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Research Article

Some More Results on Reciprocal Degree Distance Index and $\mathscr{F}$-Sum Graphs

Asima Razzaque,[1] Saima Noor,[1] Salma Kanwal,[2] and Ayesha Manzoor[2]

[1]*Basic Science Department, King Faisal University, Al Ahsa, Hofuf, Saudi Arabia*
[2]*Department of Mathematics, Lahore College for Women University, Lahore, Pakistan*

Correspondence should be addressed to Salma Kanwal; salma.kanwal055@gmail.com

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A chemical invariant of graphical structure $\mathscr{Z}$ is a unique value characteristic that remains unchanged under graph automorphisms. In the study of QSAR/QSPR, like many other chemical invariants, reciprocal degree distance has played a significant role to estimate the bioactivity of several compounds in chemistry. Reciprocal degree distance is a chemical invariant, which is the degree weighted version of Harary index, i.e., $\mathscr{R}\mathscr{D}\mathscr{D}(\mathscr{Z}) = (1/2)\sum_{\mu,\nu\in\mathscr{V}(\mathscr{Z})}(d_{\mathscr{Z}}(\mu) + d_{\mathscr{Z}}(\nu)/d_{\mathscr{Z}}(\mu,\nu))$. Eliasi and Taeri proposed four new graphic unary operations: $\mathscr{S}(\mathscr{Z}), \mathscr{R}(\mathscr{Z}), \mathscr{Q}(\mathscr{Z})$, and $\mathscr{T}(\mathscr{Z})$, frequently implemented in sum of graphs, symbolized as $\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2$, i.e., sum of two graphs $\mathscr{F}(\mathscr{Z}_1)$, $\mathscr{Z}_2; \mathscr{F}$ is one of the unary graphic operations $\mathscr{S}, \mathscr{R}, \mathscr{Q}, \mathscr{T}$. This work provides constraints for the above-mentioned invariant for this binary graphic operation F-sum of graphs.

1. Introduction

Graphic structures in this work are all simple, finite, and directionless. The collection of nodes and edges in a graph $\mathscr{Z}$ is denoted by $\mathscr{V}(\mathscr{Z})$ and $E(\mathscr{Z})$, respectively. Let $d_{\mathscr{Z}}(\mu,\nu)$ be the distance between two vertices μ and ν in $\mathscr{Z}$, given as the cardinality of lines in shortest path between the two nodes. $d_{\mathscr{Z}}(\mu)$ is the degree of a vertex μ. Chemical invariants have been proven to be beneficial in determining the correlation between a molecule's structure and its features. Quantitative structure-property relationships (QSPRs) and quantitative structure-activity relationships (QSARs) [1] are two different sorts of topological invariants. Many chemical invariants are focused on degrees, while others are focused on distances, eccentricity, connectivity, and so on.

Under these parameters, numerous graph processes are executed immediately on basic graphs to examine their features. For such composite graphs, multiple researchers estimated topological invariants and put forward a big collection of results regarding this concept as can be seen below. Another chemical invariant symbolized by $\mathscr{W}(\mathscr{Z})$, that comprises the length of shortest paths between the two nodes, and specified as the total of these lengths across all node combinations for $\mathscr{Z}$ is the *Wiener index* [2–4].

$$\mathscr{W}(\mathscr{Z}) = \sum_{\{\mu,\nu\}\subseteq\mathscr{V}(\mathscr{Z})} d_{\mathscr{Z}}(\mu,\nu). \tag{1}$$

The degree-weighted variant of the Wiener index, the *degree distance index*, was proposed in 1994. Dobrynin and Kochetova [5] were the first two to explore it. $\mathscr{D}\mathscr{D}(\mathscr{Z})$ is the degree distance invariant of a graph $\mathscr{Z}$, which is specified as

$$\mathscr{D}\mathscr{D}(\mathscr{Z}) = \sum_{\{\mu,\nu\}\subseteq\mathscr{V}(\mathscr{Z})} d_{\mathscr{Z}}(\mu,\nu)\left[d_{\mathscr{Z}}(\mu) + d_{\mathscr{Z}}(\nu)\right], \tag{2}$$

where $d_{\mathscr{Z}}(\mu,\nu)$ is the length of the shortest route among μ and ν and $d_{\mathscr{Z}}(\mu)$ is the degree of vertex μ [6, 7]. Plavsi' et al. [8] proposed the *Harary index* in 1993, symbolized by $\mathscr{H}(\mathscr{Z})$ and given by

$$\mathscr{H}(\mathscr{X}) = \frac{1}{2}\sum_{\mu,\nu\in\mathscr{V}(\mathscr{X})}\frac{1}{d_{\mathscr{X}}(\mu,\nu)}. \quad (3)$$

Das et al. [9] came up with generalized version of Harary index, i.e., *τ-Harary index*; τ is some positive real number, given as

$$\mathscr{H}(\mathscr{X}) = \frac{1}{2}\sum_{\mu,\nu\in\mathscr{V}(\mathscr{X})}\frac{1}{d_{\mathscr{X}}(\mu,\nu)+\tau}. \quad (4)$$

A new graph invariant named as *reciprocal degree distance index* was introduced by Xiong and An [10] in 2015, defined as

$$\mathscr{R}\mathscr{D}(\mathscr{X}) = \frac{1}{2}\sum_{\mu,\nu\in\mathscr{V}(\mathscr{X})}\frac{d_{\mathscr{X}}(\mu)+d_{\mathscr{X}}(\nu)}{d_{\mathscr{X}}(\mu,\nu)}. \quad (5)$$

The degree distance invariant is the weighted version of Wiener index; similarly, the reciprocal degree distance is the weighted version of Harary index. In 2015, Vijayaragavan [11] introduced reformulated reciprocal degree distance index, which is defined as

$$\mathscr{R}\mathscr{D}\mathscr{D}_{\tau}(\mathscr{X}) = \frac{1}{2}\sum_{\mu,\nu\in V(\mathscr{X})}\frac{d_{\mathscr{X}}(\mu)+d_{\mathscr{X}}(\nu)}{d_{\mathscr{X}}(\mu,\nu)+\tau}. \quad (6)$$

Many scientists and researchers have focused on composite graphs from the past thirty years. Several binary graphic operations like product Π, join +, composition ∘, corona product, cluster, wreath product, and so on were put forward by many researchers. Now we enlist briefly the work done by several researchers for the above-mentioned binary graphic processes. Yeh and Gutman [3] computed sum of distances for every pair of nodes in the above-mentioned binary graphic operations. Paulraja and Agnes [7] made computations of accurate values of degree distance invariant for Cartesian and wreath products. Moreover, the authors determined the above-mentioned invariant for certain commonly known graphic structures like torus, Hamming, hypercube, and fence structures [12], and Khalifeh et al. brought into consideration the accurate expressions for the two Zagreb invariants and computed the same invariants for certain renowned chemical structures like nano tubes and nano torus with q multi walled.

Deng et al. [13] determined first two Zagreb invariants for the four graph operations, i.e., $\mathscr{F}$−sum of graphs. Shirdel et al. [14] proposed a new invariant named as hyper-Zagreb invariant and provided computations for these invariants for the above-mentioned binary graphic invariants.

Das et al. [9] made computations of Harary invariants for same binary processes considered in [14]. During this work, the authors encountered an expression similar to Harary invariant, and they named it as second and third Harary invariants.

Eliasi and Taeri [4] introduced four new binary graphic operations and termed them as $\mathscr{F}$-sum of structures, which gave a new direction to researchers to work with. Here we give a brief overview of work done by researchers for this new kind of operation. Imran and Akhter [15, 16] provided exact expressions for maximum and minimum value of general sum connectivity invariant and forgotten invariant, respectively. Metsidik et al. [17] provided the exact expressions for hyper and reverse Wiener invariant for F-sum of structures, and further they presented exact formulas for later mentioned invariant under certain conditions on the parameters involved. Basavangoud and Ptail [18] studied hyper-Zagreb invariant for these four graphic processes termed as F-sum. Alizadeh et al. [19] put forward a new invariant obtained by deforming Harary invariant and termed it as additive weighted Harary invariant. They also investigated this newly defined invariant for many well-known graphic operations. Pattabiraman and Vijayaraganan [20] provided exact expressions in terms of other invariants for reciprocal degree invariant of several graphic binary operations like join, strong, wreath, and tensor product. Further, applying their computed results, they computed the above-mentioned invariant for fan, open, and closed fence and wheel graph. Xiong and An [10] provided exact formulas of multiplicative weighted Harary invariant for certain well-known graphic operations like join, symmetric difference, composition, and disjunction and join. Pattabiraman and Vijayaraganan [21] put forward formulas of reformulated reciprocal degree distance invariant for many existing graphic operations like strong, wreath, and tensor products and also considered join and composition. Pattabiraman [22] presented mathematical formulas for the above-mentioned invariant and its multiplicative version for tensor product of graphic structures. Su et al. [23] observed the monotonic attitude of reformulated reciprocal degree distance invariant and determined its maximum values for graphic structures with one cycle. Novelty of this work lies in the fact that binary operations give rise to new structures displaying characteristics of the factors in a more convincible way. So, determining the bounds for reciprocal degree distance invariant for the binary operation $\mathscr{F}-$ sum enabled us to evaluate the chemical properties of more complex structures using their binary components. In this work, we will determine lower and upper bounds for reciprocal degree distance invariant for $\mathscr{F}$−sum of any two graphs, where $\mathscr{F}-$ is one of $\mathscr{S}, \mathscr{R}, \mathscr{Q}, \mathscr{T}$.

2. $\mathscr{F}$-Sum of Graphs with Four Graph Operations

To begin, consider the Cartesian product $\mathscr{X}\mathscr{W}\mathscr{Y}$ of graphs $\mathscr{X}$ and $\mathscr{Y}$. $\backslash\{a=c$ and $bd\in\mathscr{E}(\mathscr{Y})\backslash\}$ or $\backslash\{b=d$ and $ac\in\mathscr{E}(\mathscr{X})\backslash\}$ for the set of vertices $\mathscr{V}(\mathscr{X})\times\mathscr{V}(\mathscr{Y})$; then, $(a,b)(c,d)$ is a line of $\mathscr{X}\mathscr{W}\mathscr{Y}$. If the separation between the two vertices in $\mathscr{X}\mathscr{W}Y$ is $d_{\mathscr{X}\mathscr{W}\mathscr{Y}}((a,b),(c,d))$, where

$$d_{\mathscr{X}\mathscr{W}\mathscr{Y}}((a,b),(c,d)) = d_{\mathscr{X}}(a,c)+d_{\mathscr{Y}}(b,d), \quad (7)$$

$\mathscr{X}\mathscr{W}\mathscr{Y}$ contains a vertex (a,b) of degree

$$d_{\mathscr{X}\mathscr{W}\mathscr{Y}}(a,b) = d_{\mathscr{X}}(a)+d_{\mathscr{Y}}(b). \quad (8)$$

Eliasi and Taeri [4] pioneered idea of $\mathscr{F}$-sums in 2009, referred to as graphic binary operations, in which $\mathscr{F}(\mathscr{X}_1)$ and $\mathscr{X}_2$ are the operands. $\mathscr{X}_1 +_{\mathscr{F}} \mathscr{X}_2$ represents $\mathscr{F}$-sum of two graphs $\mathscr{X}_1$ and $\mathscr{X}_2$, and $\mathscr{F}$ is one of the graph operations $\mathcal{S}, \mathcal{R}, \mathcal{Q}$, or $\mathcal{T}$. Figure 1 likewise shows the $\mathcal{F}$-sum of C_3 and P_2. The graph operations $\mathcal{S}, \mathcal{R}, \mathcal{Q}, \mathcal{T}$ are defined as follows:

(1) $\mathcal{S}(\mathcal{X})$ subdivided structure is generated by inserting a node in all lines of $\mathcal{X}$.

(2) If two original nodes are neighbors, line connecting them is drawn in $\mathcal{S}(\mathcal{X})$. By combining each combination of connected black vertices, $\mathcal{R}(\mathcal{X})$ is generated from $\mathcal{S}(\mathcal{X})$.

(3) In the same way, two white vertices in $\mathcal{S}(\mathcal{X})$ are linked if their corresponding edges are neighbors in $\mathcal{X}$. By combining each combination of associated white vertices, $\mathcal{Q}(\mathcal{X})$ is achieved.

(4) The combination of two graphs $\mathcal{X}$ and $\mathcal{Y}$ is represented as $\mathcal{X} \cup \mathcal{Y}$ and has the identical vertex set $\mathcal{V}$ and lines $E(\mathcal{X}) \cup \mathcal{E}(\mathcal{Y})$. The total graph $\mathcal{T}(\mathcal{X})$ is the combination of $\mathcal{R}(\mathcal{X})$ and $\mathcal{Q}(\mathcal{X})$ in this example.

The above four graph operations can be visualized in Figure 2, and they are used on the C_3 cycle.

For the above-mentioned four graph processes, multiple writers generated a variety of chemical invariants. The Wiener index of all these graph operations was estimated by Eliasi and Taeri [4]. Two optimum values for degree distance-based invariant for this new type of graphic binary operation, $\mathcal{F}$-sums, were considered by the authors in their work [6]. Findings of Eliasi and Taeri [4] were utilized to determine length of shortest paths existing in nodes of graphic structures involved in $\mathcal{F}$-sums. The authors in [15, 16] investigated novel results for these four graph operations using the forgotten index and sum-connectivity index. For each of these procedures, they provided the sharp value for the $\mathcal{F}$ invariant and its exact upper and lower values attained for the sum connectivity invariant. Khalifeh et al. computed accurate formulas for both Zagreb invariants for several of the above-mentioned graphic binary operations. The authors in [13] investigated these sums to determine exact solutions for Zagreb invariants. Investigation of hyper-Zagreb invariants and co-invariants was made in [14], for various graphic binary operations. For these four graph operations, they derived the exact formula for hyper-Zagreb invariant.

Take $|\mathcal{V}_2| = n_2$ replicas of the graph $\mathcal{F}(\mathcal{X}_1)$ and mark them with the nodes of graph $\mathcal{X}_2$ for the $\mathcal{F}$-sum of graphs $\mathcal{X}_1$ and $\mathcal{X}_2$. The vertices of $\mathcal{X}_1 +_{\mathcal{F}} \mathcal{X}_2$ can be divided into two categories: black vertices $(\mathcal{V}_1)$ and white vertices $(\mathcal{E}_1)$. Just black nodes with the identical name in $\mathcal{F}(\mathcal{X}_1)$ and tags that are adjacent in $\mathcal{X}_2$ are now joined. Here we state results needed in next section.

Lemma 1 (see [4]). *For two graphs $\mathcal{X}_1$, $\mathcal{X}_2$, let $\nu = (\nu_1, \nu_2)$ be a node of $\mathcal{X}_1 +_{\mathcal{F}} \mathcal{X}_2$. Then, we have*

(a) If $\nu_1 \in \mathcal{V}_1$ (if ν is a black node), ultimately for every $\mu = (\mu_1, \mu_2) \in \mathcal{V}(\mathcal{X}_1 +_{\mathcal{F}} \mathcal{X}_2)$, then

$$d_{\mathcal{X}_1 +_{\mathcal{F}} \mathcal{X}_2}(\mu, \nu) = d_{\mathcal{F}(\mathcal{X}_1)}(\mu_1, \nu_1) + d_{\mathcal{X}_2}(\mu_2, \nu_2). \quad (9)$$

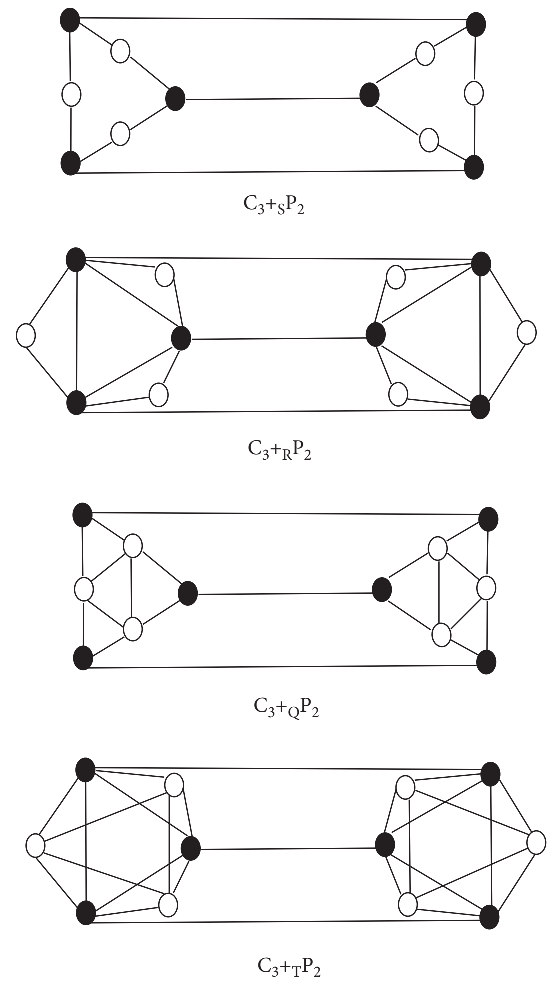

FIGURE 1: $\mathcal{F}$-sums of C_3 and P_2, $C_3 +_{\mathcal{F}} P_2$.

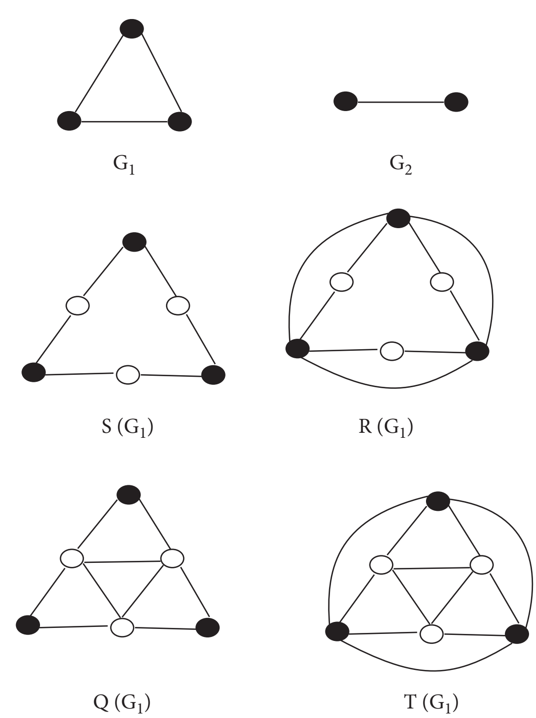

FIGURE 2: $\mathcal{F}(G_1 = C_3)$ for $\mathcal{F} = \mathcal{S}, \mathcal{R}, \mathcal{Q}, \mathcal{T}$ and $G_2 = P_2$.

(b) *If* $\nu_1 \in \mathscr{E}_1$, *then for all* $\mu = (\mu_1, \mu_2) \in \mathscr{V}(\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2)$, *with* $\mu_2 \neq \nu_2, \mu_1 = \mu_1^1 \nu_1^1 \in \mathscr{E}_1$ *and* $\mu_1^1, \nu_1^1 \in \mathscr{V}_1$ *(if* ν *and* μ *are white nodes in* $\mathscr{F}(\mathscr{Z}_1)$*), we have*

$$d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\mu, \nu) = 1 + d_{\mathscr{Z}_2}(\mu_2, \nu_2) + \min\left\{d_{\mathscr{F}(\mathscr{Z}_1)}\left(\mu_1^1, \nu_1\right), d_{\mathscr{F}(\mathscr{Z}_1)}\left(\nu_1^1, \nu_1\right)\right\}. \tag{10}$$

(c) *If* $\nu_1 \in \mathscr{E}_1$, *then for all* $\mu = (\mu_1, \mu_2) \in V(\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2)$, *where* $\mu_2 = \nu_2$ *and* $\mu_1 \in \mathscr{E}_1$ *(that is,* ν *and* μ *are white vertices in the same copy of* $\mathscr{F}(\mathscr{Z}_1)$*), we have*

$$d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\mu, \nu) = d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1, \nu_1). \tag{11}$$

Lemma 2 (see [4]). *Let* $\mathscr{Z}_1$ *and* $\mathscr{Z}_2$ *be two graphs,* $\mu_1, \nu_1 \in \mathscr{E}_1, \mu_2, \nu_2 \in \mathscr{V}_2$, *and* $\mathscr{F} = \mathscr{S}$ *or* $\mathscr{R}$. *Then, for* $\mu = (\mu_1, \mu_2)$ *and* $\nu = (\nu_1, \nu_2)$ *in* $\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2$ *with* $\mu_2 \neq \nu_2$, *we have*

$$d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\mu, \nu) = \begin{cases} 2 + d_{\mathscr{Z}_2}(\mu_2, \nu_2), & \mu_1 = \nu_1, \\ d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1, \nu_1) + d_{\mathscr{Z}_2}(\mu_2, \nu_2), & \mu_1 \neq \nu_1. \end{cases} \tag{12}$$

Lemma 3 (see [4]). *Let* $\mathscr{Z}_1$ *and* $\mathscr{Z}_2$ *be two graphs,* $\mu_1, \nu_1 \in E_1, \mu_2, \nu_2 \in \mathscr{V}_2$, *and* $\mathscr{F} = \mathscr{S}$ *or* $\mathscr{R}$. *Then, for* $\mu = (\mu_1, \mu_2)$ *and* $\nu = (\nu_1, \nu_2)$ *in* $\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2$ *with* $\mu_2 \neq \nu_2$, *then*

$$d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\mu, \nu) = \begin{cases} 2 + d_{\mathscr{Z}_2}(\mu_2, \nu_2), & \mu_1 = \nu_1, \\ 1 + d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1, \nu_1) + d_{\mathscr{Z}_2}(\mu_2, \nu_2), & \mu_1 \neq \nu_1. \end{cases} \tag{13}$$

Lemma 4 (see [6]). *Let* $\mathscr{Z}_1$ *and* $\mathscr{Z}_2$ *be two graphs and* $\mu = (\mu_1, \mu_2)$ *be a vertex of* $G_1 +_{\mathscr{F}} \mathscr{Z}_2$; *then,*

(a) *If* $\mu_1 \in \mathscr{V}_1$ *and* $\mu_2 \in \mathscr{V}_2$ *(i.e.,* μ *is a black vertex), then we have*

$$d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\mu) = d_{\mathscr{F}(1)}(\mu_1) + d_{\mathscr{Z}_2}(\mu_2). \tag{14}$$

(b) *If* $\mu_1 \in \mathscr{E}_1$ *and* $\mu_2 \in \mathscr{V}_2$ *(that is,* μ *is a white vertex), then we have*

$$d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\mu) = d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1). \tag{15}$$

Lemma 5 (see [6]). *Let* $\mathscr{Z}$ *be a graph. Then,*

(a) *If* $\mu_1 \in \mathscr{V}(\mathscr{Z})$, *then we have*

$$d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1) = k \cdot d_{\mathscr{Z}}(\mu_1), \tag{16}$$

where

$$k = \begin{cases} 1, & \mathscr{F} = \mathscr{S} \text{ or } \mathscr{Q}, \\ 2, & \mathscr{F} = \mathscr{R} \text{ or } \mathscr{T}. \end{cases} \tag{17}$$

(b) *If* $\mu_1 = \mu_1' \mu_1'' \in \mathscr{E}(\mathscr{Z})$, *then*

$$\begin{aligned} &d_{\mathscr{S}(\mathscr{Z})}(\mu_1) = d_{\mathscr{R}(\mathscr{Z})}(\mu_1) = 2, \\ &d_{\mathscr{Q}(\mathscr{Z})}(\mu_1) = d_{\mathscr{T}(\mathscr{Z})}(\mu_1) = d_{L(\mathscr{Z})}(\mu_1) + 2, \end{aligned} \tag{18}$$

where

$$d_{L(\mathscr{Z})}(\mu_1) = d_{\mathscr{Z}}(\mu_1') + d_{\mathscr{Z}}(\mu_1''). \tag{19}$$

3. Reciprocal Degree Distance Index of Four Operations on Graphs

We present constraints for the reciprocal degree distance of $\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2$ within this part, assuming $\mathscr{F} = \mathscr{S}, \mathscr{R}, \mathscr{Q}$, and $\mathscr{T}$.

3.1. Reciprocal Degree Distance for $\mathscr{F}$*-Sum of Graphs, Where* $\mathscr{F} = \mathscr{S}$ *or* $\mathscr{R}$.

First, we present bounds for the reciprocal degree distance of $\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2$ in terms of reciprocal degree distance and Harary index of the components $\mathscr{Z}_2, \mathscr{F}(\mathscr{Z}_1)$, where $\mathscr{F} = \mathscr{S}$ or $\mathscr{R}$.

Theorem 1. *For two graphs* $\mathscr{Z}_1$, $\mathscr{Z}_2$, *bounds for* $\mathscr{RDD}(\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2); \mathscr{F} = \mathscr{S}, \mathscr{R}$ *are as follows:*

$$\begin{aligned} &|\mathscr{V}_2|^2 \mathscr{RDD}_{\mathscr{D}}(\mathscr{F}(\mathscr{Z}_1)) + 4|\mathscr{E}_1|\mathscr{H}_2(\mathscr{Z}_2) + 4|\mathscr{E}_2| \\ &\quad |\mathscr{V}_2|\mathscr{H}_{\mathscr{D}}(\mathscr{F}(\mathscr{Z}_1)) - 4|\mathscr{E}_2||\mathscr{V}_2|\mathscr{H}_{\mathscr{D}}[\mathscr{Z}_1(\mathscr{F}(\mathscr{Z}_1))] \\ &\quad \leq \mathscr{RDD}(\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2) \\ &\quad \leq |\mathscr{V}_2|^2 \mathscr{RDD}_1(\mathscr{F}(\mathscr{Z}_1)) + 4|\mathscr{E}_1|\mathscr{H}_2(\mathscr{Z}_2) + 4|\mathscr{E}_2| \\ &\quad |\mathscr{V}_2|\mathscr{H}_1(\mathscr{F}(\mathscr{Z}_1)) \\ &\quad - 4|\mathscr{E}_2||\mathscr{V}_2|\mathscr{H}_1[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}_1))], \end{aligned} \tag{20}$$

where $\mathscr{D}(\mathscr{F}(\mathscr{Z}_1))$ *is the diameter and* $\mathscr{E}_1$ *denotes subfamily of white nodes of* $\mathscr{F}(\mathscr{Z}_1)$. *Also,*

$$\mathscr{H}_\tau[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}))] = \frac{1}{2} \sum_{\mu, \nu \in \mathscr{E}_1 : \mu \neq \nu} \frac{1}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu, \nu) + \tau}. \tag{21}$$

,where τ is a real number. In the specific case, when$\tau = 1, 2$ or $D(\mathscr{Z}_2)$, equality holds for $\mathscr{Z}_2 \cong K_{|\mathscr{Z}_2|}$.

Proof. Let $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$ be two nodes in $\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2$. We explore the following three possibilities based on the colors of μ and ν.

Case 1. Assume black color nodes are $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$; it means that $\mu, \nu \in \mathscr{V}_1 \times \mathscr{V}_2$. Using (a) in Lemma 1 and Lemma 4, we have

$$d_{\mathcal{X}_1 +_{\mathcal{F}} \mathcal{X}_2}(\mu,\nu) = d_{\mathcal{F}(\mathcal{X}_1)}(\mu_1,\nu_1) + d_{\mathcal{X}_2}(\mu_2,\nu_2),$$

$$\begin{aligned} A &= \frac{1}{2}\sum_{\mu,\nu\in\mathcal{V}(\mathcal{X}_1+_{\mathcal{F}}\mathcal{X}_2)} \frac{d_{\mathcal{X}_1+_{\mathcal{F}}\mathcal{X}_2}(\mu) + d_{\mathcal{X}_1+_{\mathcal{F}}\mathcal{X}_2}(\nu)}{d_{\mathcal{X}_1+_{\mathcal{F}}\mathcal{X}_2}((\mu_1,\mu_2),(\nu_1,\nu_2))} \\ &= \frac{1}{2}\sum_{(\mu_1,\mu_2),(\nu_1,\nu_2)\in\mathcal{V}(\mathcal{X}_1+_{\mathcal{F}}\mathcal{X}_2)} \frac{d_{\mathcal{F}(\mathcal{X}_1)}(\mu_1) + d_{\mathcal{X}_2}(\mu_2) + d_{\mathcal{F}(\mathcal{X}_1)}(\nu_1) + d_{\mathcal{X}_2}(\nu_2)}{d_{\mathcal{F}(\mathcal{X}_1)}(\mu_1,\nu_1) + d_{\mathcal{X}_2}(\mu_2,\nu_2)} \\ &\leq \frac{1}{2}\sum_{\mu_1,\nu_1\in\mathcal{V}_1}\sum_{\mu_2,\nu_2\in\mathcal{V}_2} \frac{d_{\mathcal{F}(\mathcal{X}_1)}(\mu_1) + d_{\mathcal{X}_2}(\mu_2) + d_{\mathcal{F}(\mathcal{X}_1)}(\nu_1) + d_{\mathcal{X}_2}(\nu_2)}{d_{\mathcal{F}(\mathcal{X}_1)}(\mu_1,\nu_1) + 1} \end{aligned} \quad (22)$$

(as $d_{\mathcal{X}_2}(\mu_2,\nu_2) \geq 1$ and $\mu_2 \neq \nu_2$)

$$\begin{aligned} &\leq \frac{1}{2}\sum_{\mu_2,\nu_2\in\mathcal{V}_2}\left(\sum_{\mu_1,\nu_1\in\mathcal{V}_1} \frac{d_{\mathcal{F}(G_1)}(\mu_1) + d_{\mathcal{F}(\mathcal{X}_1)}(\nu_1)}{d_{\mathcal{F}(\mathcal{X}_1)}(\mu_1,\nu_1) + 1}\right) \\ &\quad + \frac{1}{2}\left(\sum_{\mu_1,\nu_1\in\mathcal{V}_1} \frac{1}{d_{\mathcal{F}(\mathcal{X}_1)}(\mu_1,\nu_1) + 1}\right)\left(\sum_{\mu_2,\nu_2\in\nu_2} d_{\mathcal{X}_2}(\mu_2) + d_{\mathcal{X}_2}(\nu_2)\right), \\ A &\leq \frac{|\mathcal{V}_2|^2}{2}\left(\sum_{\mu_1,\nu_1\in\mathcal{V}_1} \frac{d_{\mathcal{F}(\mathcal{X}_1)}(\mu_1) + d_{\mathcal{F}(\mathcal{X}_1)}(\nu_1)}{d_{\mathcal{F}(\mathcal{X}_1)}(\mu_1,\nu_1) + 1}\right) \\ &\quad + 4|\mathcal{E}_2||\mathcal{V}_2|\left(\frac{1}{2}\sum_{\mu_1,\nu_1\in\mathcal{V}_1} \frac{1}{d_{\mathcal{F}(\mathcal{X}_1)}(\mu_1,\nu_1) + 1}\right). \end{aligned} \quad (23)$$

Case 2. Assume that $\mu = (\mu_1,\mu_2)$ and $\nu = (\nu_1,\nu_2)$ are of dissimilar colors, so $\mu \in \mathcal{V}_1 \times \mathcal{V}_2$ and $\nu \in \mathcal{E}_1 \times \mathcal{V}_2$. In the present condition, by (a) in Lemma 1 and Lemma 4, when black node is μ and white is ν, eventually we get

$$d_{\mathcal{X}_1 +_{\mathcal{F}} \mathcal{X}_2}(\mu,\nu) = d_{\mathcal{F}(\mathcal{X}_1)}(\mu_1,\nu_1) + d_{\mathcal{X}_2}(\mu_2,\nu_2),$$

$$\begin{aligned} B' &= \frac{1}{2}\sum_{(\mu_1,\mu_2)\in\mathcal{V}_1\times\mathcal{V}_2}\ \sum_{(\nu_1,\nu_2)\in\mathcal{E}_1\times\mathcal{V}_2} \frac{d_{\mathcal{X}_1+_{\mathcal{F}}\mathcal{X}_2}(\mu) + d_{\mathcal{X}_1+_{\mathcal{F}}\mathcal{X}_2}(\nu)}{d_{\mathcal{X}_1+_{\mathcal{F}}\mathcal{X}_2}((\mu_1,\mu_2),(\nu_1,\nu_2))} \\ &= \frac{1}{2}\sum_{\mu_2,\nu_2\in\mathcal{V}_2}\sum_{\mu_1\in\mathcal{V}_1}\sum_{\nu_1\in\mathcal{E}_1} \frac{d_{\mathcal{F}(\mathcal{X}_1)}(\mu_1) + d_{\mathcal{X}_2}(\mu_2) + d_{\mathcal{F}(\mathcal{X}_1)}(\nu_1)}{d_{\mathcal{F}(\mathcal{X}_1)}(\mu_1,\nu_1) + d_{\mathcal{X}_2}(\mu_2,\nu_2)} \\ &\leq \frac{1}{2}\sum_{\mu_2,\nu_2\in\mathcal{V}_2}\sum_{\mu_1\in\mathcal{V}_1}\sum_{\nu_1\in\mathcal{E}_1} \frac{d_{\mathcal{F}(\mathcal{X}_1)}(\mu_1) + d_{\mathcal{X}_2}(\mu_2) + d_{\mathcal{F}(\mathcal{X}_1)}(\nu_1)}{d_{\mathcal{F}(\mathcal{X}_1)}(\mu_1,\nu_1) + 1} \end{aligned} \quad (24)$$

(as $d_{\mathcal{X}_2}(\mu_2,\nu_2) \geq 1$ and $\mu_2 \neq \nu_2$)

$$\leq \frac{1}{2} \sum_{\mu_2,\nu_2 \in \mathscr{V}_2} \left(\sum_{\mu_1 \in \mathscr{V}_1} \sum_{\nu_1 \in \mathscr{E}_1} \frac{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{Z}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1) + 1} \right)$$

$$+ \frac{1}{2} \left(\sum_{\mu_1 \in \mathscr{V}_1} \sum_{\nu_1 \in \mathscr{E}_1} \frac{1}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1) + 1} \right) \left(\sum_{\mu_2,\nu_2 \in \mathscr{V}_2} d_{\mathscr{Z}_2}(\mu_2) \right),$$

$$\acute{B} \leq \frac{|\mathscr{V}_2|^2}{2} \left(\sum_{\mu_1 \in \mathscr{V}_1} \sum_{\nu_1 \in \mathscr{E}_1} \frac{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{Z}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1) + 1} \right)$$

$$+ 2|\mathscr{V}_2||\mathscr{E}_2| \left(\frac{1}{2} \sum_{\mu_1 \in \mathscr{V}_1} \sum_{\nu_1 \in \mathscr{E}_1} \frac{1}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1) + 1} \right). \tag{25}$$

As a result, for vertices of multiple colors, the above expression would be

$$B = 2\acute{B} \leq |\mathscr{V}_2|^2 \left(\sum_{\mu_1 \in \mathscr{V}_1} \sum_{\nu_1 \in \mathscr{E}_1} \frac{d_{\mathscr{F}(G_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{Z}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1) + 1} \right)$$

$$+ 4|\mathscr{V}_2||\mathscr{E}_2| \left(\frac{1}{2} \sum_{\mu_1 \in \mathscr{V}_1} \sum_{\nu_1 \in \mathscr{E}_1} \frac{1}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1) + 1} \right). \tag{26}$$

Case 3. Assume that $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$ are white, that is, $\mu \in \mathscr{E}_1 \times \mathscr{V}_2$ and $\nu \in \mathscr{E}_1 \times \mathscr{V}_2$. Let

$$C = \frac{1}{2} \sum_{(\mu_1,\mu_2),(\nu_1,\nu_2) \in \mathscr{E}_1 \times \mathscr{V}_2} \frac{d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\mu) + d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\nu)}{d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}((\mu_1,\mu_2),(\nu_1,\nu_2))}. \tag{27}$$

We divided this total into two parts $C = \widehat{Ç} + \check{Ç}$, and we have

$$\widehat{Ç} = \frac{1}{2} \sum_{(\mu_1,\mu_2),(\nu_1,\nu_2) \in \mathscr{E}_1 \times \mathscr{V}_2 ; \mu_1 = \nu_1, \mu_2 \neq \nu_2} \frac{d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\mu) + d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\nu)}{d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}((\mu_1,\mu_2),(\nu_1,\nu_2))},$$

$$\check{Ç} = \frac{1}{2} \sum_{(\mu_1,\mu_2),(\nu_1,\nu_2) \in \mathscr{E}_1 \times \mathscr{V}_2 ; u_1 \neq \nu_1} \frac{d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\mu) + d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\nu)}{d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}((\mu_1,\mu_2),(\nu_1,\nu_2))}. \tag{28}$$

By Lemmas 2, 4, and 5, we have

$$\widehat{Ç} = \frac{1}{2} \sum_{\mu_1 \in \mathscr{E}_1} \sum_{\mu_2,\nu_2 \in \mathscr{V}_2 ; \mu_2 \neq \nu_2} \frac{d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\mu) + d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\nu)}{2 + d_{\mathscr{Z}_2}(\mu_2,\nu_2)}$$

$$= \frac{1}{2} \sum_{\mu_1 \in \mathscr{E}_1} \sum_{u_2,\nu_2 \in \mathscr{V}_2 ; \mu_2 \neq \nu_2} \frac{2 + 2}{2 + d_{\mathscr{Z}_2}(\mu_2,\nu_2)}$$

$$= 2 \sum_{\mu_1 \in \mathscr{E}_1} \left(\sum_{\mu_2,\nu_2 \in \mathscr{V}_2 ; \mu_2 \neq \nu_2} \frac{1}{2 + d_{\mathscr{Z}_2}(\mu_2,\nu_2)} \right) = 4|\mathscr{E}_1| \mathscr{H}_2(\mathscr{Z}_2), \tag{29}$$

$$\check{Ç} = \frac{1}{2} \sum_{\mu_1,\nu_1 \in \mathscr{E}_1 ; \mu_1 \neq \nu_1} \sum_{\mu_2,\nu_2 \in \mathscr{V}_2} \frac{d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\mu) + d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\nu)}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1) + d_{\mathscr{Z}_2}(\mu_2,\nu_2)}$$

$$\leq \frac{1}{2} \sum_{\mu_1,\nu_1 \in \mathscr{E}_1 ; \mu_1 \neq \nu_1} \sum_{\mu_2,\nu_2 \in \mathscr{V}_2} \frac{d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\mu) + d_{\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2}(\nu)}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1) + 1}$$

(as $d_{\mathscr{Z}_2}(\mu_2,\nu_2) \geq 1$ and $\mu_2 \neq \nu_2$)

$$\leq \frac{|\mathscr{V}_2|^2}{2} \left(\sum_{\mu_1,\nu_1 \in \mathscr{E}_1 ; \mu_1 \neq \nu_1} \frac{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{Z}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1) + 1} \right), \tag{30}$$

so

$$C = \widehat{Ç} + \check{Ç}$$

$$\leq 4|\mathscr{E}_1| \mathscr{H}_2(\mathscr{Z}_2)$$

$$+ \frac{|\mathscr{V}_2|^2}{2} \left(\sum_{\mu_1,\nu_1 \in \mathscr{E}_1 ; \mu_1 \neq \nu_1} \frac{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{Z}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1) + 1} \right). \tag{31}$$

Therefore, by the above calculations and the definition of reciprocal degree distance,

$$\begin{aligned}
\mathscr{RDD}\left(\mathscr{L}_1 +_{\mathscr{F}} \mathscr{L}_2\right) &= A + B + C \\
&\leq \frac{|\mathscr{V}_2|^2}{2}\left(\sum_{\mu_1,\nu_1\in\mathscr{V}_1} \frac{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{L}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1,\nu_1) + 1}\right) \\
&\quad + 4|\mathscr{E}_2||\mathscr{V}_2|\left(\frac{1}{2}\sum_{\mu_1,\nu_1\in\mathscr{V}_1} \frac{1}{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1,\nu_1) + 1}\right) \\
&\quad + |\mathscr{V}_2|^2\left(\sum_{\mu_1\in\mathscr{V}_1}\sum_{\nu_1\in\mathscr{E}_1} \frac{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{L}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1,\nu_1) + 1}\right) \\
&\quad + 4|\mathscr{V}_2||\mathscr{E}_2|\left(\frac{1}{2}\sum_{\mu_1\in\mathscr{V}_1}\sum_{\nu_1\in\mathscr{E}_1} \frac{1}{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1,\nu_1) + 1}\right) \\
&\quad + 4|\mathscr{E}_1|\mathscr{H}_2(\mathscr{L}_2) + \frac{|\mathscr{V}_2|^2}{2}\left(\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1} \frac{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{L}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1,\nu_1) + 1}\right),
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
|\mathscr{V}_2|^2\mathscr{RDD}_1\left(\mathscr{F}(\mathscr{L}_1)\right) &= \frac{|\mathscr{V}_2|^2}{2}\left(\sum_{\mu_1,\nu_1\in\mathscr{V}_1} \frac{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{L}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1,\nu_1) + 1}\right) \\
&\quad + |\mathscr{V}_2|^2\left(\sum_{\mu_1\in\mathscr{V}_1}\sum_{\nu_1\in\mathscr{E}_1} \frac{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{L}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1,\nu_1) + 1}\right) \\
&\quad + \frac{|\mathscr{V}_2|^2}{2}\left(\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1} \frac{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{L}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1,\nu_1) + 1}\right), \\
4|\mathscr{E}_2||\mathscr{V}_2|\mathscr{H}_1\left(\mathscr{F}(\mathscr{L}_1)\right) &- 4|\mathscr{E}_2||\mathscr{V}_2|\left(\frac{1}{2}\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1} \frac{1}{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1,\nu_1) + 1}\right) \\
&= 4|\mathscr{E}_2||\mathscr{V}_2|\left(\frac{1}{2}\sum_{\mu_1,\nu_1\in\mathscr{V}_1} \frac{1}{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1,\nu_1) + 1}\right) + 4|\mathscr{V}_2||\mathscr{E}_2|\left(\frac{1}{2}\sum_{\mu_1\in\mathscr{V}_1}\sum_{\nu_1\in\mathscr{E}_1} \frac{1}{d_{\mathscr{F}(\mathscr{L}_1)}(\mu_1,\nu_1) + 1}\right).
\end{aligned} \tag{33}$$

Put these values in equation (32), and we obtain

$$\mathscr{RDD}(\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2) \leq |\mathscr{V}_2|^2 \mathscr{RDD}_1(\mathscr{F}(\mathscr{Z}_1)) + 4|\mathscr{E}_1|\mathscr{H}_2(\mathscr{Z}_2) + 4|\mathscr{E}_2||\mathscr{V}_2| \mathscr{H}_1(\mathscr{F}(\mathscr{Z}_1)) - 4|\mathscr{E}_2||\mathscr{V}_2|\left(\frac{1}{2}\sum_{u_1,v_1\in\mathscr{E}_1;u_1\neq v_1}\frac{1}{d_{\mathscr{F}(\mathscr{Z}_1)}(u_1,v_1)+1}\right)$$
$$\mathscr{RDD}(\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2) \leq |\mathscr{V}_2^2|\mathscr{RDD}_1(\mathscr{F}(\mathscr{Z}_1)) + 4|\mathscr{E}_1|\mathscr{H}_2(\mathscr{Z}_2) + 4|\mathscr{E}_2||\mathscr{V}_2| \mathscr{H}_1(\mathscr{F}(\mathscr{Z}_1)) - 4|\mathscr{E}_2||\mathscr{V}_2|\mathscr{H}_1[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}_1))]. \quad (34)$$

As $d_{\mathscr{Z}_2}(u_2,v_2)\leq\mathscr{D}(\mathscr{Z}_2)$, in the same manner, we have

$$\mathscr{RDD}_{\mathscr{Z}_1+_{\mathscr{F}}\mathscr{Z}_2} \geq |\mathscr{V}_2|^2 \mathscr{RDD}_{\mathscr{D}}(\mathscr{F}(\mathscr{Z}_1)) + 4|\mathscr{E}_1|\mathscr{H}_2(\mathscr{Z}_2) + 4|\mathscr{E}_2||\mathscr{V}_2| \mathscr{H}_{\mathscr{D}}(\mathscr{F}(\mathscr{Z}_1)) - 4|\mathscr{E}_2||\mathscr{V}_2|\mathscr{H}_{\mathscr{D}}[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}_1))]. \quad (35)$$

3.2. Reciprocal Degree Distance for $\mathscr{F}$-Sum of Graphs, Where $\mathscr{F} = \mathscr{Q}$ or $\mathscr{T}$. We next present boundaries for the reciprocal degree distance invariant of $\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2$ in terms of other graphic invariants.

Theorem 2. *For two graphs $\mathscr{Z}_1$, $\mathscr{Z}_2$, $\mathscr{F} = \mathscr{Q}$ or $\mathscr{T}$. Assume $\mathscr{F}(\mathscr{Z}_1)$ is of diameter $\mathscr{D}(\mathscr{F}(\mathscr{Z}_1))$ and the collection of white nodes is symbolized by $\mathscr{E}_1$. Then,*

$$\begin{aligned}&|\mathscr{V}_2|^2\mathscr{RDD}_{\mathscr{D}}(\mathscr{F}(\mathscr{Z}_1)) + 4|\mathscr{E}_2||\mathscr{V}_2|H_{\mathscr{D}}(\mathscr{F}(\mathscr{Z}_1)) - 4|\mathscr{E}_2||\mathscr{V}_2| H_{\mathscr{D}}[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}_1))] + |\mathscr{V}_2|^2\mathscr{RDD}_2[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}_1))] - |\mathscr{V}_2|^2 \mathscr{RDD}_1[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}_1))] + |\mathscr{V}_2|\mathscr{RDD}[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}_1))] \leq \mathscr{RDD}(\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2)\\ &\leq |\mathscr{V}_2|^2\mathscr{RDD}_1(\mathscr{F}(\mathscr{Z}_1)) + 4|\mathscr{E}_2||\mathscr{V}_2|H_1(\mathscr{F}(\mathscr{Z}_1)) - 4|\mathscr{E}_2||\mathscr{V}_2| H_1[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}_1))] + |\mathscr{V}_2|^2\mathscr{RDD}_{D+1}[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}_1))] - |\mathscr{V}_2|^2 \mathscr{RDD}_D[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}_1))] + |\mathscr{V}_2|\mathscr{RDD}[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}_1))],\end{aligned} \quad (36)$$

where

$$\mathscr{RDD}_\tau[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}))] = \frac{1}{2}\sum_{\mu,\nu\in\mathscr{E}_1}\frac{d_\mu + d_\nu}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu,\nu)+\tau},$$
$$H_\tau[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}))] = \frac{1}{2}\sum_{\mu,\nu\in\mathscr{E}_1;\mu\neq\nu}\frac{1}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu,\nu)+\tau}. \quad (37)$$

In the present condition, $\tau = 1, 2, \mathscr{D}(\mathscr{Z}_2)$ or $\mathscr{D}(\mathscr{Z}_2)+1$ and τ must be a real number. Bounds are attained by $\mathscr{Z}_2 \cong K_{|\mathscr{Z}_2|}$.

Proof. Assume that A, B, and C are the same as in Theorem 1. The quantities of A and B do not really change in the present situation. As a result, we merely compute the result of C. Consider

$$C = \frac{1}{2}\sum_{(\mu_1,\mu_2),(\nu_1,\nu_2)\in\mathscr{E}_1\times\mathscr{V}_2}\frac{d_{\mathscr{Z}_1+_F\mathscr{Z}_2}(\mu) + d_{\mathscr{Z}_1+_F\mathscr{Z}_2}(\nu)}{d_{\mathscr{Z}_1+_F\mathscr{Z}_2}((\mu_1,\mu_2),(\nu_1,\nu_2))}. \quad (38)$$

We divide this total into three parts $C = \hat{Ç} + \check{Ç} + \acute{Ç}$ and

$$\hat{Ç} = \frac{1}{2}\sum_{(\mu_1,\mu_2),(\nu_1,\nu_2)\in\mathscr{E}_1\times\mathscr{V}_2;\mu_1=\nu_1,\mu_2\neq\nu_2}\frac{d_{\mathscr{Z}_1+_{\mathscr{F}}\mathscr{Z}_2}(\mu) + d_{\mathscr{Z}_1+_{\mathscr{F}}\mathscr{Z}_2}(\nu)}{d_{\mathscr{Z}_1+_{\mathscr{F}}\mathscr{Z}_2}((\mu_1,\mu_2),(\nu_1,\nu_2))},$$
$$\check{Ç} = \frac{1}{2}\sum_{(\mu_1,\mu_2),(\nu_1,\nu_2)\in\mathscr{E}_1\times\mathscr{V}_2;\mu_1\neq\nu_1,\mu_2=\nu_2}\frac{d_{\mathscr{Z}_1+_{\mathscr{F}}\mathscr{Z}_2}(\mu) + d_{\mathscr{Z}_1+_{\mathscr{F}}\mathscr{Z}_2}(\nu)}{d_{\mathscr{Z}_1+_{\mathscr{F}}\mathscr{Z}_2}((\mu_1,\mu_2),(\nu_1,\nu_2))},$$
$$\acute{Ç} = \frac{1}{2}\sum_{(\mu_1,\mu_2),(\nu_1,\nu_2)\in\mathscr{E}_1\times\mathscr{V}_2;\mu_1\neq\nu_1,\mu_2\neq\nu_2}\frac{d_{\mathscr{Z}_1+_{\mathscr{F}}\mathscr{Z}_2}(\mu) + d_{\mathscr{Z}_1+_{\mathscr{F}}\mathscr{Z}_2}(\nu)}{d_{\mathscr{Z}_1+_{\mathscr{F}}\mathscr{Z}_2}((\mu_1,\mu_2),(\nu_1,\nu_2))}. \quad (39)$$

By Lemmas 3, 4, and 5, we have

$$
\begin{aligned}
\widehat{Ç} &= \frac{1}{2}\sum_{\mu_1\in\mathscr{E}_1}\sum_{\mu_2,\nu_2\in\mathscr{V}_2;\mu_2\neq\nu_2}\frac{d_{\mathscr{X}_1+_{\mathscr{F}}\mathscr{X}_2}(\mu)+d_{\mathscr{X}_1+_{\mathscr{F}}\mathscr{X}_2}(\nu)}{2+d_{\mathscr{X}_2}(\mu_2,\nu_2)}\\
d_{\mathscr{Q}(\mathscr{X})}(\mu) &= d_{\mathscr{T}(\mathscr{X})}(\mu)=d_{L(\mathscr{X})}(\mu)+2\\
d_{L(\mathscr{X})}(\mu) &= d_{\mathscr{X}}(\mu')+d_{\mathscr{X}}(\mu'')-2\\
&= \frac{1}{2}\sum_{\mu_1\in\mathscr{E}_1}\sum_{\mu_2,\nu_2\in\mathscr{V}_2;\mu_2\neq\nu_2}\frac{d_{L(\mathscr{X}_1)}(\mu_1)+d_{L(\mathscr{X}_1)}(\nu)+4}{2+d_{\mathscr{X}_2}(\mu_2,\nu_2)}\\
&= \frac{1}{2}\sum_{\mu_1\in\mathscr{E}_1}\sum_{\mu_2,\nu_2\in\mathscr{V}_2;\mu_2\neq\nu_2}\frac{d_{L(\mathscr{X}_1)}(\mu_1)+d_{L(\mathscr{X}_1)}(\nu)}{2+d_{\mathscr{X}_2}(\mu_2,\nu_2)}\\
&\quad + 4\sum_{\mu_1\in\mathscr{E}_1}\left(\frac{1}{2}\sum_{\mu_2,\nu_2\in\mathscr{V}_2;\mu_2\neq\nu_2}\frac{1}{2+d_{\mathscr{X}_2}(\mu_2,\nu_2)}\right),\\
&= \frac{1}{2}\left(\sum_{\mu_1\in\mathscr{E}_1}2d_{L(\mathscr{X}_1)}(\mu_1)\right)\left(\sum_{\mu_2,\nu_2\in\nu_2;\mu_2\neq\nu_2}\frac{1}{2+d_{\mathscr{X}_2}(\mu_2,\nu_2)}\right)\\
&\quad + 4|\mathscr{E}_1|\mathscr{H}_2(\mathscr{X}_2)\\
&= 2\left(\sum_{\mu_1\in\mathscr{E}_1}d_{L(\mathscr{X}_1)}(\mu_1)\right)\mathscr{H}_2(\mathscr{X}_2)+4|\mathscr{E}_1|\mathscr{H}_2(\mathscr{X}_2)\\
\sum_{\mu\in\mathscr{E}_1}d_{L(\mathscr{X})}(\mu) &= \sum_{\mu=\mu'\mu''\in\mathscr{E}_1}d_{\mathscr{X}}\mu_1'+d_{\mathscr{X}}(\mu_1'')-2\\
&= 2\left(\sum_{\mu_1=\mu_1'\mu_1''\in\mathscr{E}_1}d_{\mathscr{X}_1}(\mu_1')+d_{\mathscr{X}_1}(\mu_1'')-2\right)\\
&\mathscr{H}_2(\mathscr{X}_2)+4|\mathscr{E}_1|\mathscr{H}_2(\mathscr{X}_2)\\
&= 2\left(M_1(\mathscr{X}_1)-2|\mathscr{E}_1|\right)\mathscr{H}_2(\mathscr{X}_2)+4|\mathscr{E}_1|\mathscr{H}_2(\mathscr{X}_2)\\
&= 2\mathscr{H}_2(\mathscr{X}_2)M_1(\mathscr{X}_1)-4|\mathscr{E}_1|\mathscr{H}_2(\mathscr{X}_2)+4|\mathscr{E}_1|\mathscr{H}_2(\mathscr{X}_2),\\
\widehat{Ç} &= 2\mathscr{H}_2(\mathscr{X}_2)M_1(\mathscr{X}_1).
\end{aligned}
\tag{40}
$$

Again by Lemmas 3 and 4,

$$\begin{aligned}
\check{Ç} &= \frac{1}{2}\sum_{\mu_2\in\mathscr{V}_2}\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1}\frac{d_{F(\mathscr{X}_1)}(\mu_1)+d_{\mathscr{F}(\mathscr{X}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1,\nu_1)} \\
&= \frac{|\mathscr{V}_2|}{2}\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1}\frac{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1)+d_{\mathscr{F}(\mathscr{X}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1,\nu_1)} \\
&= |\mathscr{V}_2|\mathscr{RDD}[\mathscr{E}_1(\mathscr{F}(\mathscr{X}_1))], \\
\acute{Ç} &= \frac{1}{2}\sum_{\mu_2,\nu_2\in\mathscr{V}_2;\mu_2\neq\nu_2}\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1}\frac{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1)+d_{\mathscr{F}(\mathscr{X}_1)}(\nu_1)}{1+d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1,\nu_1)+d_{\mathscr{X}_2}(\mu_2,\nu_2)} \\
&\leq \frac{1}{2}\sum_{\mu_2,\nu_2\in\mathscr{V}_2;\mu_2\neq\nu_2}\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1}\frac{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1)+d_{\mathscr{F}(\mathscr{X}_1)}(\nu_1)}{1+d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1,\nu_1)+1} \\
&\leq \frac{1}{2}\sum_{\mu_2,\nu_2\in\mathscr{V}_2;\mu_2\neq\nu_2}\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1}\frac{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1)+d_{\mathscr{F}(\mathscr{X}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1,\nu_1)+1} \\
&\quad+\frac{1}{2}\sum_{\mu_2,\nu_2\in\mathscr{V}_2;\mu_2\neq\nu_2}\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1}\frac{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1)+d_{\mathscr{F}(\mathscr{X}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1,\nu_1)+2} \\
&\quad-\frac{1}{2}\sum_{\mu_2,\nu_2\in\mathscr{V}_2;\mu_2\neq\nu_2}\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1}\frac{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1)+d_{\mathscr{F}(\mathscr{X}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1,\nu_1)+1} \\
&\leq \frac{|\mathscr{V}_2|^2}{2}\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1}\frac{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1)+d_{\mathscr{F}(\mathscr{X}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1,\nu_1)+1} \\
&\quad+\frac{|\mathscr{V}_2|^2}{2}\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1}\frac{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1)+d_{\mathscr{F}(\mathscr{X}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1,\nu_1)+2} \\
&\quad-\frac{|\mathscr{V}_2|^2}{2}\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1}\frac{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1)+d_{\mathscr{F}(\mathscr{X}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1,\nu_1)+1} \\
&\leq \frac{|\mathscr{V}_2|^2}{2}\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1}\frac{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1)+d_{\mathscr{F}(\mathscr{X}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1,\nu_1)+1} \\
&\quad+|\mathscr{V}_2|^2\mathscr{RDD}_2[\mathscr{E}_1(\mathscr{F}(\mathscr{X}_1))]-|\mathscr{V}_2|^2\mathscr{RDD}_2[\mathscr{E}_1(\mathscr{F}(\mathscr{X}_1))].
\end{aligned} \tag{41}$$

So, value of C is

$$\begin{aligned}
C &= \hat{Ç}+\check{Ç}+\acute{Ç} \\
&\leq \frac{|\mathscr{V}_2|^2}{2}\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1}\frac{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1)+d_{\mathscr{F}(\mathscr{X}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{X}_1)}(\mu_1,\nu_1)+1} \\
&\quad+|\mathscr{V}_2|^2\mathscr{RDD}_2[\mathscr{E}_1(\mathscr{F}(\mathscr{X}_1))]-|\mathscr{V}_2|^2\mathscr{RDD}_2[\mathscr{E}_1(\mathscr{F}(\mathscr{X}_1))] \\
&\quad+|\mathscr{V}_2|\mathscr{RDD}[\mathscr{E}_1(\mathscr{F}(\mathscr{X}_1))]+2\mathscr{H}_2(\mathscr{X}_2)M_1(\mathscr{X}_1).
\end{aligned} \tag{42}$$

As a result, by the concept of reciprocal degree distance, we get

$$\begin{aligned}
\mathscr{RDD}(\mathscr{Z}_1 +_{\mathscr{F}} \mathscr{Z}_2) &= A + B + C \\
&\leq |\mathscr{V}_2| \mathscr{RDD}[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}_1))] + 2\mathscr{H}_2(\mathscr{Z}_2) M_1(\mathscr{Z}_1) \\
&- |\mathscr{V}_2|^2 \mathscr{RDD}_2[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}_1))] + |\mathscr{V}_2|^2 \mathscr{RDD}_2[\mathscr{E}_1(\mathscr{F}(\mathscr{Z}_1))] \\
&+ \frac{|\mathscr{V}_2|^2}{2}\left(\sum_{\mu_1,\nu_1\in\mathscr{V}_1} \frac{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{Z}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1)+1}\right) \\
&+ |\mathscr{V}_2|^2\left(\sum_{\mu_1\in\mathscr{V}_1}\sum_{\nu_1\in\mathscr{E}_1} \frac{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{Z}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1)+1}\right) \\
&+ \frac{|\mathscr{V}_2|^2}{2}\left(\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1} \frac{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{Z}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1)+1}\right) \\
&+ 4|\mathscr{E}_2||\mathscr{V}_2|\left(\frac{1}{2}\sum_{(\mu_1,\nu_1)\in\mathscr{V}_1} \frac{1}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1)+1}\right) \\
&+ 4|\mathscr{V}_2||\mathscr{E}_2|\left(\frac{1}{2}\sum_{\mu_1\in\mathscr{V}_1}\sum_{\nu_1\in\mathscr{E}_1} \frac{1}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1)+1}\right),
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
|\mathscr{V}_2|^2 \mathscr{RDD}_1(\mathscr{F}(\mathscr{Z}_1)) &= \frac{|\mathscr{V}_2|^2}{2}\left(\sum_{(\mu_1,\nu_1)\in\mathscr{V}_1} \frac{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{Z}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1)+1}\right) \\
&+ |\mathscr{V}_2|^2\left(\sum_{\mu_1\in\mathscr{V}_1}\sum_{\nu_1\in\mathscr{E}_1} \frac{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{Z}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1)+1}\right) \\
&+ \frac{|\mathscr{V}_2|^2}{2}\left(\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1} \frac{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1) + d_{\mathscr{F}(\mathscr{Z}_1)}(\nu_1)}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1)+1}\right), \\
& 4|\mathscr{E}_2||\mathscr{V}_2|\mathscr{H}_1(\mathscr{F}(\mathscr{Z}_1)) - 4|\mathscr{E}_2||\mathscr{V}_2|\left(\frac{1}{2}\sum_{\mu_1,\nu_1\in\mathscr{E}_1;\mu_1\neq\nu_1} \frac{1}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1)+1}\right) \\
&= 4|\mathscr{E}_2||\mathscr{V}_2|\left(\frac{1}{2}\sum_{\mu_1,\nu_1\in\mathscr{V}_1} \frac{1}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1)+1}\right) \\
&+ 4|\mathscr{V}_2||\mathscr{E}_2|\left(\frac{1}{2}\sum_{\mu_1\in\mathscr{V}_1}\sum_{\nu_1\in\mathscr{E}_1} \frac{1}{d_{\mathscr{F}(\mathscr{Z}_1)}(\mu_1,\nu_1)+1}\right).
\end{aligned} \tag{44}$$

When we plug these numbers into equation (43), we get

$$\begin{aligned}\mathcal{RDD}\left(\mathcal{X}_1+_{\mathcal{F}}\mathcal{X}_2\right)\leq{}&\left|\mathcal{V}_2\right|^2\mathcal{RDD}_1\left(\mathcal{F}\left(\mathcal{X}_1\right)\right)+4\left|\mathcal{E}_2\right|\left|\mathcal{V}_2\right|\mathcal{H}_1\left(\mathcal{F}\left(\mathcal{X}_1\right)\right)\\&-4\left|\mathcal{E}_2\right|\left|\mathcal{V}_2\right|\mathcal{H}_1\left[\mathcal{E}_1\left(\mathcal{F}\left(\mathcal{X}_1\right)\right)\right]+\left|\mathcal{V}_2\right|^2\mathcal{RDD}_2\left[\mathcal{E}_1\left(\mathcal{F}\left(\mathcal{X}_1\right)\right)\right]\\&-\left|\mathcal{V}_2\right|^2\mathcal{RDD}_1\left[\mathcal{E}_1\left(\mathcal{F}\left(\mathcal{X}_1\right)\right)\right]+\left|\mathcal{V}_2\right|\mathcal{RDD}\left[\mathcal{E}_1\left(\mathcal{F}\left(\mathcal{X}_1\right)\right)\right].\end{aligned}\tag{45}$$

As $d_{\mathcal{X}_2}(\mu_2,\nu_2)\leq\mathcal{D}(\mathcal{X}_2)$, in the same manner, we get

$$\begin{aligned}\mathcal{RDD}\left(\mathcal{X}_1+_{\mathcal{F}}\mathcal{X}_2\right)\geq{}&\left|\mathcal{V}_2\right|^2\mathcal{RDD}_D\left(\mathcal{F}\left(\mathcal{X}_1\right)\right)+4\left|\mathcal{E}_2\right|\left|\mathcal{V}_2\right|\mathcal{H}_{\mathcal{D}}\left(\mathcal{F}\left(\mathcal{X}_1\right)\right)\\&-4\left|\mathcal{E}_2\right|\left|\mathcal{V}_2\right|\mathcal{H}_{\mathcal{D}}\left[\mathcal{E}_1\left(\mathcal{F}\left(\mathcal{X}_1\right)\right)\right]+\left|\mathcal{V}_2\right|^2\mathcal{RDD}_{D+1}\left[\mathcal{E}_1\left(\mathcal{F}\left(\mathcal{X}_1\right)\right)\right]\\&-\left|\mathcal{V}_2\right|^2\mathcal{RDD}_{\mathcal{D}}\left[\mathcal{E}_1\left(\mathcal{F}\left(\mathcal{X}_1\right)\right)\right]+\left|\mathcal{V}_2\right|\mathcal{RDD}\left[\mathcal{E}_1\left(\mathcal{F}\left(\mathcal{X}_1\right)\right)\right].\end{aligned}\tag{46}$$

4. Conclusion

In this work, we presented constraints (lower and upper) for the reciprocal degree distance of $\mathcal{X}_1+_{\mathcal{F}}\mathcal{X}_2$, assuming $\mathcal{F}=\mathcal{S},\mathcal{R}$ in form of Theorem 1 and for $\mathcal{F}=\mathcal{Q},\mathcal{T}$ as in Theorem 2. We also observed that these bounds are expressions involving several other chemical invariants.

Several tools of graph theory are being used to formulate mathematical structure of many phenomena of chemistry. Solutions of these molecular problems are being considered using some nontrivial graph theoretical ideas. Combination of mathematics, chemistry and information science is giving rise to an emerging field of research chem- informatics, that is under kind consideration of many new researchers. In the future, we are interested in computing the bounds for other degree-based chemical invariants for $\mathcal{F}$-sums of graphs like redefined Zagreb indices. Further, we plan to order the $\mathcal{F}$-sums of graphs with respect to the above-mentioned chemical invariants giving first, second, and third maxima or minima.

Data Availability

No data other than research papers cited as references are used and that supports this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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