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Research Article

Estimation and Confidence Intervals of a New PCI C_{Npmc} for Logistic-Exponential Process Distribution

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The process capability index has been introduced as an effective tool used in industries to aid in the assessment of process performance as well as to measure how much the product meets the costumer expectations. We are aware that classical process capability indices provide better results when the quality characteristic of the processes follows normal distribution. However, these classical indices may not provide accurate results for evaluating nonnormally distributed process which in turn may hinder the decision-making. In this article, we consider a new process capability index C_{Npmc} which is based on cost function and is applicable both for normally and nonnormally distributed processes. In order to estimate the process capability index C_{Npmc} when the process follows logistic-exponential distribution, we have used ten classical methods of estimation, and the performances of these classical estimates of the index C_{Npmc} are compared in terms of their mean squared errors through a simulation study. Next, we construct five bootstrap confidence intervals of the process capability index C_{Npmc} and compare them in terms of their average width and coverage probabilities. Finally, two data sets related to electronic industries are reanalyzed to show the applicabilities of the proposed methods.

1. Introduction

Vannman [1] constructed a superstructure PCI which was referred to as $C_p(u,v)$ and is defined by

$$C_p(u,v) = \frac{d - u|\mu - m|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}, u, v > 0, \quad (1)$$

where μ and σ are the process mean and standard deviation, T is the target value and $d = (\mathrm{USL} - \mathrm{LSL})/2$, and $m = (\mathrm{LSL} + \mathrm{USL})/2$. Let USL and LSL be the upper and lower specification limits. The four basic indices, C_p, C_{pk}, C_{pm}, and C_{pmk}, are special cases of $C_p(u,v)$, by letting $u = 0$ or 1 and $v = 0$ or 1. Later, in the year 1997, Chen and Pearn [2] generalized the work of Vannman [1] and proposed a new quantile-based PCI superstructure index $C_{Np}(u,v)$ for any underlying distribution which is defined as

$$C_{Np}(u,v) = \frac{d - u|M - m|}{3\sqrt{((F_{99.865} - F_{0.135})/6)^2 + v(M - T)^2}}, \quad (2)$$

where F_α is the αth percentile and M is the median of the distribution. It has been observed that for skewed distributions, process median M is a more robust measure than the process mean μ. Considering the values of $(u,v) = (0,\ 0)$, $(1,0)$, $(0,1)$, and $(1,1)$, we can obtain the following four indices for any underlying distribution, such as

$$C_{Np} = \frac{U - L}{F_{99.865} - F_{0.135}}, \quad (3)$$

$$C_{Npk} = \min\left\{\frac{U-M}{((F_{99.865}-F_{0.135})/2)}, \frac{M-L}{((F_{99.865}-F_{0.135})/2)}\right\}, \quad (4)$$

$$C_{Npm} = \frac{U-L}{6\sqrt{((F_{99.865}-F_{0.135})/6)^2+(M-T)^2}}, \quad (5)$$

$$C_{Npmk} = \min\left\{\frac{U-M}{3\sqrt{((F_{99.865}-F_{0.135})/6)^2+(M-T)^2}}, \frac{M-L}{3\sqrt{((F_{99.865}-F_{0.135})/6)^2+(M-T)^2}}\right\}. \quad (6)$$

Clearly, when the underlying distribution is normal, then $\mu = M$ and $\sigma = (F_{99.865} - F_{0.135})/6$, and hence, the index $C_{Np}(u, v)$ reduces to $C_p(u, v)$.

In this paper, we propose a cost-effective PCI, say C_{Npmc}, using the tolerance cost function as suggested by Jeang et al. [3] which is obtained by replacing the denominator of C_{Npm} in Equation (3) by sum of quality loss, $((F_{99.865} - F_{0.135})/6)^2 + (M - T)^2$, and tolerance cost, $C_M(t)$, i.e., $[6\sqrt{((F_{99.865} - F_{0.135})/6)^2 + (M - T)^2 + C_M(t)}]$ which in turn we obtain a new index, defined as

$$C_{Npmc} = \frac{U-L}{6\sqrt{((F_{99.865}-F_{0.135})/6)^2+(M-T)^2+C_M(t)}}, \quad (7)$$

where $C_M(t) = C_0 + C_1 \exp\{-C_2 t\}$, C_0, C_1, C_2 are the coefficients for the tolerance cost function and t is the process tolerance.

In order to estimate the parameters of a model, we find multiple papers on various estimation techniques in the literature. But when attempting to estimate a model's parameters and PCIs, the maximum likelihood (ML) method is frequently utilized as a starting point. Comparative studies of various methods of estimation have been carried out for different models. It has been observed that a particular estimation procedure outperforms the others for a particular model. In the premise of this, in this paper, we consider nine estimators besides ML estimators for estimating the PCI, C_{Npmc}, under logistic-exponential distribution (LED), namely, least squares estimators (LSE), weighted least squares estimators (WLSE), maximum product spacing estimators (MPSE), minimum spacing absolute distance estimators (MDE), minimum spacing absolute-log distance estimators (MLDE), percentile estimators (PCE), Cramèr-von Mises estimators (CME), Anderson-Darling estimators (ADE), and right-tail Anderson-Darling estimators (RADE). Through a Monte-Carlo simulation analysis, the effectiveness of the estimators is evaluated in relation to their respective mean squared errors (MSEs). Point estimation might not offer accurate estimates of the PCIs, nevertheless, because of errors in the estimators. Therefore, the interval estimation methods of PCIs are used to evaluate variability or divergence in the estimates. Many methods, including the bootstrap approach, have recently been developed for building confidence intervals (CIs) for processes with nonnormal distributions. In this regard, readers may refer to the works of Leiva et al. [4], Pearn et al. [5, 6], Kashif et al. [7, 8], Weber et al. [9], Rao et al. [10], and Alomani et al. [11], to name a few. Further, Saha et al. [12] in their studies focussed on parametric estimation, bootstrap confidence interval, and highest posterior density credible interval of the index C_{pk} using normal distribution. Therefore, based on the aforementioned ten traditional estimation techniques, we consider five bootstrap confidence intervals (BCIs), namely, the standard bootstrap ($\mathscr{SB}$), percentile bootstrap ($\mathscr{PB}$), Student's t bootstrap ($\mathscr{STB}$), bias-corrected percentile bootstrap ($\mathscr{BCPB}$), and bias-corrected accelerated bootstrap ($\mathscr{BCAB}$). Estimated coverage probabilities (CPs) and average widths (AWs) are taken into account when assessing BCIs.

The goal of this paper is to develop a guideline for the choice of best estimation method that produces better estimates and CI for C_{Npmc} when the processes follow LED. Thus far, we have not come across any report for calculating PCI, C_{Npmc}, where five BCIs based on the ten traditional estimating techniques for the LED are taken into account. It is our endeavour to fill this gap through this work.

The remainder of the paper is structured as follows: sensitivity analysis has been carried out in Section 2. In Section 3, we introduce the PCI C_{Npmc} for LED. In Section 4, we describe the considered methods of estimation for C_{Npmc}. In Section 5, five BCIs ($\mathscr{SB}$, $\mathscr{PB}$, $\mathscr{STB}$, $\mathscr{BCPB}$, and $\mathscr{BCAB}$) have been discussed for the PCI C_{Npmc} based on considered estimation methods. A simulation study has been conducted and is discussed in Section 6 in order to evaluate the performance of the estimation methods and BCIs under various scenarios. Section 7 presents empirical applications employing data sets connected to the electronic industries. The concluding remarks and future works are provided in Section 8.

2. Sensitivity Analysis

According to Flaig [13] and Saha et al. [14], the study of net sensitivity ($\mathscr{NS}$) using a distribution function for a specific PCI is defined as

$$\begin{aligned}\mathscr{NS} &= \frac{1}{p_0}\lim_{\varepsilon\longrightarrow 0}\left[\frac{\{F(U)-F(L)\}-\{F(U-\varepsilon)-F(L-\varepsilon)\}}{\varepsilon}\right] \\ &= \frac{f(U)-f(L)}{p_0}.\end{aligned} \quad (8)$$

When $\mathscr{NS}$ values are positive, the distribution is observed to become less resilient (or more sensitive) at upper specification limit compared to lower specification; conversely, when $\mathscr{NS}$ values are negative, the distribution is shown to become less sensitive. The distribution in the context of PCI is less sensitive/more resilient if the $\mathscr{NS}$ values are low (in absolute sense). Table 1 has the $\mathscr{NS}$ values for the gamma, Weibull, and LE distributions. The mathematical NS values are expressed as defective per million (dpm) values. Table 1 shows that the Weibull and gamma

Table 1: Analysis of LE distribution's sensitivity.

Model	$f(\mathscr{USL})$	$f(\mathscr{LSL})$	$\mathscr{NS}(dpm)$
Gamma ($\varrho = 0.25,\ \vartheta = 4.00$)	3.029586×10^{-11}	0.08878047	−93453.13
Weibull ($\varrho = 0.25,\ \vartheta = 4.00$)	3.029586×10^{-11}	0.08878047	−155134.80
Logistic-exponential ($\varrho = 0.25,\ \vartheta = 4.00$)	0.01709995	0.002673143	15186.11

distributions are more sensitive (or less robust) than LED for $(L, U) = (0.50, 5.50)$ and $p0 = 0.95$, respectively.

3. C_{Npmc} for Logistic-Exponential Distribution

Logistic-exponential distribution (LED) was proposed by Lan and Leemis [15]. This distribution exhibits increasing, decreasing, bathtub (BT), and upside-down bathtub (UBT) shaped hazard rate function, and it is quite useful in product and process control and reliability analysis as all products or items exhibit at least one of the aforementioned characteristics of the hazard functions. The probability density, cumulative distribution, and quantile functions of the LED are

$$f(z \mid \varrho, \vartheta) = \frac{\vartheta.\varrho\left(e^{\vartheta z} - 1\right)^{(\varrho-1)} e^{\vartheta z}}{\left\{1 + \left(e^{\vartheta z} - 1\right)^{\varrho}\right\}^2}, z > 0, \varrho, \vartheta > 0, \tag{9}$$

$$F(z \mid \varrho, \vartheta) = \frac{\left(e^{\vartheta z} - 1\right)^{\varrho}}{1 + \left(e^{\vartheta z} - 1\right)^{\varrho}}, z > 0, \varrho, \vartheta > 0, \tag{10}$$

$$\xi(\gamma \mid \varrho, \vartheta) = \frac{1}{\vartheta} \log\left[1 + \left(\frac{\gamma}{1-\gamma}\right)^{1/\varrho}\right], 0 < \gamma < 1, \tag{11}$$

where ϱ is the shape parameter and ϑ is scale parameter of the two parameter LED, respectively. This distribution is a generalization of exponential distribution, and it can be obtained by taking $\varrho = 1$. This distribution is in the bathtub and upside-down bathtub classes for $0 < \varrho < 1$ and $\varrho > 1$, respectively. Then, the index C_{Npm} (see Chen and Pearn [2]) and the proposed index C_{Npmc} for the two parameter LED are, respectively, given as

$$C_{Npm} = \frac{U - L}{6\sqrt{\left((\xi(\gamma_3 \mid \varrho, \vartheta) - \xi(\gamma_1 \mid \varrho, \vartheta))/6\right)^2 + \left(\xi(\gamma_2 \mid \varrho, \vartheta) - T\right)^2}},$$
$$C_{Npmc} = \frac{U - L}{6\sqrt{\left((\xi(\gamma_3 \mid \varrho, \vartheta) - \xi(\gamma_1 \mid \varrho, \vartheta))/6\right)^2 + \left(\xi(\gamma_2 \mid \varrho, \vartheta) - T\right)^2 + C_M(t)}}, \tag{12}$$

where γ_ith is the quantile of the two parameter LED with parameters (ϱ, ϑ).

4. Estimation of C_{Npmc}

This section deals with the estimation of unknown parameters of the model using ten methods of estimation, namely, MLE, LSE, WLSE, PCE, CME, MPSE, MDE, MLDE, ADE, and RADE and the corresponding estimator of C_{Npmc}.

4.1. Maximum Likelihood Estimators. Let $z_1, z_2, \cdots, z_n$ be a random sample of size n drawn from two parameter LED (9), and then, the likelihood function is given by

$$L(\varrho, \vartheta) = \prod_{i=1}^{n} f(z_i; \varrho, \vartheta) = \prod_{i=1}^{n} \frac{\vartheta.\rho\left(e^{\vartheta z_i} - 1\right)^{(\varrho-1)} e^{\vartheta z_i}}{\left\{1 + \left(e^{\vartheta z_i} - 1\right)^{\varrho}\right\}^2}. \tag{13}$$

Taking logarithm on both the sides of Equation (13), we have

$$\log L(\varrho, \vartheta) = n \log(\vartheta) + n \log(\varrho) + (\varrho - 1)\sum_{i=1}^{n} \log\left(e^{\vartheta z_i} - 1\right) + \vartheta \sum_{i=1}^{n} z_i - 2\sum_{i=1}^{n} \log\left\{1 + \left(e^{\vartheta z_i} - 1\right)^{\varrho}\right\}. \tag{14}$$

The MLEs of ϱ and ϑ, say $\widehat{\varrho}_{mle}$ and $\widehat{\vartheta}_{mle}$, respectively, can be obtained as an iterative solutions of the following two equations:

$$\frac{\partial \log L(\varrho, \vartheta)}{\partial \varrho} = \frac{n}{\rho} + \sum_{i=1}^{n} \log\left(e^{\vartheta z_i} - 1\right) - 2\sum_{i=1}^{n} \frac{\left(e^{\vartheta z_i} - 1\right)^{\varrho} \log\left(e^{\vartheta z_i} - 1\right)}{1 + \left(e^{\vartheta z_i} - 1\right)^{\varrho}} = 0, \tag{15}$$

$$\frac{\partial \log L(\varrho, \vartheta)}{\partial \vartheta} = \frac{n}{\vartheta} + \sum_{i=1}^{n} \frac{(\varrho - 1) z_i . e^{\vartheta z_i}}{e^{\vartheta z_i} - 1} + \sum_{i=1}^{n} z_i - 2\sum_{i=1}^{n} \frac{\varrho\left(e^{\vartheta z_i} - 1\right)^{\varrho-1} x_i . e^{\vartheta z_i}}{1 + \left(e^{\vartheta z_i} - 1\right)^{\varrho}} = 0. \tag{16}$$

Equations (15) and (16) can be solved for ϱ and ϑ using any numerical iterative procedure. Since the MLEs of$\varrho$and$\vartheta$are not in the closed forms, therefore, we have used nonlinear minimization (NLM) (see Dennis and Schnabel [16]) technique by using some initial guess value for the parameters, say$\varrho = 0.01$and$\vartheta = 0.01$, and obtaining the estimates of ϱ and ϑ as $\widehat{\varrho}_{mle}$ and $\widehat{\vartheta}_{mle}$, respectively. Consequently, the MLE of C_{Npmc} can be obtained as

$$\widehat{C}_{Npmc}^{mle} = \frac{U-L}{6\sqrt{\left(\left(\xi\left(\gamma_3 \mid \widehat{\varrho}_{mle}, \widehat{\vartheta}_{mle}\right) - \xi\left(\gamma_1 \mid \widehat{\varrho}, \widehat{\vartheta}\right)\right)/6\right)^2 + \left(\xi\left(\gamma_2 \mid \widehat{\varrho}_{mle}, \widehat{\vartheta}_{mle}\right) - T\right)^2 + C_M(t)}}, \tag{17}$$

where the MLE of the quantile function with parameters (ϱ, ϑ) is

$$\xi\left(\gamma_i | \widehat{\varrho}_{mle}, \widehat{\vartheta}_{mle}\right) = \frac{1}{\widehat{\vartheta}_{mle}} \log\left\{1 + \log\left(1 + \frac{\gamma_i}{1-\gamma_i}\right)\right\}^{1/\widehat{\varrho}_{mle}}, i = 1, 2, 3. \tag{18}$$

4.2. Least Squares Estimators. We minimize the following function with respect to ρ and ϑ for obtaining the LSEs denoted by $\widehat{\varrho}_{lse}$ and $\widehat{\vartheta}_{lse}$.

$$S(\varrho, \vartheta) = \sum_{i=1}^{n}\left[F(z_{i:n}|\varrho, \vartheta) - \frac{i}{n+1}\right]^2, \tag{19}$$

where $F(\cdot)$ is the CDF, given in Equation (10) and $z_{i:n}$, $i = 1, \cdots, n$ is the ith order statistic of a random sample $z_1, z_2, \cdots, z_n$. Equivalently, they can be obtained by solving

$$\begin{aligned} &\sum_{i=1}^{n}\left[F(z_{i:n}|\varrho, \vartheta) - \frac{i}{n+1}\right]\varsigma_1(z_{i:n}|\varrho, \vartheta) = 0, \\ &\sum_{i=1}^{n}\left[F(z_{i:n}|\varrho, \vartheta) - \frac{i}{n+1}\right]\varsigma_2(z_{i:n}|\varrho, \vartheta) = 0, \end{aligned} \tag{20}$$

where

$$\varsigma_1(z_{i:n}|\varrho, \vartheta) = \left[1 + \left(e^{\vartheta z_{i:n}} - 1\right)^{\varrho}\right]^{-2}\left(e^{\vartheta z_{i:n}} - 1\right)^{\varrho}\log\left(e^{\vartheta z_{i:n}} - 1\right), \tag{21}$$

$$\varsigma_2(z_{i:n}|\varrho, \vartheta) = \left[1 + \left(e^{\vartheta z_{i:n}} - 1\right)^{\varrho}\right]^{-2}\left[\rho z_{i:n} e^{\vartheta z_{i:n}}\left(e^{\vartheta z_{i:n}} - 1\right)^{\varrho-1}\right]. \tag{22}$$

Substituting the LSEs, we can get the estimator of C_{Npmc} as

$$\widehat{C}_{Npmc}^{lse} = \frac{U-L}{6\sqrt{\left(\left(\xi\left(\gamma_3 \mid \widehat{\varrho}_{lse}, \widehat{\vartheta}_{lse}\right) - \xi\left(\gamma_1 \mid \widehat{\varrho}, \widehat{\vartheta}_{lse}\right)\right)/6\right)^2 + \left(\xi\left(\gamma_2 \mid \widehat{\varrho}_{lse}, \widehat{\vartheta}_{lse}\right) - T\right)^2 + C_M(t)}}. \tag{23}$$

4.3. Weighted Least Squares Estimators. The WLSEs, $\widehat{\varrho}_{wlse}$ and $\widehat{\vartheta}_{wlse}$, can be obtained by minimizing the following function:

$$W(\varrho, \vartheta) = \sum_{i=1}^{n} w_i\left[F(z_{i:n}|\varrho, \vartheta) - \frac{i}{n+1}\right]^2. \tag{24}$$

The estimators $\widehat{\varrho}_{wlse}$ and $\widehat{\vartheta}_{wlse}$ of the parameters ϱ and ϑ can be obtained by solving the following nonlinear equations:

$$\begin{aligned} &\sum_{i=1}^{n} w_i\left[F(z_{i:n}|\varrho, \vartheta) - \frac{i}{n+1}\right]\varsigma_1(z_{i:n}|\varrho, \vartheta) = 0, \\ &\sum_{i=1}^{n} w_i\left[F(z_{i:n}|\varrho, \vartheta) - \frac{i}{n+1}\right]\varsigma_2(z_{i:n}|\varrho, \vartheta) = 0, \end{aligned} \tag{25}$$

where $w_i = ((n+1)^2(n+2))/(i(n-i+1))$, $\varsigma_1(z_{i:n}|\varrho, \vartheta)$, and $\varsigma_2(z_{i:n}|\varrho, \vartheta)$ are defined in Equations (21) and (22), respectively. Substituting the WLSEs, we can get the estimator of C_{Npmc} as

$$\widehat{C}_{Npmc}^{wlse} = \frac{U-L}{6\sqrt{\left(\left(\xi\left(\gamma_3 \mid \widehat{\varrho}_{wlse}, \widehat{\vartheta}_{wlse}\right) - \xi\left(\gamma_1 \mid \widehat{\varrho}, \widehat{\vartheta}_{wlse}\right)\right)/6\right)^2 + \left(\xi\left(\gamma_2 \mid \widehat{\varrho}_{wlse}, \widehat{\vartheta}_{wlse}\right) - T\right)^2 + C_M(t)}}. \tag{26}$$

4.4. Percentile Estimators. The percentile estimates $\widehat{\varrho}_{pce}$ and $\widehat{\vartheta}_{pce}$ of the parameters ϱ and ϑ can be obtained by minimizing the following function with respect to ϱ and ϑ:

$$P(\varrho, \vartheta) = \sum_{j=1}^{n} \left\{ z_{(j:n)} - \frac{1}{\vartheta} \log \left[1 + \left(\frac{p_j}{1 - p_j} \right)^{1/\varrho} \right] \right\}^2. \quad (27)$$

Several estimators of p_j can be used here; see, for example, Mann et al. [17]. In this paper, we have consider $p_j = j/(n+1)$. Substituting the PCEs, we can get the estimator of C_{Nmpc} as

$$\widehat{C}_{Npmc}^{pce} = \frac{U - L}{6\sqrt{\left(\left(\xi\left(\gamma_3 \mid \widehat{\varrho}_{pce}, \widehat{\vartheta}_{pce}\right) - \xi\left(\gamma_1 \mid \widehat{\varrho}, \widehat{\vartheta}_{pce}\right)\right)/6\right)^2 + \left(\xi\left(\gamma_2 \mid \widehat{\varrho}_{pce}, \widehat{\vartheta}_{pce}\right) - T\right)^2 + C_M(t)}}. \quad (28)$$

4.5. Cramèr-von Mises Estimators. The Cramèr-von Mises estimators of ϱ and ϑ, say $\widehat{\varrho}_{cme}$ and $\widehat{\vartheta}_{cme}$, can be obtained by minimizing the following function with respect to ϱ and ϑ.

$$C(\varrho, \vartheta) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(z_{i:n} | \varrho, \vartheta) - \frac{2i-1}{2n} \right]^2. \quad (29)$$

The estimators $\widehat{\varrho}_{cme}$ and $\widehat{\vartheta}_{cme}$ of the parameters ϱ and ϑ can be obtained by solving the following nonlinear equations:

$$\begin{aligned} &\sum_{i=1}^{n} \left[F(z_{i:n} | \varrho, \vartheta) - \frac{2i-1}{2n} \right] \varsigma_1(z_{i:n} | \varrho, \vartheta) = 0, \\ &\sum_{i=1}^{n} \left[F(z_{i:n} | \varrho, \vartheta) - \frac{2i-1}{2n} \right] \varsigma_2(z_{i:n} | \varrho, \vartheta) = 0, \end{aligned} \quad (30)$$

where $\varsigma_1(z_{i:n}|\varrho, \vartheta)$ and $\varsigma_2(z_{i:n}|\varrho, \vartheta)$ are given by Equations (21) and (22), respectively. Substituting the CMEs, we can get the estimator of C_{Npmc} as

$$\widehat{C}_{Npmc}^{cme} = \frac{U - L}{6\sqrt{\left(\left(\xi\left(\gamma_3 \mid \widehat{\varrho}_{cme}, \widehat{\vartheta}_{cme}\right) - \xi\left(\gamma_1 \mid \widehat{\varrho}, \widehat{\vartheta}_{cme}\right)\right)/6\right)^2 + \left(\xi\left(\gamma_2 \mid \widehat{\varrho}_{cme}, \widehat{\vartheta}_{cme}\right) - T\right)^2 + C_M(t)}}. \quad (31)$$

4.6. Maximum Product of Spacing Estimators. Maximum product of spacing (MPS) method was proposed by Cheng and Amin [18, 19] which can be used as an alternative to ML method of estimation for estimating parameters of continuous univariate distributions. Define the uniform spacings of a random sample from the LED as

$$D_i(\varrho, \vartheta) = F(z_{i:n} | \varrho, \vartheta) - F(z_{i-1:n} | \varrho, \vartheta), i = 1, 2, \cdots, n, \quad (32)$$

where $z_{i:n}$, $i = 1, 2, \cdots, n$ is the ith order statistic of a random sample $z_1, z_2, \cdots, z_n$. Note that $z_{0:n} = 0$ and $z_{n+1:n} = 1$. The MPSEs, $\widehat{\varrho}_{mpse}$ and $\widehat{\vartheta}_{mpse}$, of the parameters ϱ and ϑ can be obtained by maximizing the following geometric mean of the spacing function with respect to ϱ and ϑ

$$G(\varrho, \vartheta) = \left[\prod_{i=1}^{n+1} D_i(\varrho, \vartheta) \right]^{1/n+1}, \quad (33)$$

or, equivalently, by maximizing the function

$$H(\varrho, \vartheta) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log D_i(\varrho, \vartheta). \quad (34)$$

The estimators $\widehat{\varrho}_{mpse}$ and $\widehat{\vartheta}_{mpse}$ of the parameters ϱ and ϑ can be obtained by solving the nonlinear equations:

$$\begin{aligned} \frac{\partial}{\partial \varrho} H(\varrho, \vartheta) &= \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(\varrho, \vartheta)} [\varsigma_1(z_{i:n} \mid \varrho, \vartheta) - \varsigma_1(z_{i-1:n} | \varrho, \vartheta)] = 0, \\ \frac{\partial}{\partial \vartheta} H(\varrho, \vartheta) &= \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(\varrho, \vartheta)} [\varsigma_2(z_{i:n} \mid \varrho, \vartheta) - \varsigma_2(z_{i-1:n} \mid \varrho, \vartheta)] = 0, \end{aligned} \quad (35)$$

where $\varsigma_1(z_{i:n}|\varrho, \vartheta)$ and $\varsigma_2(z_{i:n}|\varrho, \vartheta)$ are given by Equations

(21) and (22), respectively. Substituting the MPSEs, we can get the estimator of C_{Npmc} as

$$\widehat{C}_{Npmc}^{mpse} = \frac{U-L}{6\sqrt{\left(\left(\xi\left(\gamma_3 \mid \widehat{\varrho}_{mpse}, \widehat{\vartheta}_{mpse}\right) - \xi\left(\gamma_1 \mid \widehat{\varrho}, \widehat{\vartheta}_{mpse}\right)\right)/6\right)^2 + \left(\xi\left(\gamma_2 \mid \widehat{\varrho}_{mpse}, \widehat{\vartheta}_{mpse}\right) - T\right)^2 + C_M(t)}}. \tag{36}$$

4.7. Minimum Spacing Absolute Distance Estimators. Torabi [20] proposed the minimum spacing absolute distance estimators (MDE) of the parameters of a distribution. Thus, MDE of parameters ϱ and ϑ can be obtained by minimizing the following function:

$$T(\varrho, \vartheta) = \sum_{i=1}^{n+1} \left| D_i(\varrho, \vartheta) - \frac{1}{n+1} \right|, \tag{37}$$

with respect to ϱ and ϑ, respectively. The estimators $\widehat{\varrho}_{mde}$ and $\widehat{\vartheta}_{mde}$ of the parameters ϱ and ϑ can be obtained by solving the following nonlinear equations:

$$\begin{aligned}
\frac{\partial}{\partial \varrho} T(\varrho, \vartheta) &= \sum_{i=1}^{n+1} \frac{D_i(\varrho, \vartheta) - (1/(n+1))}{|D_i(\varrho, \vartheta) - (1/(n+1))|} \left[\varsigma_1\left(z_{(i:n)} | \varrho, \vartheta\right) - \varsigma_1\left(z_{(i-1:n)} | \varrho, \vartheta\right)\right] = 0,\\
\frac{\partial}{\partial \vartheta} T(\varrho, \vartheta) &= \sum_{i=1}^{n+1} \frac{D_i(\varrho, \vartheta) - (1/(n+1))}{|D_i(\varrho, \vartheta) - (1/(n+1))|} \left[\varsigma_2\left(z_{(i:n)} | \varrho, \vartheta\right) - \varsigma_2\left(z_{(i-1:n)} | \varrho, \vartheta\right)\right] = 0,
\end{aligned} \tag{38}$$

where $D_i(\varrho, \vartheta) \neq 1/(n+1)$, $\varsigma_1(\cdot|\varrho, \vartheta)$, and $\varsigma_2(\cdot|\varrho, \vartheta)$ are defined in Equations (21) and (22), respectively. Substituting the MDEs, we can get the estimator of C_{Npmc} as

$$\widehat{C}_{Npmc}^{mde} = \frac{U-L}{6\sqrt{\left(\left(\xi\left(\gamma_3 \mid \widehat{\varrho}_{mde}, \widehat{\vartheta}_{mde}\right) - \xi\left(\gamma_1 \mid \widehat{\varrho}, \widehat{\vartheta}_{mde}\right)\right)/6\right)^2 + \left(\xi\left(\gamma_2 \mid \widehat{\varrho}_{mde}, \widehat{\vartheta}_{mde}\right) - T\right)^2 + C_M(t)}}. \tag{39}$$

4.8. Minimum Spacing Absolute Log-Distance Estimators. Torabi [20] proposed the minimum spacing absolute-log distance estimators (MLDE). The MLDEs of the parameters ϱ and ϑ can be obtained by minimizing the function:

$$T(\varrho, \vartheta) = \sum_{i=1}^{n+1} \left| \log D_i(\varrho, \vartheta) - \log \frac{1}{n+1} \right|. \tag{40}$$

The estimators $\widehat{\varrho}_{mlde}$ and $\widehat{\vartheta}_{mlde}$ of the parameters ϱ and ϑ can be obtained by solving the following nonlinear equations:

$$\begin{aligned}
\frac{\partial}{\partial \varrho} T(\varrho, \vartheta) &= \sum_{i=1}^{n+1} \frac{\log D_i(\varrho, \vartheta) - \log(1/(n+1))}{|\log D_i(\varrho, \vartheta) - \log(1/(n+1))|} \frac{1}{D_i(\varrho, \vartheta)}\\
&\quad \cdot \left[\varsigma_1\left(z_{(i:n)} | \varrho, \vartheta\right) - \varsigma_1\left(z_{(i-1:n)} | \varrho, \vartheta\right)\right] = 0,\\
\frac{\partial}{\partial \vartheta} T(\varrho, \vartheta) &= \sum_{i=1}^{n+1} \frac{\log D_i(\varrho, \vartheta) - \log(1/(n+1))}{|\log D_i(\varrho, \vartheta) - \log(1/(n+1))|} \frac{1}{D_i(\varrho, \vartheta)}\\
&\quad \cdot \left[\varsigma_2\left(z_{(i:n)} | \varrho, \vartheta\right) - \varsigma_2\left(z_{(i-1:n)} | \varrho, \vartheta\right)\right] = 0,
\end{aligned} \tag{41}$$

where $\log D_i(\varrho, \vartheta) \neq \log(1/(n+1))$, $\varsigma_1(\cdot|\varrho, \sigma)$, and $\varsigma_2(\cdot|\varrho, \vartheta)$ are defined in Equations (21) and (22), respectively. Substituting the MLDEs, we can get the estimator of C_{Npmc} as

$$\widehat{C}_{Npmc}^{mlde} = \frac{U-L}{6\sqrt{\left(\left(\xi\left(\gamma_3 \middle| \widehat{\varrho}_{mlde}, \widehat{\vartheta}_{mlde}\right) - \xi\left(\gamma_1 \mid \widehat{\varrho}, \widehat{\vartheta}_{mlde}\right)\right)/6\right)^2 + \left(\xi\left(\gamma_2 \mid \widehat{\varrho}_{mlde}, \widehat{\vartheta}_{mlde}\right) - T\right)^2 + C_M(t)}}. \tag{42}$$

Table 2: Average estimates of different methods of estimation and the corresponding MSEs for LE distribution with true value of C_{Npmc}.

Sample size	Estimates of C_{npmc} and corresponding MSEs									
	MLE MSE	LSE MSE	WLSE MSE	PCE MSE	CME MSE	MPSE MSE	MDE MSE	MLDE MSE	ADE MSE	RADE MSE
	$\varrho = 8.0$, $\vartheta = 0.25$, $C_{Npm} = 2.441746$, $C_{Npmc} = 1.048471$									
20	1.025881	1.034055	1.036051	1.023361	1.046793	1.025216	1.036768	1.036082	1.038867	1.038775
	0.001660	0.002651	0.001895	0.002259	0.002696	0.001581	0.002165	0.001946	0.002921	0.002699
50	1.026142	1.042598	1.045120	1.035420	1.048458	1.037592	1.045534	1.045186	1.045911	1.045822
	0.000639	0.000913	0.000691	0.001011	0.000786	0.000567	0.000748	0.000721	0.001164	0.001124
100	1.026748	1.046519	1.047701	1.040851	1.048984	1.048984	1.047769	1.047746	1.047868	1.047824
	0.000384	0.000686	0.000447	0.000714	0.000677	0.000241	0.000513	0.000494	0.000887	0.000834
	$\varrho = 8.0$, $\vartheta = 0.75$, $C_{Npm} = 0.945505$, $C_{Npmc} = 0.733128$									
20	0.741886	0.732809	0.732933	0.732543	0.733337	0.732512	0.733889	0.733672	0.742344	0.742132
	$7.97 \times e^{-05}$	$8.84 \times e^{-05}$	$7.99 \times e^{-05}$	$8.93 \times e^{-05}$	$8.37 \times e^{-05}$	$7.87 \times e\ -05$	$8.24 \times e\ -05$	$8.04 \times e^{-05}$	$9.13 \times e^{-05}$	$9.05 \times e^{-05}$
50	0.741626	0.7329117	0.733004	0.7327142	0.733106	0.732725	0.737978	0.737862	0.748842	0.746732
	$3.22 \times e\ -05$	$3.69 \times e^{-05}$	$3.27 \times e^{-05}$	$3.26 \times e^{-05}$	$3.64 \times e^{-05}$	$3.10 \times e\ -05$	$3.47 \times e\ -05$	$3.39 \times e\ -05$	$3.96 \times e\ -05$	$3.81 \times e\ -05$
100	0.741629	0.744074	0.743115	0.744139	0.743268	0.732940	0.743287	0.743236	0.744174	0.744164
	$1.62 \times e\ -05$	$1.93 \times e\ -05$	$1.69 \times e\ -05$	$1.99 \times e\ -05$	$1.89 \times e\ -05$	$1.53 \times e\ -05$	$1.84 \times e^{-05}$	$1.77 \times e^{-05}$	$2.11 \times e\ -05$	$2.07 \times e\ -05$
	$\varrho = 12.0$, $\vartheta = 0.25$, $C_{Npm} = 3.281044$, $C_{Npmc} = 1.094454$									
20	1.084777	1.087469	1.085521	1.088017	1.087078	1.083222	1.085963	1.085884	1.088264	1.088155
	0.000504	0.000682	0.000529	0.000734	0.000627	0.000489	0.000604	0.000566	0.000814	0.000786
50	1.086602	1.088170	1.086690	1.088452	1.087896	1.085662	1.087442	1.087212	1.088284	1.088224
	0.000201	0.000417	0.000256	0.000449	0.000385	0.000161	0.000314	0.000288	0.000564	0.000512
100	1.091965	1.093589	1.092504	1.093751	1.093374	1.091564	1.092896	1.092764	1.094264	1.094077
	$7.77 \times e^{-05}$	$9.53 \times e\ -05$	$8.24 \times e^{-05}$	$9.66 \times e\ -05$	$9.24 \times e\ -05$	$7.16 \times e\ -05$	$8.98 \times e^{-05}$	$8.76 \times e^{-05}$	$9.97 \times e\ -05$	$9.82 \times e\ -05$
	$\varrho = 12.0$, $\vartheta = 0.75$, $C_{Npm} = 0.949039$, $C_{Npmc} = 0.734772$									
20	0.734619	0.734746	0.734641	0.734763	0.734703	0.734429	0.734679	0.734668	0.734785	0.734778
	$3.56 \times e\ -05$	$3.91 \times e\ -05$	$3.94 \times e^{-05}$	$3.66 \times e\ -05$	$3.88 \times e\ -05$	$3.42 \times e\ -05$	$3.81 \times e\ -05$	$3.74 \times e\ -05$	$3.99 \times e\ -05$	$3.97 \times e\ -05$
50	0.734666	0.734844	0.734774	0.734876	0.734812	0.734657	0.734796	0.734781	0.734884	0.734849
	$1.52 \times e\ -05$	$1.81 \times e^{-05}$	$1.59 \times e^{-05}$	$1.84 \times e\ -05$	$1.76 \times e^{-05}$	$1.41 \times e\ -05$	$1.69 \times e\ -05$	$1.64 \times e\ -05$	$1.95 \times e\ -05$	$1.89 \times e\ -05$
100	0.738505	0.734633	0.734567	0.734688	0.734585	0.734487	0.734541	0.734508	0.734759	0.734722
	$7.47 \times e^{-06}$	$7.81 \times e^{-06}$	$7.55 \times e^{-06}$	$7.88 \times e^{-06}$	$7.75 \times e\ -06$	$7.39 \times e^{-06}$	$7.69 \times e\ -06$	$7.62 \times e^{-06}$	$7.98 \times e\ -06$	$7.93 \times e^{-06}$

Table 3: AW and CPs of BCIs of C_{Npmc} by using MLEs of the parameters along with true values of C_{Npmc}.

PCI Parameters	n	$\mathscr{SB}$ AW	$\mathscr{SB}$ CP	$\mathscr{PB}$ AW	$\mathscr{PB}$ CP	$\mathscr{STB}$ AW	$\mathscr{STB}$ CP	$\mathscr{BCPB}$ AW	$\mathscr{BCPB}$ CP	$\mathscr{BCAB}$ AW	$\mathscr{BCAB}$ CP
$C_{Npm}\ c = 1.048471$	20	0.938787	0.920	0.920182	0.922	0.914283	0.919	0.889994	0.918	0.886789	0.917
$\varrho = 8.0$	50	0.826569	0.925	0.798785	0.926	0.791124	0.924	0.772134	0.922	0.768512	0.921
$\vartheta = 0.25$	100	0.456566	0.931	0.424437	0.933	0.416754	0.929	0.401326	0.929	0.398845	0.928
$C_{Npm}\ c = 0.733128$	20	0.751114	0.921	0.715634	0.924	0.702561	0.920	0.673432	0.919	0.668025	0.918
$\varrho = 8.0$	50	0.681126	0.926	0.652238	0.928	0.646724	0.924	0.606972	0.922	0.597774	0.920
$\vartheta = 0.75$	100	0.348842	0.933	0.317879	0.934	0.311174	0.932	0.284532	0.929	0.279973	0.927
$C_{Npmc} = 1.094454$	20	0.969967	0.921	0.934371	0.923	0.928884	0.920	0.901126	0.918	0.897769	0.917
$\varrho = 12.0$	50	0.850138	0.926	0.817892	0.929	0.807854	0.927	0.777878	0.926	0.769238	0.924
$\vartheta = 0.25$	100	0.482541	0.933	0.451214	0.934	0.442156	0.932	0.419992	0.930	0.415537	0.929
$C_{Npmc} = 0.734772$	20	0.748692	0.922	0.714562	0.923	0.700364	0.920	0.677649	0.919	0.669898	0.918
$\varrho = 12.0$	50	0.687779	0.924	0.657885	0.926	0.642114	0.923	0.601242	0.923	0.599986	0.922
$\vartheta = 0.75$	100	0.350127	0.935	0.318856	0.936	0.314326	0.931	0.297859	0.930	0.285437	0.929

Table 4: AW and CPs of BCIs of C_{Npmc} by using LSEs of the parameters along with true values of C_{Npmc}.

PCI Parameters	n	$\mathscr{SB}$ AW	$\mathscr{SB}$ CP	$\mathscr{PB}$ AW	$\mathscr{PB}$ CP	$\mathscr{STB}$ AW	$\mathscr{STB}$ CP	$\mathscr{BCPB}$ AW	$\mathscr{BCPB}$ CP	$\mathscr{BCAB}$ AW	$\mathscr{BCAB}$ CP
$C_{Npm}\ c = 1.048471$	20	0.946786	0.920	0.927764	0.921	0.919868	0.919	0.898888	0.919	0.897681	0.917
$\varrho = 8.0$	50	0.834427	0.924	0.802341	0.925	0.798998	0.923	0.781112	0.922	0.770983	0.921
$\vartheta = 0.25$	100	0.467789	0.933	0.437675	0.935	0.422457	0.930	0.416743	0.929	0.404075	0.928
$C_{Npmc} = 0.733128$	20	0.756754	0.923	0.727799	0.924	0.715454	0.921	0.688975	0.921	0.678964	0.918
$\varrho = 8.0$	50	0.698988	0.925	0.667893	0.927	0.648989	0.924	0.614564	0.923	0.608879	0.922
$\vartheta = 0.75$	100	0.361146	0.933	0.347986	0.935	0.323247	0.931	0.298887	0.929	0.282987	0.927
$C_{Npm}\ c = 1.094454$	20	0.989233	0.921	0.949988	0.922	0.935764	0.920	0.910068	0.918	0.907612	0.917
$\varrho = 12.0$	50	0.858967	0.925	0.827682	0.927	0.816779	0.926	0.786525	0.925	0.777779	0.922
$\vartheta = 0.25$	100	0.488897	0.934	0.460987	0.936	0.447777	0.931	0.423359	0.930	0.418756	0.929
$C_{Npm}\ c = 0.734772$	20	0.750002	0.921	0.720098	0.922	0.714563	0.920	0.689797	0.918	0.676667	0.916
$\varrho = 12.0$	50	0.690893	0.924	0.668877	0.925	0.645657	0.923	0.618877	0.923	0.616782	0.922
$\vartheta = 0.75$	100	0.367777	0.935	0.324982	0.936	0.318877	0.931	0.303067	0.929	0.288898	0.927

4.9. Anderson-Darling and Right-Tail Anderson-Darling Estimators. The Anderson-Darling estimator (see Anderson and Darling [21]) is another type of minimum distance estimators. The ADEs $\widehat{\varrho}_{ade}$ and $\widehat{\vartheta}_{ade}$ of the parameters ϱ and ϑ of L-E distribution are obtained by minimizing the function:

$$A(\varrho, \vartheta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log F\left(z_{(i:n)}|\varrho, \vartheta\right) + \log \bar{F}\left(z_{(n+1-i:n)}|\varrho, \vartheta\right)\right\}. \tag{43}$$

These estimators can also be obtained by solving the following nonlinear equations:

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\varsigma_1\left(z_{(i:n)}|\varrho, \vartheta\right)}{F(z_{i:n}|\varrho, \vartheta)} - \frac{\varsigma_1\left(z_{(n+1-i:n)}|\varrho, \vartheta\right)}{\bar{F}\left(z_{(n+1-i:n)}|\varrho, \vartheta\right)}\right] = 0,$$
$$\sum_{i=1}^{n}(2i-1)\left[\frac{\varsigma_2\left(z_{(i:n)}|\varrho, \vartheta\right)}{F\left(z_{(i:n)}|\varrho, \vartheta\right)} - \frac{\varsigma_2\left(z_{(n+1-i:n)}|\varrho, \vartheta\right)}{\bar{F}\left(z_{(n+1-i:n)}|\varrho, \vartheta\right)}\right] = 0, \tag{44}$$

where $\varsigma_1(\cdot|\varrho, \vartheta)$ and $\varsigma_2(\cdot|\varrho, \vartheta)$ are defined in Equations (21) and (22), respectively. Substituting the ADEs, we can get the estimator of C_{Npmc} as

TABLE 5: AW and CPs of BCIs of C_{Npmc} by using WLSEs of the parameters along with true values of C_{Npmc}.

PCI Parameters	n	$\mathcal{SB}$ AW	$\mathcal{SB}$ CP	$\mathcal{PB}$ AW	$\mathcal{PB}$ CP	$\mathcal{STB}$ AW	$\mathcal{STB}$ CP	$\mathcal{BCPB}$ AW	$\mathcal{BCPB}$ CP	$\mathcal{BCAB}$ AW	$\mathcal{BCAB}$ CP
C_{Npm} $c = 1.048471$	20	0.939897	0.920	0.922436	0.921	0.917765	0.918	0.891232	0.918	0.887984	0.916
$\varrho = 8.0$	50	0.835421	0.925	0.801137	0.926	0.796579	0.924	0.773365	0.922	0.768624	0.921
$\vartheta = 0.25$	100	0.459993	0.930	0.427893	0.932	0.424311	0.929	0.407863	0.929	0.401147	0.927
C_{Npm} $c = 0.733128$	20	0.754762	0.921	0.724317	0.923	0.709218	0.920	0.674456	0.918	0.669236	0.917
$\varrho = 8.0$	50	0.686782	0.925	0.657684	0.927	0.648879	0.923	0.617617	0.922	0.598834	0.920
$\vartheta = 0.75$	100	0.350782	0.933	0.323347	0.935	0.317889	0.931	0.289796	0.929	0.281124	0.928
C_{Npm} $c = 1.094454$	20	0.971124	0.921	0.936754	0.922	0.929227	0.919	0.902341	0.918	0.898954	0.917
$\varrho = 12.0$	50	0.852136	0.925	0.819797	0.928	0.813254	0.927	0.778241	0.925	0.769776	0.923
$\vartheta = 0.25$	100	0.484487	0.933	0.452783	0.935	0.443365	0.932	0.425743	0.929	0.416784	0.928
C_{Npm} $c = 0.734772$	20	0.751126	0.922	0.717698	0.924	0.704872	0.921	0.678989	0.919	0.670634	0.918
$\varrho = 12.0$	50	0.693561	0.925	0.659889	0.927	0.648844	0.924	0.610678	0.923	0.602471	0.922
$\vartheta = 0.75$	100	0.354489	0.933	0.325678	0.935	0.319982	0.932	0.299967	0.931	0.287864	0.929

TABLE 6: AW and CPs of BCIs of C_{Npmc} by using PC PCEs of the parameters along with true values of C_{Npmc}.

PCI Parameters	n	$\mathcal{SB}$ AW	$\mathcal{SB}$ CP	$\mathcal{PB}$ AW	$\mathcal{PB}$ CP	$\mathcal{STB}$ AW	$\mathcal{STB}$ CP	$\mathcal{BCPB}$ AW	$\mathcal{BCPB}$ CP	$\mathcal{BCAB}$ AW	$\mathcal{BCAB}$ CP
C_{Npm} $c = 1.048471$	20	0.951112	0.920	0.927875	0.923	0.919999	0.920	0.895684	0.918	0.889988	0.917
$\varrho = 8.0$	50	0.834321	0.924	0.812673	0.925	0.800742	0.923	0.778987	0.922	0.772244	0.921
$\vartheta = 0.25$	100	0.458789	0.933	0.432531	0.935	0.418889	0.930	0.415643	0.930	0.407825	0.929
C_{Npm} $c = 0.733128$	20	0.756667	0.922	0.728767	0.924	0.705675	0.921	0.678887	0.920	0.675433	0.918
$\varrho = 8.0$	50	0.687779	0.925	0.662312	0.926	0.649991	0.924	0.616745	0.922	0.608866	0.921
$\vartheta = 0.75$	100	0.342317	0.933	0.357769	0.935	0.319798	0.931	0.298604	0.929	0.280994	0.928
C_{Npmc} $= 1.094454$	20	0.978976	0.921	0.947893	0.923	0.932125	0.920	0.908996	0.919	0.900781	0.917
$\varrho = 12.0$	50	0.858888	0.926	0.829879	0.928	0.817766	0.925	0.780957	0.924	0.774532	0.922
$\vartheta = 0.25$	100	0.487887	0.933	0.465781	0.936	0.449999	0.931	0.423452	0.929	0.418788	0.927
C_{Npmc} $= 0.734772$	20	0.750897	0.921	0.723411	0.922	0.708889	0.920	0.689878	0.919	0.675643	0.917
$\varrho = 12.0$	50	0.697865	0.925	0.674532	0.926	0.648887	0.923	0.617684	0.922	0.607893	0.921
$\vartheta = 0.75$	100	0.366679	0.934	0.332214	0.935	0.318887	0.930	0.307984	0.929	0.288867	0.927

$$\widehat{C}_{Npmc}^{ade} = \frac{U - L}{6\sqrt{\left(\left(\xi\left(\gamma_3 \mid \widehat{\varrho}_{ade}, \widehat{\vartheta}_{ade}\right) - \xi\left(\gamma_1 \mid \widehat{\varrho}, \widehat{\vartheta}_{ade}\right)\right)/6\right)^2 + \left(\xi\left(\gamma_2 \mid \widehat{\varrho}_{ade}, \widehat{\vartheta}_{ade}\right) - T\right)^2 + C_M(t)}}. \tag{45}$$

Table 7: AW and CPs of BCIs of C_{Npmc} by using CMEs of the parameters along with true values of C_{Npmc}.

PCI Parameters	n	$\mathcal{SB}$ AW	$\mathcal{SB}$ CP	$\mathcal{PB}$ AW	$\mathcal{PB}$ CP	$\mathcal{STB}$ AW	$\mathcal{STB}$ CP	$\mathcal{BCPB}$ AW	$\mathcal{BCPB}$ CP	$\mathcal{BCAB}$ AW	$\mathcal{BCAB}$ CP
C_{Npmc} = 1.048471	20	0.942317	0.920	0.926754	0.921	0.917896	0.919	0.897683	0.918	0.889897	0.917
$\varrho = 8.0$	50	0.845328	0.924	0.812317	0.926	0.806754	0.924	0.777657	0.922	0.769897	0.921
$\vartheta = 0.25$	100	0.460987	0.932	0.428889	0.934	0.429797	0.929	0.416743	0.929	0.406754	0.928
C_{Npm} $c = 0.733128$	20	0.764351	0.921	0.727899	0.923	0.710068	0.919	0.679988	0.918	0.670894	0.916
$\varrho = 8.0$	50	0.690714	0.925	0.662324	0.927	0.646745	0.923	0.619878	0.922	0.598834	0.921
$\vartheta = 0.75$	100	0.360792	0.933	0.327677	0.936	0.319878	0.931	0.291114	0.929	0.283784	0.928
C_{Npmc} = 1.094454	20	0.973434	0.921	0.947765	0.922	0.930714	0.919	0.907886	0.918	0.901262	0.917
$\varrho = 12.0$	50	0.858867	0.925	0.826513	0.927	0.817656	0.925	0.786745	0.924	0.771135	0.922
$\vartheta = 0.25$	100	0.488794	0.932	0.456579	0.935	0.448967	0.931	0.427774	0.929	0.417798	0.928
C_{Npmc} = 0.734772	20	0.768794	0.922	0.718889	0.923	0.708798	0.921	0.682237	0.919	0.677784	0.918
$\varrho = 12.0$	50	0.697778	0.925	0.664432	0.927	0.650798	0.924	0.614532	0.924	0.606756	0.923
$\vartheta = 0.75$	100	0.360224	0.933	0.335429	0.936	0.322134	0.932	0.300785	0.930	0.289798	0.928

Table 8: AW and CPs of BCIs of C_{Npmc} by using MPSEs of the parameters along with true values of C_{Npmc}.

PCI Parameters	n	$\mathcal{SB}$ AW	$\mathcal{SB}$ CP	$\mathcal{PB}$ AW	$\mathcal{PB}$ CP	$\mathcal{STB}$ AW	$\mathcal{STB}$ CP	$\mathcal{BCPB}$ AW	$\mathcal{BCPB}$ CP	$\mathcal{BCAB}$ AW	$\mathcal{BCAB}$ CP
C_{Npmc} = 1.048471	20	0.934461	0.921	0.918792	0.922	0.913262	0.919	0.889879	0.917	0.886573	0.916
$\varrho = 8.0$	50	0.823617	0.925	0.792314	0.926	0.789436	0.923	0.769981	0.923	0.768435	0.921
$\vartheta = 0.25$	100	0.451129	0.930	0.421027	0.932	0.410782	0.929	0.399561	0.929	0.397814	0.927
C_{Npmc} = 0.733128	20	0.745137	0.921	0.707348	0.923	0.693373	0.920	0.669894	0.919	0.667124	0.917
$\varrho = 8.0$	50	0.678856	0.924	0.641276	0.925	0.635891	0.923	0.598786	0.922	0.596782	0.921
$\vartheta = 0.75$	100	0.345872	0.933	0.313396	0.934	0.307482	0.932	0.279957	0.930	0.278346	0.928
C_{Npmc} = 1.094454	20	0.967237	0.922	0.930867	0.923	0.925643	0.919	0.898978	0.919	0.895641	0.917
$\varrho = 12.0$	50	0.845632	0.925	0.813425	0.926	0.805673	0.924	0.773217	0.924	0.768891	0.923
$\vartheta = 0.25$	100	0.478567	0.933	0.444247	0.935	0.437124	0.932	0.418793	0.931	0.414168	0.929
C_{Npmc} = 0.734772	20	0.747862	0.922	0.712752	0.924	0.698985	0.920	0.673251	0.919	0.668769	0.918
$\varrho = 12.0$	50	0.683347	0.925	0.654761	0.926	0.637774	0.924	0.598984	0.923	0.596888	0.922
$\vartheta = 0.75$	100	0.346767	0.934	0.317774	0.936	0.309969	0.931	0.292235	0.931	0.283129	0.929

Similarly, the right-tail Anderson-Darling (RAD) estimators $\widehat{\varrho}_{rtade}$ and $\widehat{\vartheta}_{rtade}$ of the parameters ϱ and ϑ are obtained by minimizing the following function:

$$R(\varrho,\vartheta) = \frac{n}{2} - 2\sum_{i=1}^{n} F(z_{i:n}|\varrho,\vartheta) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log \bar{F}\left(z_{(n+1-i:n)}|\varrho,\vartheta\right). \tag{46}$$

These estimators can also be obtained by solving the nonlinear equations:

$$\begin{aligned} -2\sum_{i=1}^{n}\varsigma_1\left(x_{(i:n)}|\varrho,\vartheta\right) + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\varsigma_1\left(z_{(n+1-i:n)}|\varrho,\vartheta\right)}{\bar{F}(z_{n+1-i:n}|\varrho,\vartheta)} &= 0,\\ -2\sum_{i=1}^{n}\varsigma_2\left(x_{(i:n)}|\varrho,\vartheta\right) + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\varsigma_2\left(z_{(n+1-i:n)}|\varrho,\vartheta\right)}{\bar{F}\left(z_{(n+1-i:n)}|\varrho,\vartheta\right)} &= 0, \end{aligned} \tag{47}$$

where $\varsigma_1(\cdot|\varrho,\vartheta)$ and $\varsigma_2(\cdot|\varrho,\vartheta)$ are defined in Equations (21) and (22), respectively. Substituting the RADEs, we can get the estimator of C_{Npmc} as

Table 9: AW and CPs of BCIs of C_{Npmc} by using MDEs of the parameters along with true values of C_{Npmc}.

PCI Parameters	n	$\mathcal{SB}$ AW	CP	$\mathcal{PB}$ AW	CP	$\mathcal{STB}$ AW	CP	$\mathcal{BCPB}$ AW	CP	$\mathcal{BCAB}$ AW	CP
C_{Npmc} = 1.048471	20	0.945674	0.920	0.926784	0.922	0.918867	0.919	0.891268	0.919	0.889674	0.917
$\varrho = 8.0$	50	0.828795	0.924	0.799999	0.926	0.796548	0.923	0.778567	0.922	0.769978	0.921
$\vartheta = 0.25$	100	0.456953	0.932	0.428884	0.934	0.417878	0.930	0.407659	0.929	0.399999	0.929
C_{Npmc} = 0.733128	20	0.754543	0.923	0.723321	0.925	0.703434	0.922	0.677889	0.921	0.669988	0.919
$\varrho = 8.0$	50	0.684563	0.925	0.658794	0.927	0.647764	0.924	0.611246	0.922	0.606784	0.921
$\vartheta = 0.75$	100	0.349999	0.933	0.356743	0.934	0.317786	0.932	0.297634	0.929	0.280673	0.927
C_{Npmc} = 1.094454	20	0.976853	0.921	0.945879	0.923	0.930427	0.920	0.904563	0.918	0.899978	0.917
$\varrho = 12.0$	50	0.854459	0.926	0.821114	0.928	0.812546	0.926	0.778889	0.925	0.770989	0.923
$\vartheta = 0.25$	100	0.485657	0.933	0.459128	0.935	0.446768	0.932	0.420893	0.929	0.417775	0.928
C_{Npmc} = 0.734772	20	0.749994	0.921	0.717876	0.923	0.709785	0.920	0.686753	0.919	0.671142	0.917
$\varrho = 12.0$	50	0.688884	0.925	0.665432	0.926	0.644457	0.923	0.613763	0.922	0.606097	0.922
$\vartheta = 0.75$	100	0.362248	0.934	0.325476	0.936	0.315579	0.931	0.299867	0.930	0.286782	0.928

Table 10: AW and CPs of BCIs of C_{Npmc} by using MLDEs of the parameters along with true values of C_{Npmc}.

PCI Parameters	n	$\mathcal{SB}$ AW	CP	$\mathcal{PB}$ AW	CP	$\mathcal{STB}$ AW	CP	$\mathcal{BCPB}$ AW	CP	$\mathcal{BCAB}$ AW	CP
C_{Npmc} = 1.048471	20	0.937684	0.921	0.919981	0.922	0.916767	0.921	0.893347	0.918	0.886666	0.917
$\varrho = 8.0$	50	0.828674	0.924	0.796754	0.926	0.789899	0.924	0.772236	0.923	0.768778	0.921
$\vartheta = 0.25$	100	0.458975	0.931	0.426745	0.933	0.416785	0.929	0.402672	0.929	0.398973	0.928
C_{Npmc} = 0.733128	20	0.748978	0.921	0.712635	0.923	0.697894	0.920	0.672234	0.920	0.668795	0.919
$\varrho = 8.0$	50	0.680126	0.925	0.646756	0.927	0.637880	0.924	0.609994	0.923	0.597896	0.922
$\vartheta = 0.75$	100	0.347894	0.933	0.314457	0.935	0.310477	0.932	0.281146	0.930	0.275674	0.929
C_{Npmc} = 1.094454	20	0.975643	0.921	0.941134	0.923	0.920784	0.920	0.901114	0.919	0.897769	0.917
$\varrho = 12.0$	50	0.858763	0.925	0.832256	0.927	0.817764	0.925	0.780345	0.925	0.774532	0.924
$\vartheta = 0.25$	100	0.476785	0.933	0.449998	0.936	0.437982	0.932	0.419944	0.931	0.418976	0.929
C_{Npmc} = 0.734772	20	0.748798	0.922	0.720067	0.923	0.705621	0.920	0.677789	0.919	0.669987	0.917
$\varrho = 12.0$	50	0.691114	0.925	0.657765	0.926	0.638989	0.925	0.612934	0.923	0.600789	0.922
$\vartheta = 0.75$	100	0.350891	0.934	0.325461	0.936	0.312222	0.932	0.297896	0.931	0.289991	0.929

$$\widehat{C}_{Npmc}^{rade} = \frac{U-L}{6\sqrt{\left(\left(\xi\left(\gamma_3\middle|\widehat{\varrho}_{rade},\widehat{\vartheta}_{ade}\right)-\xi\left(\gamma_1\mid\widehat{\varrho},\widehat{\vartheta}_{rade}\right)\right)/6\right)^2+\left(\xi\left(\gamma_2\mid\widehat{\varrho}_{rade},\widehat{\vartheta}_{rade}\right)-T\right)^2+C_M(t)}}. \quad (48)$$

5. Bootstrap Confidence Intervals

In this section, we use bootstrap technique to construct confidence intervals for the PCI C_{Npmc} using all considered methods of estimation. Here, we consider five methods of bootstrap CIs, namely, (i) $\mathcal{SB}$, (ii) $\mathcal{PB}$, (iii) $\mathcal{STB}$, (iv) $\mathcal{BCPB}$, and (v) $\mathcal{BCAB}$. We discuss the algorithm for the bootstrap methods based on maximum likelihood only.

Table 11: AW and CPs of BCIs of C_{Npmc} by using ADEs of the parameters along with true values of C_{Npmc}.

PCI Parameters	n	$\mathscr{SB}$ AW	$\mathscr{SB}$ CP	$\mathscr{PB}$ AW	$\mathscr{PB}$ CP	$\mathscr{STB}$ AW	$\mathscr{STB}$ CP	$\mathscr{BCPB}$ AW	$\mathscr{BCPB}$ CP	$\mathscr{BCAB}$ AW	$\mathscr{BCAB}$ CP
C_{Npmc} = 1.048471	20	0.957612	0.920	0.928889	0.922	0.920765	0.919	0.899994	0.918	0.898756	0.917
$\varrho = 8.0$	50	0.837786	0.924	0.807683	0.926	0.799898	0.923	0.784568	0.923	0.773254	0.922
$\vartheta = 0.25$	100	0.469939	0.933	0.439856	0.935	0.425673	0.930	0.418756	0.929	0.406572	0.927
C_{Npmc} = 0.733128	20	0.762315	0.923	0.728883	0.924	0.716777	0.921	0.689996	0.920	0.679798	0.918
$\varrho = 8.0$	50	0.698879	0.925	0.668867	0.926	0.656345	0.924	0.617756	0.923	0.621146	0.921
$\vartheta = 0.75$	100	0.364534	0.934	0.348987	0.936	0.326547	0.931	0.300756	0.929	0.287648	0.928
C_{Npmc} = 1.094454	20	0.996112	0.921	0.947658	0.922	0.939867	0.920	0.914436	0.918	0.908978	0.917
$\varrho = 12.0$	50	0.858988	0.925	0.829994	0.927	0.817778	0.924	0.787645	0.923	0.778987	0.921
$\vartheta = 0.25$	100	0.490064	0.934	0.462316	0.936	0.450067	0.931	0.423364	0.930	0.419754	0.928
C_{Npmc} = 0.734772	20	0.750002	0.921	0.720098	0.923	0.714563	0.920	0.689797	0.918	0.676667	0.917
$\varrho = 12.0$	50	0.692235	0.924	0.669138	0.925	0.648799	0.924	0.619162	0.923	0.617789	0.922
$\vartheta = 0.75$	100	0.369193	0.934	0.326674	0.936	0.320978	0.931	0.306547	0.929	0.290067	0.928

Table 12: AW and CPs of BCIs of C_{Npmc} by using RADEs of the parameters along with true values of C_{Npmc}.

PCI Parameters	n	$\mathscr{SB}$ AW	$\mathscr{SB}$ CP	$\mathscr{PB}$ AW	$\mathscr{PB}$ CP	$\mathscr{STB}$ AW	$\mathscr{STB}$ CP	$\mathscr{BCPB}$ AW	$\mathscr{BCPB}$ CP	$\mathscr{BCAB}$ AW	$\mathscr{BCAB}$ CP
C_{Npmc} = 1.048471	20	0.954562	0.921	0.928787	0.923	0.923126	0.920	0.898865	0.919	0.890783	0.918
$\varrho = 8.0$	50	0.838798	0.924	0.814765	0.926	0.806534	0.924	0.779933	0.922	0.775456	0.921
$\vartheta = 0.25$	100	0.460473	0.933	0.436758	0.936	0.419989	0.931	0.417658	0.930	0.409786	0.928
C_{Npmc} = 0.733128	20	0.760891	0.922	0.729337	0.924	0.707678	0.921	0.679568	0.920	0.678796	0.918
$\varrho = 8.0$	50	0.688889	0.924	0.667659	0.926	0.652311	0.924	0.618867	0.922	0.611124	0.921
$\vartheta = 0.75$	100	0.345559	0.934	0.358978	0.937	0.320678	0.931	0.299393	0.929	0.283426	0.927
C_{Npmc} = 1.094454	20	0.979926	0.921	0.949898	0.923	0.937658	0.920	0.912223	0.918	0.904536	0.917
$\varrho = 12.0$	50	0.858888	0.924	0.829879	0.927	0.817766	0.925	0.780957	0.923	0.774532	0.922
$\vartheta = 0.25$	100	0.493321	0.934	0.467983	0.936	0.450217	0.931	0.427865	0.929	0.420407	0.927
C_{Npmc} = 0.734772	20	0.754327	0.921	0.727865	0.922	0.721543	0.920	0.692157	0.918	0.677768	0.917
ϱ=12.0	50	0.699993	0.925	0.677968	0.926	0.651124	0.924	0.618484	0.923	0.609894	0.922
$\vartheta = 0.75$	100	0.372314	0.935	0.342316	0.937	0.319227	0.931	0.308878	0.929	0.289098	0.928

(1) Let $(Z_1, Z_2, \cdots, Z_n)$ be a random sample of size n drawn from LED (ϱ, ϑ). Compute MLEs $(\widehat{\varrho}_{mle}, \widehat{\vartheta}_{mle})$ of (ϱ, ϑ). A bootstrap sample $(Z_1^*, Z_2^*, \cdots, Z_n^*)$ is obtained by multiplying $1/n$ as mass at each point from the original sample

(2) Compute the MLEs $(\widehat{\varrho}^*_{mle}, \widehat{\vartheta}^*_{mle})$ of (ϱ, ϑ) as well as $\widehat{C}^{*mle}_{Npmc}$ of C_{Npmc}. The Mth bootstrap estimator of C_{Npmc} is computed as $\widehat{C}^{*mle(M)}_{Npmc} = \widehat{C}^{mle}_{Npmc}(Z_1^{*(M)}, Z_2^{*(M)}, \cdots, Z_n^{*(M)})$

(3) There are total number of n^n resamples. From these resamples, the entire collection of R values of $\widehat{C}^{*mle}_{Npmc}$ from smallest to largest would constitute an empirical bootstrap distribution as $\{\widehat{C}^{*mle(I)}_{Npmc}, I = 1(1)R\}$

5.1. $\mathscr{SB}$. Let AM^{*mle} and SD^{*mle} be the sample mean and standard deviation of $\{\widehat{C}^{*mle(I)}_{Npmc}, I = 1(1)R\}$, i.e.,

$$AM^{*mle} = \frac{1}{R}\sum_{I=1}^{R} \widehat{C}^{*mle(I)}_{Npmc},$$
$$SD^{*mle} = \sqrt{\frac{1}{(R-1)}\sum_{I=1}^{R}\left(\widehat{C}^{*mle(I)}_{Npmc} - AM'*mle\right)^2}, \tag{49}$$

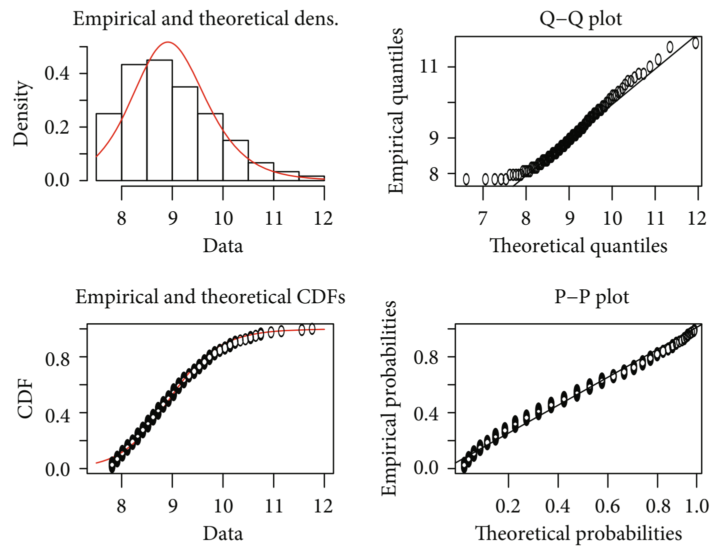

Figure 1: Histogram-density, P-P plot, Q-Q plot, and theoretical and empirical CDFs of data set I.

respectively. A $100(1-\delta)\%\mathcal{SB}$ confidence interval of C_{Npmc} is given as

$$\left\{AM^{*mle} - \mathcal{Z}_{(\delta/2)} \times SD^{*mle}, AM^{*mle} + \mathcal{Z}_{(\delta/2)} \times SD^{*mle}\right\}, \tag{50}$$

where $\mathcal{Z}_{(\delta/2)}$ is obtained by using upper $(\delta/2)$th point of the standard normal deviate.

5.2. $\mathcal{PB}$. Let $\widehat{C}_{Npmc}^{*mle(\xi)}$ be the ξ percentile of $\{\widehat{C}_{Npmc}^{*mle(I)}, I = 1(1)R\}$, i.e., $\widehat{C}_{Npmc}^{*mle(\xi)}$ is such that

$$\frac{1}{R}\sum_{I=1}^{R} \mathrm{In}\left(\widehat{C}_{Npmc}^{*(I)} \leq \widehat{C}_{Npmc}^{*(\xi)}\right) = \xi, 0 < \xi < 1, \tag{51}$$

where $\mathrm{In}(\cdot)$ is an indicator function. Then, a $100(1-\delta)\%$ $\mathcal{PB}$ confidence interval of C_{Npmc} is given as

$$\left\{\widehat{C}_{Npmc}^{*mle(R\times(\delta/2))}, \widehat{C}_{Npmc}^{*mle(R\times(1-(\delta/2)))}\right\}. \tag{52}$$

5.3. $\mathcal{STB}$. Let $\widehat{t}_{Npmc}^{*mle(\xi)}$ be the ξ percentile of $\{(\widehat{C}_{Npmc}^{*mle(I)} - \widehat{C}_{Npmc})/SD^{*mle}\}, I = 1, 2, \cdots, R$, i.e., $\widehat{t}_{Npmc}^{*mle(\xi)}$ is such that

$$\frac{1}{B}\sum_{I=1}^{R} \mathrm{In}\left(\frac{\widehat{C}_{Npmc}^{*mle(I)} - \widehat{C}_{Npmc}}{SD^{*mle}} \leq \widehat{t}_{Npmc}^{*mle(\xi)}\right) = \tau, 0 < \tau < 1, \tag{53}$$

where $\mathrm{In}(.)$ is defined above. A $100(1-\gamma)\%\mathcal{TB}$ confidence interval of $\mathcal{C}_{pc}$ is given by

$$\left\{AM^{*mle} - \widehat{t}_{Npmc}^{*mle(\xi)} \times SD^{*mle}, AM^{*mle} + \widehat{t}_{Npmc}^{*mle(\xi)} \times SD^{*mle}\right\}. \tag{54}$$

5.4. $\mathcal{BCPB}$. At first, locate the observed $\widehat{C}_{Npmc}$ in the order statistics $\{\widehat{C}_{Npmc}^{*mle(I)}, I = 1(1)R\}$. Next, we compute $G_0 = 1/R\sum_{I=1}^{R}\mathrm{In}(\widehat{C}_{Npmc}^{*mle(I)} \leq \widehat{C}_{Npmc})$ and $\Psi_0 = \Phi^{-1}(G_0)$ to calculate ψ_l and ψ_u where

$$\begin{aligned}\psi_l &= \Phi\left(2\Psi_0 - \xi_{(1-\delta/2)}\right),\\ \psi_u &= \Phi\left(2\Psi_0 + \xi_{(1-\delta/2)}\right),\end{aligned} \tag{55}$$

respectively. Then, $100(1-\delta)\%\mathcal{BCPB}$ confidence interval of C_{Npmc} is

$$\left\{\widehat{C}_{Npmc}^{*mle(R\times\psi_l)}, \widehat{C}_{Npmc}^{*mle(R\times\psi_u)}\right\}. \tag{56}$$

5.5. $\mathcal{BCAB}$. Calculate

$$\Phi_0 = \frac{\sum_{i=1}^{n}\left(\widehat{C}_{Npmc}^{mle}(\cdot) - \widehat{C}_{Npmc}^{mle}(I)\right)^3}{6\left[\sum_{i=1}^{n}\left(\widehat{C}_{Npmc}^{mle}(\cdot) - \widehat{C}_{Npmc}^{mle}(I)\right)^2\right]^{3/2}}, \tag{57}$$

where Φ_0 is called the acceleration factor and $\widehat{C}_{Npmc}^{mle}(I)$ is the MLE of C_{Npmc} based on $(n-1)$ observations after excluding the Ith observation.

$$\widehat{C}_{N}^{mle}pmc(\cdot) = \frac{1}{n}\sum_{i=1}^{n}\widehat{C}_{Npmc}^{mle}(I). \tag{58}$$

Then, a $100(1-\delta)\%\mathcal{BCAB}$ confidence interval of C_{Npmc} is given as

$$\left\{\widehat{C}_{Npmc}^{*mle(V_1)}, \widehat{C}_{Npmc}^{*mle(V_2)}\right\}, \tag{59}$$

Table 13: Model fitting summary of the considered data sets I and II.

Data set	Model	Log-likelihood	AIC	BIC	K-S statistic	K-S p value
I	L-E distribution	-151.0645	306.129	311.7040	0.076564	0.4826
II	L-E distribution	-184.7552	373.5105	378.7208	0.042123	0.9943

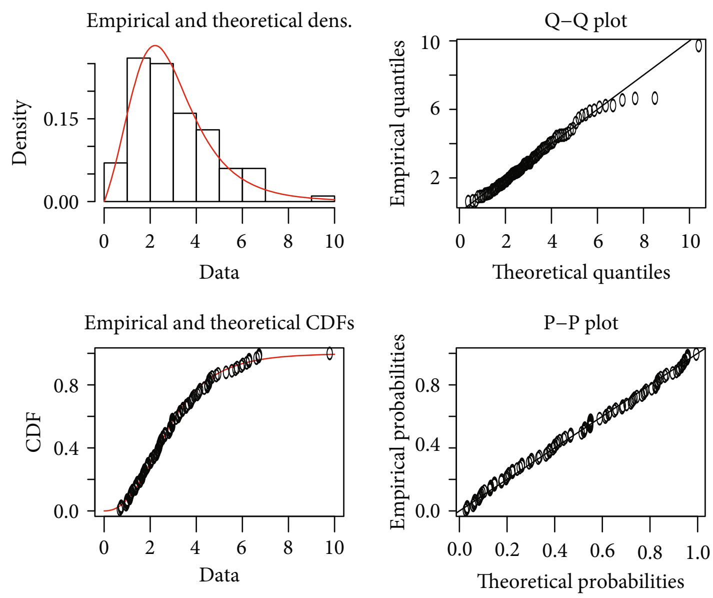

Figure 2: Histogram-density, P-P plot, Q-Q plot, and theoretical and empirical CDFs of data set II.

where $V_1 = \Phi(\Psi_0 + ((\Psi_0 + Z_{(\delta/2)})/(1 - \Phi_0(\Psi_0 + Z_{(\delta/2)}))))$ and $V_2 = \Phi(\Psi_0 + ((\Psi_0 + Z_{(1-\delta/2)})/(1 - \Phi_0(\Psi_0 + Z_{(1-\delta/2)}))))$, respectively.

6. Simulation and Discussion

Here, we conduct a simulation research to evaluate the behaviour of the different PCI C_{Npmc} estimators as described in Section 3. The performance of the estimators is compared in terms of their respective MSEs. The BCIs are compared in terms of AW and CP. The sample sizes are 20, 50, and 100 each. Additionally, we set the target value, $T = 2.50$, and the lower and upper specification limitations as 0.50 and 9.50 as well as the $(\varrho, \vartheta))$ values of (8.0,0.25), (8.0,0.75), (12.0,0.25), and (12.0,0.75), respectively. The tolerance cost function's coefficients are given as $\mathscr{C}_0 = 1.0$, $\mathscr{C}_1 = 3.0$, and $\mathscr{C}_2 = 2.0$. The values of $\mathscr{C}_M(t)$, C_{Npm}, and C_{Npmc} are then determined at $t = 0.75$. The following stages are used to present the method for obtaining the average estimations of the index C_{Npmc} and the accompanying MSEs:

(1) Draw a random sample of size n from $LED\,(\varrho, \vartheta)$

(2) Estimate the parameters ϱ, ϑ using MLE

(3) Estimate the PCI C_{Npmc} using the estimates of parameters ϱ and ϑ

(4) Repeat steps 1-3; $\mathscr{K} = 1{,}000$ times

(5) Calculate the average estimate of C_{Npmc} and MSEs based on $\mathscr{K}$ repetitions

In a similar way, we can find the estimates of C_{Npmc} and the corresponding MSEs by using LSE, WLSE, CME, PCE, MPSE, MDE, MLDE, ADE, and RADE wherein MLEs are used as initial values. The simulation results are reported in Table 2. The step-by-step procedure to obtain the considered BCIs ($\mathscr{SB}$, $\mathscr{PB}$, $\mathscr{BCPB}$, $\mathscr{TB}$, and $\mathscr{BCAB}$) is discussed in details in Section 4. For each design, $R = 1{,}000$ bootstrap samples with each of size n are drawn from the original sample using the estimates of the parameters and replicated $\mathscr{K} = 5{,}000$ times. The results of the 95% BCIs, viz., $\mathscr{SB}$, $\mathscr{PB}$, $\mathscr{BCPB}$, $\mathscr{STB}$, and $\mathscr{BCAB}$, are constructed by each of the classical methods of estimation for C_{Npmc} and are reported in Tables 3–12, respectively. The R codes for calculation of point estimate, corresponding to MSEs and AWs and CPs of BCIs of C_{Npmc}, are given as Appendix at last.

Simulated outcomes of considered estimators for the index C_{Npmc} are listed in Table 2. From Table 2, we observe that as the sample sizes increases, MSEs decrease in all the cases which eventually proves the consistency of the considered methods of estimation for our study. Simulation results of the configurations examined in our study indicate that

Table 14: Classical estimates of C_{Npmc} and C_{Npm} (MLE, LSE, WLSE, PCE, CME, MPSE, MDE, MLDE, ADE, and RADE) along with the widths of BCIs ($\mathcal{SB}$, $\mathcal{PB}$, $\mathcal{STB}$, $\mathcal{BCPB}$, and $\mathcal{BCAB}$) based on considered data sets I and II ($\mathcal{C}_0 = 1$, $\mathcal{C}_1 = 3$, $\mathcal{C}_2 = 2$, and $t = 0.50$).

Data set	Methods of estimation	Classical estimates of C_{Npm} and C_{Npmc}	Widths of BCIs				
			$\mathcal{SB}$	$\mathcal{PB}$	$\mathcal{STB}$	$\mathcal{BCPB}$	$\mathcal{BCAB}$
I	MLE	$\widehat{C}_{Npm}^{mle} = 0.528883$	0.764942	0.734325	0.684689	0.667603	0.661133
		$\widehat{C}_{Npmc}^{mle} = 0.379826$	0.664523	0.647895	0.624533	0.618794	0.615676
I	LSE	$\widehat{C}_{Npm}^{lse} = 0.521393$	0.764954	0.734338	0.684697	0.667615	0.661142
		$\widehat{C}_{Npmc}^{lse} = 0.380923$	0.664527	0.647895	0.624535	0.618794	0.61565
I	WLSE	$\widehat{C}_{Npm}^{wlse} = 0.529476$	0.764948	0.734333	0.684697	0.667611	0.661138
		$\widehat{C}_{Npmc}^{wlse} = 0.380877$	0.664524	0.647896	0.624533	0.618795	0.615676
I	PCE	$\widehat{C}_{Npm}^{pce} = 0.529489$	0.764955	0.734338	0.684696	0.667615	0.661143
		$\widehat{C}_{Npmc}^{pce} = 0.380906$	0.664528	0.647894	0.624535	0.618795	0.61566
I	CME	$\widehat{C}_{Npm}^{cme} = 0.521379$	0.764953	0.734338	0.684695	0.667614	0.661140
		$\widehat{C}_{Npmc}^{cme} = 0.372148$	0.664528	0.647895	0.624534	0.618796	0.61566
I	MPSE	$\widehat{C}_{Npm}^{mpse} = 0.521362$	0.764894	0.732563	0.684532	0.667587	0.661127
		$\widehat{C}_{Npmc}^{mpse} = 0.372137$	0.664521	0.647894	0.624532	0.618795	0.615672
I	MDE	$\widehat{C}_{Npm}^{mde} = 0.529774$	0.764952	0.734339	0.684696	0.667613	0.661141
		$\widehat{C}_{Npmc}^{mde} = 0.380944$	0.664527	0.647896	0.624534	0.618795	0.615677
I	MLDE	$\widehat{C}_{Npm}^{mlde} = 0.529684$	0.764948	0.734336	0.684696	0.667610	0.661139
		$\widehat{C}_{Npmc}^{mlde} = 0.380913$	0.664527	0.647897	0.624534	0.618795	0.615676
I	ADE	$\widehat{C}_{Npm}^{ade} = 0.531341$	0.764956	0.734341	0.684696	0.667616	0.661144
		$\widehat{C}_{Npmc}^{ade} = 0.381237$	0.664529	0.647895	0.624535	0.618797	0.615679
I	RADE	$\widehat{C}_{Npm}^{rade} = 0.531211$	0.764954	0.734339	0.684695	0.667614	0.661144
		$\widehat{C}_{Npmc}^{rade} = 0.381155$	0.664528	0.647895	0.624535	0.618796	0.615677
II	MLE	$\widehat{C}_{Npm}^{mle} = 0.594109$	0.779868	0.7508672	0.703461	0.696773	0.692257
		$\widehat{C}_{Npmc}^{mle} = 0.489147$	0.688714	0.664532	0.641178	0.637861	0.635642
II	LSE	$\widehat{C}_{Npm}^{lse} = 0.594131$	0.779868	0.7508672	0.703463	0.696773	0.692256
		$\widehat{C}_{Npmc}^{lse} = 0.489159$	0.688729	0.664546	0.641178	0.637864	0.635643
II	WLSE	$\widehat{C}_{Npm}^{wlse} = 0.594126$	0.779867	0.7508673	0.703460	0.696772	0.692256
		$\widehat{C}_{Npmc}^{wlse} = 0.489147$	0.688716	0.664533	0.641176	0.637862	0.635641
II	PCE	$\widehat{C}_{Npm}^{pce} = 0.594133$	0.779869	0.7508675	0.703464	0.696772	0.692258
		$\widehat{C}_{Npmc}^{pce} = 0.489164$	0.688733	0.664549	0.641178	0.637865	0.635645
II	CME	$\widehat{C}_{Npm}^{cme} = 0.594130$	0.779867	0.7508672	0.703461	0.696772	0.692256
		$\widehat{C}_{Npmc}^{cme} = 0.489157$	0.688724	0.664543	0.641179	0.637864	0.635644
II	MPSE	$\widehat{C}_{Npm}^{mpse} = 0.594107$	0.779866	0.7508672	0.703460	0.696771	0.692255
		$\widehat{C}_{Npmc}^{mpse} = 0.489145$	0.688712	0.664531	0.641176	0.637861	0.635641

Table 14: Continued.

Data set	Methods of estimation	Classical estimates of C_{Npm} and C_{Npmc}	Widths of BCIs $\mathscr{SB}$	$\mathscr{PB}$	$\mathscr{STB}$	$\mathscr{BCPB}$	$\mathscr{BCAB}$
II	MDE	$\widehat{C}_{Npm}^{mde} = 0.594133$	0.779869	0.7508677	0.703460	0.696774	0.692258
		$\widehat{C}_{Npmc}^{mde} = 0.489154$	0.688722	0.664539	0.641178	0.637864	0.635642
II	MLDE	$\widehat{C}_{Npm}^{mlde} = 0.594129$	0.779866	0.7508672	0.703460	0.696772	0.692257
		$\widehat{C}_{Npmc}^{mlde} = 0.489151$	0.688718	0.664537	0.641177	0.637862	0.635642
II	ADE	$\widehat{C}_{Npm}^{ade} = 0.594138$	0.779869	0.7508677	0.703464	0.696775	0.692259
		$\widehat{C}_{Npmc}^{ade} = 0.489177$	0.688752	0.664559	0.641187	0.637872	0.635649
II	RADE	$\widehat{C}_{Npm}^{rade} = 0.594136$	0.779868	0.7508676	0.703464	0.696774	0.692259
		$\widehat{C}_{Npmc}^{rade} = 0.489171$	0.688746	0.664555	0.641182	0.637868	0.635647

MPSE outperform other estimators, while second best estimator is MLE followed by WLSE using MSEs as the criteria. Thus, the order of performance of the methods of estimation in terms of MSE is MPSE < MLE < WLSE < MLDE < MDE < CME < LSE < PCE < RADE < ADE. Further, we observe that when the parameter values of ϱ increases, the value of the index C_{Npmc} increases. Results of the estimated AW and CPs of BCIs of the index C_{Npmc} using all the considered methods of estimation (MLE, LSE, WLSE, CME, PCE, MPSE, MDE, MLDE, ADE, and RADE) are listed in Tables 3–5, respectively. The comparisons of BCIs are made on the basis of lower average width and higher coverage probabilities. We take into account the nominal value as 95% for comparing coverage probabilities. Results in Tables 3–12 indicate that CI of $\mathscr{BCAB}$ provides smaller AW, while CI of $\mathscr{PB}$ provides higher CPs for all configurations and for all methods of estimation considered in the study. Moreover, we can say that, for almost all sample sizes, among the five methods of BCIs, the simulation results show the following order from the least in terms of the AW: $\mathscr{BCAB} < \mathscr{BCPB} < \mathscr{STB} < \mathscr{PB} < \mathscr{SB}$ for all settings considered in this study. Therefore, we conclude that $\mathscr{BCAB}$ method is superior to all other considered BCIs for LE distribution. Also, it has been observed that in most of the situations in simulation study, the AW of BCIs are small by using MPSE method than the other considered methods.

7. Applications

In this section, two electronic industry-related data sets are reanalyzed for illustrative purposes. In order to check the validity of the proposed model, one sample Kolmogorov-Smirnov (K-S) statistic along with its p values and two information theoretic criteria such as AIC and BIC are used. The associated unknown parameters of the model are estimated using the likelihood method. The steps listed below are used to determine the K-S statistic's p values:

(i) Fit the chosen distribution to the data

(ii) Compute the corresponding K-S statistic

(iii) 10,000 identical samples to the size of the data from the fitted distribution are simulated

(iv) Calculate the K-S statistic using the associated 10,000 values

(v) Using step (iv), create a histogram of the 10,000 values

(vi) By contrasting the histogram with the recorded statistic from step (ii), one can determine the p value

7.1. Data Set I: Electronic Telecommunication Amplifier Data. The data set I relates to the quality of the electronic communication amplifiers. The data was collected by Juran Institute [22] and reanalyzed by Peng [23]. The key quality characteristic of the data set is the gain of decibels with production specification limits $(L, T, U) = (7.75, 10, 12.25)$. In case of L-E distribution, MLEs of the parameters ϱ and ϑ are $\widehat{\varrho} = 0.07745682$ and $\widehat{\vartheta} = 13.33916572$. The model fitting summary (viz., log-likelihood, AIC, BIC, empirical and theoretical densities and CDFs, P-P plot, and Q-Q plot; see Figure 1) is reported in Table 13. The data set is given below:
1,10.4,8.8,9.7,7.8,9.9,11.7,8.0,9.3,9.0,8.2,8.9,10.1,9.4,9.2,7.9, 9.5,10.9,7.8,8.3,9.1,8.4,9.6,11.1,7.9,8.5,8.7,7.8,10.5,8.5,11.5,8.0, 7.9,8.3,8.7,10.0,9.4,9.0,9.2,10.7,9.3,9.7,8.7,8.2,8.9,8.6,9.5,9.4,8.8, 8.3,8.4,9.1,10.1,7.8,8.1,8.8,8.0,9.2,8.4,7.8,7.9,8.5,9.2,8.7,10.2,7.9, 9.8,8.3,9.0,9.6,9.9,10.6,8.6,9.4,8.8,8.2,10.5,9.7,9.1,8.0,8.7,9.8,8.5, 8.9,9.1,8.4,8.1,9.5,8.7,9.3,8.1,10.1,9.6,8.3,8.0,9.8,9.0,8.9,8.1,9.7, 8.5,8.2,9.0,10.2,9.5,8.3,8.9,9.1,10.3,8.4,8.6,9.2,8.5,9.6,9.0,10.7,8.6, 10.0,8.8,8.6.

7.2. Data Set II: Data Set Relates to Electronic Industry. The data set II is taken from Leiva et al. [4] which represents the ball size of wire bonding for an electronic connection from the integrated circuit apparatus to the lead frame. Here, the process was monitored with LSL = 0.5 mil and USL = 8.0 mil (1 mil = 1/1000 in =0.0254 mm). Here, we have considered the target value $T = 3.0$ mil. In case of L-E distribution, MLEs of the parameters ϱ and ϑ are $\widehat{\varrho} = 0.2561459$ and $\widehat{\vartheta} = 2.0632703$. The model fitting summary (viz., log-likelihood, AIC, BIC, empirical and theoretical densities and CDFs, P-P plot, and Q-Q plot; see Figure 2) is reported in Table 13. The data set is given below:

2.891,4.035,4.495,2.890,2.312,3.158,5.228,3.334,5.896, 5.639,3.842,1.590,1.954,1.842,0.680,2.752,1.301,2.260,0.889, 2.381,0.619,2.788,1.050,3.750,3.508,6.123,6.549,5.954,2.207, 4.417,4.805,1.516,2.227,2.797,1.636,1.066,0.940,4.101,4.542, 1.295,1.770,3.492,5.706,3.722,6.644,2.472,1.383,4.494,1.694, 2.892,2.111,3.591,2.093,3.222,2.891,2.582,0.665,3.234,1.102, 1.083,1.508,1.811,2.803,6.659,0.923,6.229,3.177,2.333,1.311, 4.419,2.495,0.921,4.061,9.725,1.600,4.281,3.360,1.131,1.618, 4.489,3.696,1.982,2.413,5.480,1.992,2.573,1.845,4.620,6.221, 1.694,4.882,1.380,3.982,2.260,2.366,2.899,3.782,2.336,1.175, 3.055.

In Table 14, we report the point estimates of C_{Npmc} and C_{Npm} and the width of BCIs based on different methods of estimation. For both the data sets, we observe that the width of $\mathscr{BCAB}$ interval is minimum as compared to width of other BCIs. Similar trend is exhibited in the simulation study. Further, we observe that among all methods of estimation, MPSE performs better than other methods of estimation for data sets I and II, respectively.

8. Conclusions and Future Works

In this work, we evaluate five BCIs of the PCI C_{Npmc} using MLEs, LSEs, WLSEs, CMEs, MPEs, MDEs, MLDEs, PCEs, ADEs, and RADEs. Theoretical comparisons of the cited methods will be tedious; therefore, in order to compare the performance of estimators, we undertake simulation study using different sample sizes and varied combination of parameters. We compare the performance of the estimators in respect of MSE. The performance of BCIs for the index C_{Npmc} is compared in respect of AW and CPs. Results from the simulation study indicate that MPSEs outperform other estimators while the second best estimator is MLEs followed by WLSEs. Further, CIs of BCAB outperform other CIs with respect to AW and CPs for considered methods of estimation. The results of the data analysis portray similar trend as in case of simulation study. Further, the considered index may be applicable to other areas associated with quality control such as process monitoring and acceptance sampling. Over and above, results and methods discussed in this study may be utilized by industries for decision-making. The present study can also be extended to neutrosophic statistics when the data comes from the production process or when a product lot is incomplete, incredible, and indeterminate (see Aslam and Albassam [24] and Aslam et al. [25]).

Appendix

The R Codes for Calculation of Point Estimate, Corresponding to MSEs and AWs and CPs of BCIs of C_{Npmc}, are given below:

```
g = function(L,T,U);
{
QU = (1/a)*log(1 + (0.99865/(1-0.99865))^(1/b));
QM = (1/a)*log(1 + (0.5/(1-0.5))^(1/b));
QL = (1/a)*log(1 + (0.00135/(1-0.00135))^(1/b));
QSe = (QU-QL)/6; Qe = sqrt(QSe^2 + (QM-T)^2);
c0 =1;   c1 = 3;   c2 = 2;   t = .75;   ;Cost=c0+c1*exp(-c2*t);
Qec=sqrt(QSe^2+(QM-T)^2+Cost);
cnpm=(U-L)/(6*Qe);cnpm;cnpmc=(U-L)/(6*Qec);cnpmc
#Random number generation from LED ($\varrho$;$\vartheta$)
# maximum likelihood estimate .... # Maximum product spacings estimate
}
reep=t(replicate(1000,g()))
# Obtain Average estimate and corresponding MSE
#Repeat function
fb=function(th)
{
a=th[1];b=th[2]
l=n*log(a)+n*log(b)+(b-1)*sum(log(exp(a*x)-1))-a*sum(x)
-2*sum(log(1+(exp(a*x)-1)^b))
return(-l)
}
z=nlm(fb,c(0.8,0.8));z
a_cap=z$estimate[1];a_cap
b_cap=z$estimate[2];b_cap
fd=function(th)
{a_cap=th[1];b_cap=th[2]
l2=n*log(a_cap)+n*log(b_cap)+(b_cap-1)*sum(log(exp(a_cap*x)-1))-a_cap*sum(x)
-2*sum(log(1+(exp(a_cap*x)-1)^b_cap))
return(-l2)}
a_cap_boot_mle=mean(a_cap_boot);
b_cap_boot_mle=mean(b_cap_boot);
cnpmc_boot=sort(cnpmc_boot);
cnpmc_boot_mle=mean(cnpmc_boot);
cnpmc_boot_sd=sd(cnpmc_boot);
# Standard bootstrap,..., bias-corrected accelerated bootstrap
```

Data Availability

The link of the dataset used in this study is included within the article.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors have contributed equally and agreed to the published version of the manuscript.

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