell [w(x_c)] = \sum_{k \in I(x)} \beta_k \ell [\varphi \left( \| x_c - x_k \|_2 \right)], \quad x_c \in \Xi. \tag{9}
\]

Now, using the (7)–(9), we can write as follows:

\[
\sum_{k \in I(x_c)} \beta_k \ell [\varphi \left( \| x_c - x_k \|_2 \right)] = \sum_{j \in I(x)} a_j w_j
\]

\[
= \sum_{j \in I(x)} a_j \left( \sum_{k \in I(x)} \beta_k \varphi \left( \| x_j - x_k \|_2 \right) \right). \tag{10}
\]

To compute the unknown weights \( a_j \), the system algebraic (10) can be written as

\[
\begin{bmatrix}
\varphi \| x_1 - x_1 \| & \varphi \| x_2 - x_1 \| & \cdots & \varphi \| x_1 I(x), x_1 \| - x_1 \| \\
\varphi \| x_1 - x_2 \| & \varphi \| x_2 - x_2 \| & \cdots & \varphi \| x_2 I(x), x_2 \| - x_2 \|\\
\cdots & \cdots & \cdots & \cdots \\
\varphi \| x_1 I(x), x_1 \| - x_1 \| & \varphi \| x_2 I(x), x_2 \| - x_2 \|
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_{I(x)}
\end{bmatrix} = \begin{bmatrix}
\ell \varphi \| x_c - x_1 \| \\
\ell \varphi \| x_c - x_2 \| \\
\vdots \\
\ell \varphi \| x_c I(x) \|
\end{bmatrix}. \tag{11}
\]

3. Mesh-free Numerical Scheme Based on LRBF-FD Approximation for the Parabolic Singularly Perturbed Models

Herein, we will develop a mesh-free scheme based on forward finite difference (FFD) and LRBF-FD for the simulation of parabolic SPMs (1). The steps of the scheme are as follows

3.1. Semidiscretization. In this scheme, the first step is semidiscretization of model (1) in time space. For this purpose, define \( t_n = n \tau, \quad n = 0, 1, \ldots, K, \) where \( \tau = T / K \) is the time step. Now, approximate time derivative by FFD and apply Crank–Nicolson (CN) on other part of the model, then we have

\[
\epsilon \frac{w_{n+1}^{x} + w_{n}^{x} - c^{n+1} w_{n+1}^{x} + w_{n}^{x}}{2} + \mu a^{n+1} w_{n+1}^{x} + w_{n}^{x} - b^{n+1} w_{n+1}^{x} - w_{n}^{x} = f(x, t^{n+1}) + s^{n}, \quad 0 \leq n \leq K - 1, \tag{13}
\]

where \( w^{n+1} = w(x, t^{n+1}), a^{n+1} = a(x, t^{n+1}), b^{n+1} = b(x, t^{n+1}), c^{n+1} = c(x, t^{n+1}), \) and \( s^n \) is the truncation error (T. E.) in semidiscrete problem.

With initial and BCs

\[
\begin{align*}
\epsilon \tau w_{x}^{n+1} + \tau a^{n+1} w_{x}^{n+1} - (2b^{n+1} + \tau c^{n+1}) w^{n+1} &= -\epsilon \tau w_{x}^{n} + \tau a^{n+1} w_{x}^{n} + (\tau c^{n+1} - 2b^{n+1}) w^{n} + 2f(x, t^{n+1}) + 2\tau s^{n}. \tag{15}
\end{align*}
\]
Equation (15) is the system of linear ordinary differential equations (ODEs) for each value of $n$ and if we drop out the small $T. E. s_n$, we have

$$
\varepsilon \tau w^{n+1}_{xx} + \tau \mu a^{n+1} w^{n+1}_x - (2b^{n+1} + \tau c^{n+1}) w^{n+1} = -\varepsilon \tau w^n_{xx} - \tau \mu a^{n+1} w^n_x + \left( \tau c^{n+1} - 2b^{n+1} \right) w^n + 2 \tau f(x, t^{n+1}).
$$

(16)

3.2. Fully Discretization. For fully discretization, we used LRBF-FD approximation. First of all, discretize the given domain by the uniform node $x_j, \ j = 1, 2, \ldots, M$. Then, discretize the equation (16) at $x = x_i$ and the spatial derivatives are approximated as follows:

$$
\varepsilon \tau w^{n+1}_{xx}(x_i) \approx \alpha^{(1)}_i w^{n+1}_i + \alpha^{(2)}_i w^{n+1}_j + \cdots + \alpha^{(M)}_i w^{n+1}_M,
$$

(17)

$$
\varepsilon \tau w^{n+1}_{xx}(x_i) \approx \alpha^{(1)}_i w^{n+1}_i + \alpha^{(2)}_i w^{n+1}_j + \cdots + \alpha^{(M)}_i w^{n+1}_M.
$$

(17)

After that use the above approximation in (16), we have the following system of linear equations:

$$
\varepsilon \tau \sum_{j=1}^{M} \alpha^{(2)}_{ij} w^{n+1}_j + \tau \mu a^{n+1} \sum_{j=1}^{M} \alpha^{(1)}_{ij} w^{n+1}_j - (2b^{n+1} + \tau c^{n+1}) w^{n+1}_i = -\varepsilon \tau \sum_{j=1}^{M} \alpha^{(2)}_{ij} w^{n}_j - \tau \mu a^{n+1} \sum_{j=1}^{M} \alpha^{(1)}_{ij} w^n_j + \left( \tau c^{n+1} - 2b^{n+1} \right) w^n_i)
$$

(18)

The above system can be written in matrix form as

$$
[A]_{M \times M} [W]_{M \times 1} = [B]_{M \times 1},
$$

(19)

where the elements of the matrices are

$$
A_{ij} = \begin{cases} 
\varepsilon \tau \alpha^{(2)}_{ij} + \tau \mu a^{n+1} \alpha^{(1)}_{ij}, & i \neq j, \\
\varepsilon \tau \alpha^{(2)}_{ij} + \tau \mu a^{n+1} \alpha^{(1)}_{ij} - (2b^{n+1} + \tau c^{n+1}), & i = j,
\end{cases}
$$

(20)

$$
B_i = -\varepsilon \tau \sum_{j=1}^{M} \alpha^{(2)}_{ij} w^{n}_j - \tau \mu a^{n+1} \sum_{j=1}^{M} \alpha^{(1)}_{ij} w^n_j + \left( \tau c^{n+1} - 2b^{n+1} \right) w^n_i + 2 \tau f(x, t^{n+1}).
$$
Input $k_{\min}$, $k_{\max}$, increment $k = 1$;
while ($k < k_{\min}$ or $k > k_{\max}$), produce interpolation matrix say $A$; use $[U, S, V] = \text{svd}(A)$; $K = \text{max}(S)/\text{min}(S)$
if $K < \text{min}K$
shape = shape – increment;
else if $K > \text{max}K$
shape = shape + increment;
end
end

Algorithm 1: Local RBF-FD scheme algorithm.

<table>
<thead>
<tr>
<th>Time</th>
<th>$L_\infty$ Errors</th>
<th>$\epsilon = 2^{-10}$</th>
<th>$\epsilon = 2^{-15}$</th>
<th>$\epsilon = 2^{-20}$</th>
<th>$\epsilon = 2^{-25}$</th>
<th>$\epsilon = 2^{-30}$</th>
<th>$\epsilon = 2^{-35}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2.809E – 04</td>
<td>1.869E – 04</td>
<td>9.654E – 05</td>
<td>2.411E – 06</td>
<td>1.466E – 06</td>
<td>1.469E – 06</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>2.300E – 03</td>
<td>1.500E – 03</td>
<td>8.173E – 04</td>
<td>2.411E – 05</td>
<td>3.545E – 06</td>
<td>3.547E – 06</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$L_2$</td>
<td>1.281E – 04</td>
<td>1.525E – 05</td>
<td>7.483E – 06</td>
<td>2.625E – 07</td>
<td>1.446E – 07</td>
<td>1.458E – 07</td>
</tr>
<tr>
<td>0.5</td>
<td>5.800E – 03</td>
<td>3.700E – 05</td>
<td>2.100E – 03</td>
<td>2.491E – 05</td>
<td>4.891E – 06</td>
<td>4.897E – 06</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>1.060E – 03</td>
<td>6.400E – 03</td>
<td>3.900E – 03</td>
<td>1.261E – 04</td>
<td>5.795E – 06</td>
<td>5.805E – 06</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>1.980E – 02</td>
<td>1.130E – 02</td>
<td>7.300E – 03</td>
<td>2.386E – 04</td>
<td>6.660E – 06</td>
<td>6.678E – 06</td>
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<tr>
<td></td>
<td>$L_2$</td>
<td>1.100E – 03</td>
<td>1.205E – 04</td>
<td>6.556E – 05</td>
<td>2.146E – 06</td>
<td>2.954E – 07</td>
<td>3.006E – 07</td>
</tr>
</tbody>
</table>

Table 1: Maximum absolute $L_\infty$ and $L_2$ errors at different times and parameters with $M = 150$, $n_s = 3$.

<table>
<thead>
<tr>
<th>Time</th>
<th>$\epsilon = 2^{-10}$</th>
<th>$\epsilon = 2^{-15}$</th>
<th>$\epsilon = 2^{-20}$</th>
<th>$\epsilon = 2^{-25}$</th>
<th>$\epsilon = 2^{-30}$</th>
<th>$\epsilon = 2^{-35}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.476</td>
<td>0.435</td>
<td>0.459</td>
<td>0.410</td>
<td>0.371</td>
<td>0.420</td>
</tr>
<tr>
<td>0.5</td>
<td>1.57</td>
<td>1.34</td>
<td>1.43</td>
<td>1.33</td>
<td>1.34</td>
<td>1.54</td>
</tr>
<tr>
<td>1.0</td>
<td>2.67</td>
<td>2.78</td>
<td>2.73</td>
<td>2.79</td>
<td>2.56</td>
<td>2.91</td>
</tr>
</tbody>
</table>

Table 2: CPU time at different time and $\epsilon$.

Now, apply the BCs and solved the obtained linear system of order $(M - 2) \times (M - 2)$ by the LU decomposition method.

For simulation purpose, we used the uniform nodes instead of nonuniform nodes since uniform nodes gives better results than nonuniform nodes for the proposed algorithm.

4. MATLAB Implementation Steps of the Developed Scheme

The implementation steps of the scheme in MATLAB are as follows:

A1. Partition the spatial domain $(0, 1)$ into grid points with uniform step length and the grid points are $x_j = j - 1/N - 1, j = 1, 2, ..., N$ and enter the constant shape parameter
A2. Compute the local first and second order approximation named as sparse differentiation matrices $a^{(1)}$ and $a^{(2)}$
A3. Enter the linear system (19) to get the solution at grid point
A4. Solve the entered system in MATLAB via LU decomposition method and iterates up to $T$ (final time)

5. Numerical Simulation and Discussion

This section ponders some particular SPMs of (1) for simulation purpose. To check the chastity and competence of the developed numerical scheme, maximum absolute error $L_\infty$ and relative error $L_2$ are computed by the formulas:

$$L_\infty = \left\| u^{\text{analytic}} - u^{\text{appx}} \right\|_{\infty} = \max_j \left| u_j^{\text{analytic}} - u_j^{\text{appx}} \right|,$$

$$L_2 = \Delta x \sum_{k=1}^{N} \left( u_k^{\text{analytic}} - u_k^{\text{appx}} \right)^2,$$

where $u^{\text{analytic}}$ and $u^{\text{appx}}$ denote analytic and approximated solutions respectively.

The simulation work has been done in the MATLAB environment on desktop computer (Intel(R) Core(TM) i3-6100T CPU, 4 GB RAM).

The shape parameter used in RBFs is calculated by the following algorithm [37]. (Algorithm 1)
5.1. SP Model 1 (Reaction-Diffusion, For $\mu = 0$). Consider the one parameter SP reaction-diffusion model as $w_t - \varepsilon w_{xx} + (1 + xe^{-t})w = f(x, t)$, $w(x, 0) = 0$, $w(0, t) = 0$, $w(1, t) = 0$, over the domain $(x, t) \in (0, t) \times (0, 1)$ which is particular case of the model (1). The function $f(x, t)$ is taken from exact solution given by [38].

$$w(x, t) = t \left( \frac{e^{-\sqrt{\varepsilon}t} + e^{-(1-x)/\sqrt{\varepsilon}}}{1 + e^{-1/\sqrt{\varepsilon}}} - \cos^2(\pi x) \right). \quad (22)$$

The results of the model are presented in Tables 1 and 2 and Figure 2. The Table 1 submits the maximum absolute $L_{\infty}$ and $L_2$ errors at different time and parameter up to $\varepsilon = 2^{-35}$. The Table concludes that as we decrease the parameter the errors decrease. For $\varepsilon = 2^{-30}$ at time $t = 1.0$, the present $L_{\infty}$ error with $M = 150$, $\varepsilon = 2^{-30}$ is $6.660E - 06$ while the same error with $N = 256$, $6.72E - 06$ (hybrid method) and $2.80E - 04$ (backward Euler method in time and central difference approximation in space) [38]. This concludes that the proposed scheme produces good results with small number of grid points. Table 2 reports the CPU time at different time and perturbation parameter $\varepsilon$. Figure 2 depicts the physical behavior of the model at different times with parameter $\varepsilon = 2^{-10}$. The figure shows that, as time increases, the solution decreases and comes closer and closer near $x = 0.5$.

5.2. SP Model 2 (Reaction-Diffusion, for $\mu = 0$). Consider the one parameter SP reaction-diffusion model as $w_t - \varepsilon w_{xx} + (1 + xe^{-t})w = f(x, t)$, $w(x, 0) = 0$, $w(0, t) = 0$, $w(1, t) = 0$, over the domain $(x, t) \in (0, t) \times (0, 1)$ which is same as the above example except the known function $f(x, t)$. The function $f(x, t)$ is taken from exact solution given by [38].

\begin{figure}[h!]
\centering
\includegraphics[width=\textwidth]{solution_behavior.png}
\caption{Solution behavior of the mode at different time with parameter $\varepsilon = 2^{-10}$ and $M = 150$.}
\end{figure}
Table 3: Maximum absolute $L_\infty$ and $L_2$ errors at different times and parameters with $M = 150$, $\eta_1 = 3$.

<table>
<thead>
<tr>
<th>Time</th>
<th>Errors</th>
<th>$\epsilon = 2^{-10}$</th>
<th>$\epsilon = 2^{-15}$</th>
<th>$\epsilon = 2^{-20}$</th>
<th>$\epsilon = 2^{-25}$</th>
<th>$\epsilon = 2^{-30}$</th>
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</thead>
<tbody>
<tr>
<td>0.1</td>
<td>$L_\infty$</td>
<td>9.634E−05</td>
<td>1.007E−04</td>
<td>7.792E−05</td>
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<td>1.395E−05</td>
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</tr>
<tr>
<td></td>
<td>$L_2$</td>
<td>5.337E−06</td>
<td>1.120E−06</td>
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<tr>
<td>0.3</td>
<td>$L_\infty$</td>
<td>6.895E−04</td>
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<td>$L_2$</td>
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</tr>
<tr>
<td>0.5</td>
<td>$L_\infty$</td>
<td>1.600E−03</td>
<td>1.800E−03</td>
<td>1.600E−03</td>
<td>4.166E−05</td>
<td>3.763E−05</td>
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<tr>
<td></td>
<td>$L_2$</td>
<td>9.198E−05</td>
<td>1.666E−05</td>
<td>1.468E−05</td>
<td>1.591E−06</td>
<td>1.594E−06</td>
<td>1.597E−06</td>
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<tr>
<td>0.7</td>
<td>$L_\infty$</td>
<td>2.800E−03</td>
<td>2.900E−03</td>
<td>2.800E−03</td>
<td>6.528E−05</td>
<td>3.986E−05</td>
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</tr>
<tr>
<td></td>
<td>$L_2$</td>
<td>1.535E−04</td>
<td>2.711E−05</td>
<td>2.518E−05</td>
<td>1.802E−06</td>
<td>1.734E−06</td>
<td>1.738E−06</td>
</tr>
<tr>
<td>1.0</td>
<td>$L_\infty$</td>
<td>4.700E−03</td>
<td>4.600E−03</td>
<td>4.800E−03</td>
<td>1.292E−04</td>
<td>3.854E−05</td>
<td>3.818E−05</td>
</tr>
<tr>
<td></td>
<td>$L_2$</td>
<td>2.587E−04</td>
<td>4.282E−05</td>
<td>4.248E−05</td>
<td>2.055E−06</td>
<td>1.792E−06</td>
<td>1.736E−06</td>
</tr>
</tbody>
</table>

Table 4: CPU time at different time and $\epsilon$.

<table>
<thead>
<tr>
<th>Time</th>
<th>$\epsilon = 2^{-10}$</th>
<th>$\epsilon = 2^{-15}$</th>
<th>$\epsilon = 2^{-20}$</th>
<th>$\epsilon = 2^{-25}$</th>
<th>$\epsilon = 2^{-30}$</th>
<th>$\epsilon = 2^{-35}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.456</td>
<td>0.440</td>
<td>0.448</td>
<td>0.408</td>
<td>0.367</td>
<td>0.418</td>
</tr>
<tr>
<td>0.5</td>
<td>1.32</td>
<td>1.32</td>
<td>1.34</td>
<td>1.28</td>
<td>1.29</td>
<td>1.52</td>
</tr>
<tr>
<td>1.0</td>
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<td>2.67</td>
<td>2.65</td>
<td>2.67</td>
<td>2.47</td>
<td>2.88</td>
</tr>
</tbody>
</table>

Figure 3: Solution behavior of the mode at different time with parameter $\epsilon = 2^{-20}$ and $M = 150$. 
\[ w(x, t) = \left(1 - e^{-t}\right) \left(\frac{e^{-x/\sqrt{\epsilon}} + e^{-(1-x)/\sqrt{\epsilon}}}{1 + e^{-1/\sqrt{\epsilon}}} - \cos^2(\pi x)\right). \]  

(23)

Tables 3 and 4 and Figure 3 offer the results of the model. Table 3 submits the maximum absolute $L_{\infty}$ and $L_2$ errors at different time and parameter up to $\epsilon = 2^{-35}$. The table concludes that, as we decrease the parameter the errors decrease. For $\epsilon = 2^{-30}$, at time $t = 1.0$, the present $L_{\infty}$ error with $M = 150$, $\epsilon = 2^{-30}$ is $3.854 \times 10^{-5}$, while the same error with $N = 256$, $5.67 \times 10^{-3}$ (hybrid method) [38]. This concludes that the proposed scheme produces good results with small number of grid points. Table 4 reports the CPU time at different time and perturbation parameter $\epsilon$. Figure 3 depicts the physical behavior in 2D and 3D of the model at different times with parameter $\epsilon = 2^{-20}$. The figure shows that, as time increases the solution decreases and comes closer and closer between $x = 0.4$ to $x = 0.6$. Second, the solution becomes steeper near the right and left boundaries.

5.3. SP Model 3. Consider the two parameter SP reaction-diffusion model with initial and BCs as [39, 40].

\[ \epsilon w_{xx}(x, t) + \mu (1 + x)w_x(x, t) - w_t(x, t) - w(x, t) = 16x^2 (1 - x)^2, \]  

(24)

and $w(x, 0) = 0$, $w(0, t) = 0$, $w(1, t) = 0$ over the domain $(x, t) \in (0, t) \times (0, T]$. The exact solution of the model is not given in literature but the solution behaviour is given in [39, 40] up to time $t = 1.0$. Here, we present numerical results in 2D and 3D form of the model in Figure 4 with parameter $\epsilon = 10^{-4}$, $\mu = 10^{-10}$, $M = 150$. The figure exhibits that as time increases up to $t = 1.0$ the solution decreases and...
Figure 5: Solution behavior of the mode at different time with parameter \( \varepsilon = 10^{-4}, \mu = 10^{-10} \) and \( M = 150 \).
the above model, we do not know the analytical solution; and closer between the right and left boundaries. Since for and goes up to

and $w(0,x) = 0$, $w(0,t) = 0$, $w(1,t) = 0$ over the domain $(x,t) \in (0,1) \times (0,T]$. The exact solution of the model is not given in literature but the solution behaviour is given in [40, 41] up to time $t = 1.0$. Here, we present numerical results in 2D and 3D form of the model in Figure 5 with paramter $\varepsilon = 10^{-4}$, $\mu = 10^{-10}$, $M = 150$. The figure exhibits that, as time increases up to $t = 2.0$, the solution decreases and goes up to $-0.274287$. Second, the solution comes closer and closer between the right and left boundaries. Since for the above model, we do not know the analytical solution; therefore to explain the performance of the proposed scheme, we depicted the solution behavior of the model.

6. Concluding Remarks

In this article, the authors developed a mesh-free numerical scheme based on FFD and LRBF-FD for the simulation of unsteady-state 1D parabolic singularly perturbed CDR models with two parameters and variable coefficients. The novelty of the work is that we solved unsteady-state singularly perturbed problems with two parameters and variable coefficients. The proposed numerical scheme is tested by taking several numerical examples and the results found satisfactory. The proposed mesh-free scheme is new for the 1D parabolic singularly perturbed CDR models and provides similar results as given in [38–41]. The developed scheme easily worked for small perturbed parameter $\varepsilon = 10^{-4}$ and variable coefficients and produced comparable results with [38–41]. The main advantage of the scheme is that it is easy to implement and an attractive alternative to the FDM and FEM which are classical solution schemes for the higher dimensional nonlinear PDEs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References


5.4. SP Model 4. Consider the two parameter SP reaction-diffusion model with initial and BCs as [40, 41].


