Hindawi
Journal of Mathematics
Volume 2022, Article ID 3203108, 5 pages
https://doi.org/10.1155/2022/3203108

Research Article

Strongly Multiplicative Labeling of Diamond Graph, Generalized Petersen Graph, and Some Other Graphs

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Received 1 April 2022; Accepted 29 April 2022; Published 5 July 2022

Academic Editor: A. Ghareeb

A finite, simple graph of order k is said to be a strongly multiplicative graph when all vertices of the graph are labeled by positive integers $1, 2, 3, \ldots, k$ such that the induced edge labels of the graph, obtained by the product of labels of end vertices of edges, are distinct. In this paper, we show that the diamond graph Br_n for ? ≥ 3, umbrella graph $U_{m,n}$, and generalized Petersen graph $\mathrm{GP}(n, k)$, for $n \geq 3$ and $1 \leq k < (n/2)$, admit strongly multiplicative labeling. Moreover, strongly multiplicative labeling of a double comb graph and sunflower planar graph has also been investigated and elaborated as well with different examples.

1. Introduction

Graph labeling is an important technique in discrete mathematics. Applications of graph labeling can be found everywhere in life, from communication networks to possible moves in a board game. In the 18th century, Swiss mathematician Leonhard Euler invented the concept of graph theory. Some applications of graph labeling include network flow, fuzzy graph theory, coding theory, robotics, channel assignment, robotics, and many more. In 1967, a basic idea of graph labeling was presented by Rosa [1]. A comprehensive study of labeling of different graphs is presented by Gallian [2].

Strongly multiplicative labeling first appeared in the work of Beineke and Hedge [3]. Adiga et al. [4] computed an upper bound for the maximum number of edges sharper than that provided in [3] in a strongly multiplicative graph of order n. In 2010, Vaidya and Kanani generalized that the graphs constructed by the arbitrary super-subdivision of cycle, path, star, and tadpole graphs are strongly multiplicative [5, 6]. Punitha et al. [7] investigated the strongly multiplicative labeling of circulant graphs. For some interesting results on labeling, we refer the readers to [8–14].

In this paper, strongly multiplicative labeling of different types of graphs is investigated. First, we plot the graph of a diamond graph, umbrella graph, generalized Petersen graph, double comb graph, and sunflower planar graph and then label the vertices of these graphs according to the condition of strongly multiplicative labeling.

2. Materials and Methods

In the section, we give simple definitions of different standard graphs which are under consideration for strongly multiplicative labeling.

Definition 1. A graph $G = (V; E)$ contains set of vertices (points or nodes) V and set of edges (links or lines) E. These objects are called lines and points.

We will consider connected, simple, undirected, and finite graph $G = (V(G), E(G))$, where the size of the graph is $|E(G)| = q$ and the order is $|V(G)| = p$.

Definition 2. A graph $G = (V(G), E(G))$ having p vertices is known as a strongly multiplicative graph when all vertices of graph are labeled by p consecutive numbers of positive integers 1, 2, 3, . . ., *p;* then, the edge labels that are induced on by the product of labels of the end vertices are different.

Definition 3. A diamond network (Br_n) for $n \geq 3$ is gained by connection of a single vertex x to all other vertices v_i, $i = 1, 2, \ldots, n$, of the triangular ladder graph TL_n.

Definition 4. The umbrella graph $U_{m,n}$ (where $m > 2, n > 1$) is the graph found by connecting the pendant edge of path P_n at the essential vertex of the fan graph which is plotted at the center such as $F_m = P_m + K_1$.

Definition 5. The generalized Petersen graph $\mathrm{GP}(n,k)$ also represented by $P(n,k)$ for $n \geq 3$ and $1 \leq k < (n/2)$ which is also a connected and cubic graph with a vertex set $\{v_i, w_i | i = 1, 2, \ldots, n\}$ and the edge set is $\{v_i v_{i+1}, v_i w_i, w_i w_{i+k} | i = 1, 2, \ldots, n\}$.

These graphs were first presented by Coxeter (1950) and then named by a mathematician Watkins (1969). The generalized Petersen graph $\mathrm{GP}(n,k)$ is isomorphic to $\mathrm{GP}(n, n-k)$. Due to this isomorphic relation, $k < (n/2)$ is defined with no restriction of generality.

Definition 6. A double comb graph (DC_n) is found from a path P_n by connecting two pendant vertices with each vertex of P_n. It is also represented by $P_n \odot 2K_1$.

Definition 7. A sunflower planar graph is denoted by Sf_n which is achieved from a wheel graph having vertices $v_0, v_1, v_2, v_3, \ldots, v_m$ (v_0 is essential vertex whereas $v_1, v_2, v_3, \ldots, v_m$ are rim vertices) and other vertices $u_1, u_2, u_3, \ldots, u_m$ such that u_j is joined to v_j and $v_j + 1$ (mod$n$).

3. Main Results

Theorem 1. *The diamond graph (Br_n) admits strongly multiplicative labeling.*

Proof. The vertex set of Br_n is $V = \{x, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{n-r}\}$ and the edge set is $E = \{v_i v_{i+1} | i = 1, 2, \ldots, n-1\} \cup \{u_i u_{i+1} | i = 1, 2, \ldots, n-2\} \cup \{u_i v_i, u_i v_{i+1} | i = 1, 2, \ldots, n-1\} \cup \{x v_i | i = 1, 2, \ldots, n\}$.

It is to be noted that $|V(Br_n)| = 2n$ and $|E(Br_n)| = 5n - 5$.

Let $v_i, u_i v_i, u_i$, and x be the vertices of the diamond graph. Here, the vertices u_i are adjacent to v_i and v_{i+1}. The mapping of vertices is defined as follows:

$$f\colon V(Br_n) \longrightarrow \{1, 2, \ldots, 2n\}, \tag{1}$$

where

$$\begin{aligned} f(v_i) &= 2i - 1, \quad \forall 1 \leq i \leq n, \\ f(v_i) &= 2i + 2, \quad \forall 1 \leq i \leq n - 1, \end{aligned} \tag{2}$$

and $f(x) = 2i$.

According to this mapping, the products of label of two end vertices are the induced edge labels, which are also distinct, i.e., no two edges have same labels.

Hence, Br_n is a strongly multiplicative graph.

Strongly multiplicative labeling of Br_7 is elaborated in Figure 1. □

Theorem 2. *The umbrella graph $U(m,n)$ admits strongly multiplicative labeling.*

Proof. Proof. The vertex set of $U(m,n)$ $U(m,n)$ is $V = \{u_1, u_2, u_3, \ldots, u_m, u_0 = v_1, v_2, v_3, \ldots, v_n\}$ and the edge set is as follows:

$$\begin{aligned} E = &\{v_1 u_s\colon 1 \leq s \leq m\} \cup \{u_s u_{s+1}\colon 1 \leq s \leq m - 1\} \\ &\cup \{u_s u_{s+1}\colon\ 1 \leq s \leq n - 1\}. \end{aligned} \tag{3}$$

It is to be noted that $|V(U(m,n))| = m + n$ and $|E(U(m,n))| = 2m + n - 2$.

Let $v_1, v_2, v_3, \ldots, v_n$ and $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the graph as specified in Figure 2. The labeling of the graph must be strongly multiplicative. The mapping of the vertices is defined as follows:

$$f\colon V(U(m,n)) \longrightarrow \{1, 2, \ldots, m + n\}, \tag{4}$$

where

$$\begin{aligned} f(u_i) &= i, \quad \forall 1 \leq i \leq m, \\ f(v_i) &= i + m, \quad \forall 1 \leq i \leq n. \end{aligned} \tag{5}$$

According to this mapping, the products of label of two end vertices are the induced edge labels which are also distinct, i.e., no two edges have same labels.

Hence, $U(m,n)$ is a strongly multiplicative graph.

It is to be noted that the above defined mapping is proved for even and odd values of n.

In Figure 2, strongly multiplicative labeling of $U(7,4)$ is shown. □

Theorem 3. *The generalized Petersen graph $GP(n,k)$ admits strongly multiplicative labeling.*

Proof. The generalized Petersen graph $\mathrm{GP}(n,k)$ $GP(n,\ k)$ for $n \geq 3$ and $1 \leq k < (n/2)$ is a cubic graph which is also connected.

The vertex set of $\mathrm{GP}(n,k)$ $GP(n,\ k)$ is $V = \{v_i, u_i\colon 1 \leq i \leq n\}$. Here, $\{v_1, v_2, \ldots, v_n\}$ are the vertices of the inner cycle and $\{u_1, u_2, \ldots, u_n\}$ are the vertices of the outer cycle of the generalized Peterson graph. The set of edges of $\mathrm{GP}(n,k) GP(n,\ k)$ is as follows:

$$E = \{u_i v_i, u_i u_{i+1}, v_i v_{i+k}\colon\ 1 \leq i \leq n\}. \tag{6}$$

It is to be noted that $|V(\mathrm{GP}(n,k))| = 2n$ and $|E(\mathrm{GP}(n,k))| = 3n$.

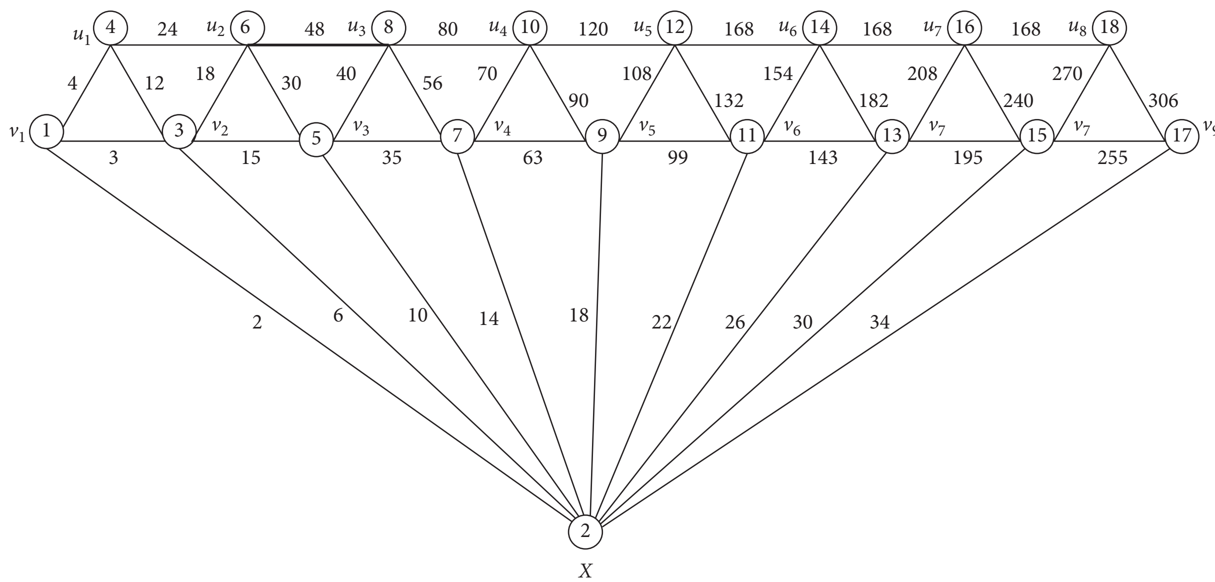

Figure 1: Representation of Br_9 with strongly multiplicative labeling.

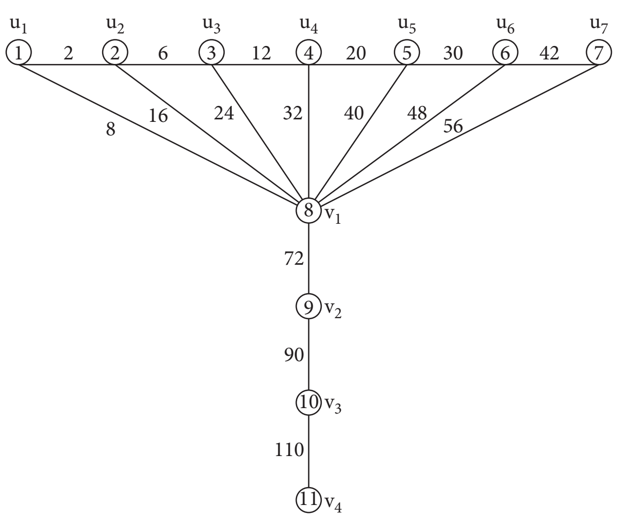

Figure 2: Representation of $U(7, 4)$ with strongly multiplicative labeling.

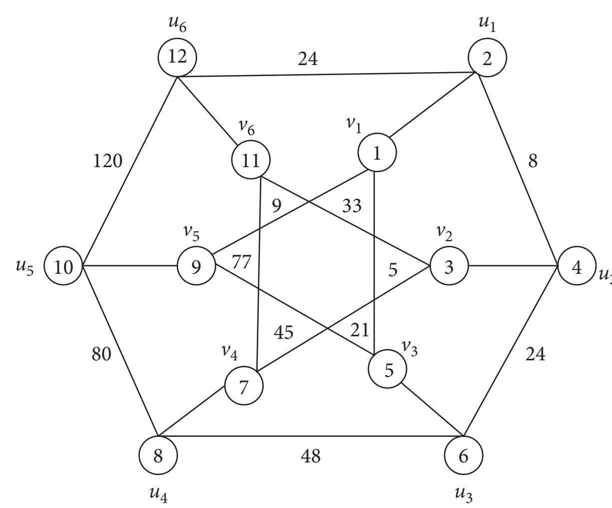

Figure 3: Representation of GP(6, 2) with strongly multiplicative labeling.

Vertices of the generalized Petersen graph are labeled as follows:

$$f\colon V(\mathrm{GP}(n, k)) \longrightarrow \{1, 2, \ldots, 2n\}, \tag{7}$$

where

$$\begin{aligned} f(v_i) &= 2i - 1, \quad \forall 1 \leq i \leq n, \\ f(u_i) &= 2i, \quad \forall 1 \leq i \leq n. \end{aligned} \tag{8}$$

According to this mapping, the products of label of two end vertices are the induced edge labels which are also distinct, i.e., no two edges have same labels.

Hence, $\mathrm{GP}(n, k)$ is strongly multiplicative graph.

In Figure 3, strongly multiplicative labeling of the Durer graph GP(6, 2) is shown. □

Theorem 4. *The double comb graph* (DC_n) *admits strongly multiplicative labeling.*

Proof. The vertex set of DC_n is $V = \{v_m, u_m, w_m\colon 1 \leq m \leq n\}$ and the edge set is as follows:

$$\begin{aligned} E = \big[\{(u_m u_{m+1})\colon 1 \leq m \leq n-1\} \cup \{(u_m v_m)\colon 1 \leq m \leq n\} \\ \cup \{(u_m w_m)\colon 1 \leq m \leq n\}\big]. \end{aligned} \tag{9}$$

It is to be noted that $|V(\mathrm{DC}_n)| = 3n$ and $|E(\mathrm{DC}_n)| = 3n - 1$.

Let $u_1, u_2, u_3, \ldots, u_m$, $v_1, v_2, v_3, \ldots, v_m$, and $w_1, w_2, w_3, \ldots, w_m$ be the vertices of a path. The mapping of vertices is defined as follows:

$$f\colon V(\mathrm{DC}_n) \longrightarrow \{1, 2, \ldots, 3n\}, \tag{10}$$

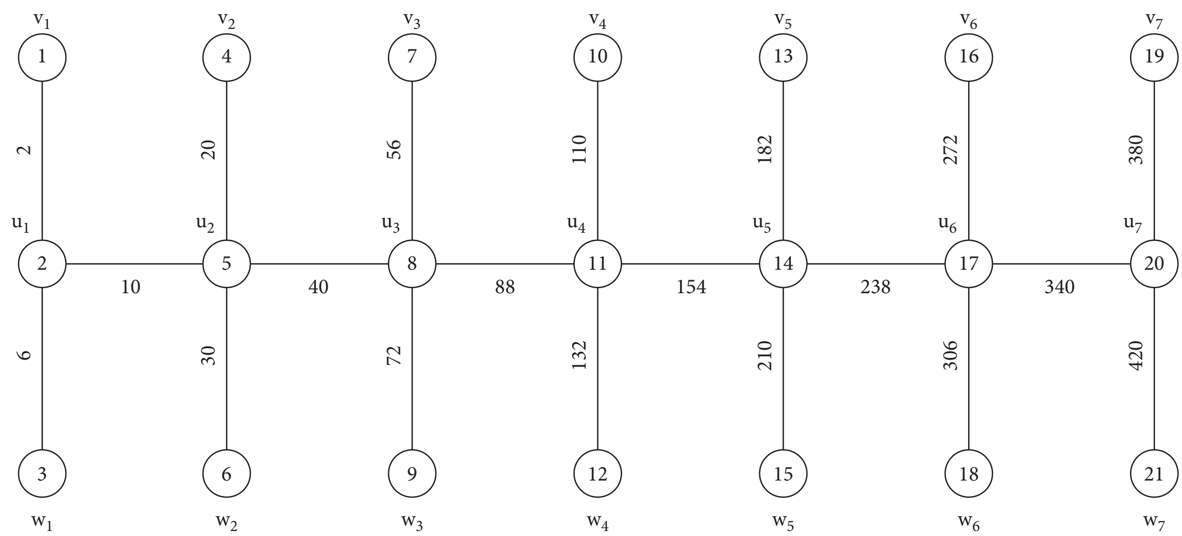

FIGURE 4: Representation of $(\mathrm{DC}_7)(DC_7)$ with strongly multiplicative labeling.

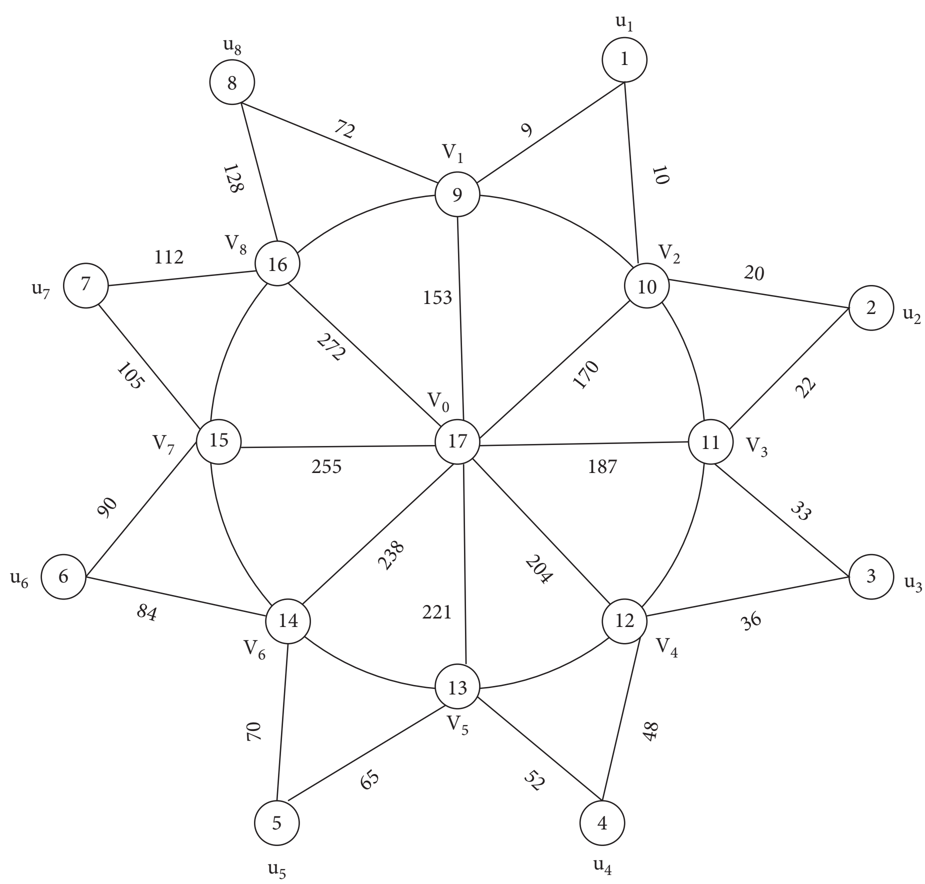

FIGURE 5: Representation of $(Sf_8)(Sf_8)$ with strongly multiplicative labeling.

where

$$
\begin{aligned}
f(w_i) &= 3i, \quad \forall 1 \leq i \leq n, \\
f(u_i) &= 3i - 1, \quad \forall 1 \leq i \leq n, \\
f(v_i) &= 3i - 2, \quad \forall 1 \leq i \leq n.
\end{aligned}
\tag{11}
$$

According to this mapping, the products of label of two end vertices are the induced edge labels which are also distinct, i.e., no two edges have same labels.

Hence, the double comb graph (DC_n) is a strongly multiplicative graph.

In Figure 4, strongly multiplicative labeling of (DC_7) is elaborated. □

Theorem 5. *The sunflower planar graph* (Sf_n) *admits a strongly multiplicative graph.*

Proof. The vertex set of the sunflower planar graph is $V = \{v_0, v_m, u_m\colon 1 \leq m \leq n\}$ and the edge set is as follows:

$$E = \{v_0 v_m\colon 1 \leq m \leq n\} \cup \{v_m v_{m+1}, v_n v_1\colon 1 \leq m \leq n-1\} \cup \{u_m v_m, u_m v_{m+1}\colon 1 \leq m \leq n\}. \tag{12}$$

It is to be noted that $|V(Sf_n)| = 2n+1$ and $|E(Sf_n)| = 4n$.

Let $v_0, v_1, v_2, v_3, \ldots, v_m$ and $u_1, u_2, u_3, \ldots, u_m$ be the connected vertices of a graph. In order to get strongly multiplicative labeling of the graph, the mapping of vertices is defined as follows:

$$f\colon V(Sf_n) \longrightarrow \{1, 2, \ldots, 2n+1\}, \tag{13}$$

where

$$\begin{aligned} f(u_i) &= i, \quad \forall 1 \leq i \leq n, \\ f(v_i) &= i + n, \quad \forall 1 \leq i \leq n, \\ f(v_0) &= i + 2n. \end{aligned} \tag{14}$$

According to this mapping, the products, of label of two end vertices are the induced edge labels which are also distinct i.e., no two edges have same labels.

Hence, (Sf_n) is a strongly multiplicative graph.

In Figure 5, strongly multiplicative labeling of the sunflower planar graph (Sf_8) is shown. □

4. Conclusion

In this paper, strongly multiplicative labeling of certain well-known families of graphs is analyzed. First, we label families of order k of the diamond graph, umbrella graph, generalized Petersen graph, double comb graph, and sunflower planar graph. We label the vertices of these graphs according to the condition of strongly multiplicative labeling. By multiplying labels of two adjacent vertices, we get the value of their connected edge. By doing this technique, we generated some important results for any order of different types of graphs which are presented in this work. By using these results, strongly multiplicative labeling of any order n of graph can be calculated. These results give a basic idea to compute strongly multiplicative labeling of the diamond graph, umbrella graph, generalized Petersen graph, double comb graph, and sunflower planar graph.

4.1. Open Problems. Problems of finding strongly multiplicative labeling of transformation graphs, Mobius ladder, polygonal chains, stacked prism graph, and many others are open for future research.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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