Hindawi
Journal of Mathematics
Volume 2022, Article ID 3208164, 6 pages
https://doi.org/10.1155/2022/3208164

Research Article

A Note on the Signed Clique Domination Numbers of Graphs

Baogen Xu, Ting Lan, and Mengmeng Zheng

Department of Mathematics, East China Jiaotong University, Nanchang 330013, China

Correspondence should be addressed to Baogen Xu; baogenxu@163.com

Received 19 May 2022; Accepted 5 July 2022; Published 17 September 2022

Academic Editor: Serkan Araci

Let $G = (V, E)$ be a graph. A function $f\colon E \longrightarrow \{-1, +1\}$ is said to be a signed clique dominating function (SCDF) of G if $\sum_{e\in E(K)} f(e) \geq 1$ holds for every nontrivial clique K in G. The signed clique domination number of G is defined as $\gamma_{\mathrm{scl}}'(G) = \min\{\sum_{e\in E(G)} f(e) | f \text{ is an SCDF of} G\}$. In this paper, we investigate the signed clique domination numbers of join of graphs. We correct two wrong results reported by Ao et al. (2014) and Ao et al. (2015) and determine the exact values of the signed clique domination numbers of $P_m \vee \overline{K_n}$ and $C_m \vee K_n$.

1. Introduction

We use Bondy and Murty [1] for terminology and notation not defined here and consider only simple and undirected graphs.

The theory of domination is an important content in graphs, and its applications are more and more widely. In 1995, Dunbar et al. [2] first introduced the signed domination of graphs, and now, there are a lot of variations, such as the total domination [3]. However, most of them belong to the vertex domination of graphs, and a few of the edge dominations were studied. In 2001, B. Xu puts forward the signed edge domination of graphs [4]. Since then, many variations based on edge domination have become more and more abundant, such as signed total domination [5], Roman domination [6], signed cycle domination [7], and signed clique domination [8]. The emergence of these concepts enriched and completed the domination theory of graphs. In this paper, we consider the signed clique domination numbers of joint graphs.

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Every maximal complete subgraph K of graph G is called a clique of G; that is, there are no other complete subgraphs contain K. A clique K is called nontrivial if $K \neq K_1$.

Given a graph $G = (V, E)$, for any vertex $v \in V$, let $E(v)$ be the set of edges in G incident to v. If $A \subseteq V$, $B \subseteq V$, and $A \cap B = \emptyset$, then we write $E(A, B) = \{uv \in E | u \in A, v \in B\}$.

For any two disjoint graphs G_1 and G_2, then $G_1 \vee G_2$ denotes the joint graph of G_1 and G_2, where

$$V(G_1 \vee G_2) = V(G_1) \cup V(G_2),$$
$$E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1), v \in V(G_2)\}. \tag{1}$$

For ease of description, let $G = (V, E)$ be a graph, $S \subseteq E$ and $f\colon E \longrightarrow R$ be a real-valued function defined on E, then we put $f(S) = \sum_{e\in S} f(e)$.

Definition 1 (see [8]). Let $G = (V, E)$ be a graph, a function $f\colon E \longrightarrow \{-1, +1\}$ is said to be a signed clique dominating function (SCDF) of G if $f(E(K)) \geq 1$ holds for every nontrivial clique K in G. The signed clique domination number of G is defined as

$$\gamma_{\mathrm{scl}}'(G) = \min\{f(E) | f \text{ is an SCDF of } G\}. \tag{2}$$

If f is an SCDF such that $\gamma_{\mathrm{scl}}'(G) = f(E(G))$, then the function f is said to be a minimum SCDF of G.

Lemma 1 (see [8]). For any graph G, $\gamma_{\mathrm{scl}}'(G) \equiv |E(G)|$ (mod2).

Lemma 2 (see [8]). *For any connected graph G of order n, if $|E(G)| = m$ and $\omega(G) \leq 4$. Then,*

$$\gamma_{\mathrm{scl}}'(G) \geq 2n - m - 2. \tag{3}$$

This lower bound is the best possible.

For the joint graphs of the two graphs, we have the following results.

Lemma 3 (see [9]).
(1) For any two positive integers $m \geq 3$ and $n \geq 2$, then

$$\gamma_{\mathrm{scl}}'(C_m \vee nK_2) = \begin{cases} 3-n, & \text{when } m = 3, \\ m+n, & \text{when } m \geq 4. \end{cases} \tag{4}$$

(2) For any two positive integers $m \geq 4$ and $n \geq 4$, then

$$\gamma_{\mathrm{scl}}'(C_m \vee C_n) = \begin{cases} m+n+1, & \text{when } m \text{ and } n \text{ are odd}, \\ m+n, & \text{otherwise}. \end{cases} \tag{5}$$

Lemma 4 (see [10]). *For any positive integer $(n > i > 3)$, then*

$$\gamma_{\mathrm{scl}}'(C_i \vee \overline{K_{n-i}}) = \begin{cases} 6-n, & \text{when } i = 3, \\ i, & \text{when } i \geq 4 \text{ and } i \equiv 0 \,(\mathrm{mod}\, 2), \\ n-2, & \text{when } i \geq 4 \text{ and } i \equiv 1 \,(\mathrm{mod}\, 2). \end{cases} \tag{6}$$

However, we find the following two lemmas are wrong.

Lemma 5 (see [10]). *For any positive integer $(n > i > 0)$, then*

$$\gamma_{\mathrm{scl}}'(P_i \vee \overline{K_{n-i}}) = \begin{cases} n-1, & \text{when } i = 1, \\ \left(\lfloor \frac{i}{2} \rfloor - \lceil \frac{i}{2} \rceil\right)(n-i) + (i-1), & \text{when } i \geq 2. \end{cases} \tag{7}$$

For example, in fact, when $n = 6$ and $i = 5$, then $\gamma_{\mathrm{scl}}'(P_5 \vee \overline{K_1}) \leq 1$. The labeling of $P_5 \vee \overline{K_1}$ is shown in Figure 1, and when $n = 8$ and $i = 6$, then $\gamma_{\mathrm{scl}}'(P_6 \vee \overline{K_2}) \leq 3$. The labeling of $P_6 \vee \overline{K_2}$ is shown in Figure 2 where unlabeled edges are assigned as -1.

Lemma 6 (see [11]). *For any positive integer $m \geq 3$ and $n \geq 3$, then*

$$\gamma_{\mathrm{scl}}'(K_n \vee C_m) = \begin{cases} 2(6-n)\lceil \frac{m}{2} \rceil - (n+1)m + \frac{n(n-1)}{2}, & \text{when } n = 3, 4, 5, \\ -(n+1)m + 2n + 2 + \frac{(-1)^{\lfloor n/2 \rfloor + 1} + 1}{2}, & \text{when } n \geq 6. \end{cases} \tag{8}$$

In fact, the above conclusion is not true for $n = 3$ or 4, and m is odd. For example, we may see $\gamma_{\mathrm{scl}}'(K_3 \vee C_5) \leq -1$ and $\gamma_{\mathrm{scl}}'(K_4 \vee C_5) \leq -9$. The labeling of $K_3 \vee C_5$ and $K_4 \vee C_5$ is shown in Figures 3 and 4, respectively, where unlabeled edges are assigned as -1.

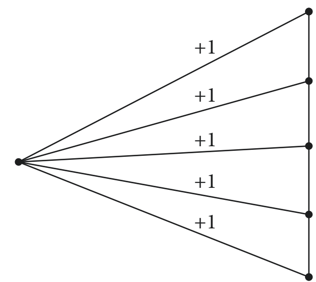

Figure 1: $P_5 \vee \overline{K_1}$.

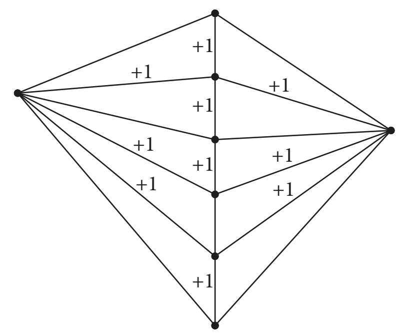

Figure 2: $P_6 \vee \overline{K_2}$.

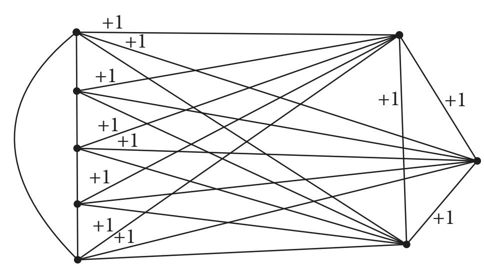

Figure 3: $K_3 \vee C_5$.

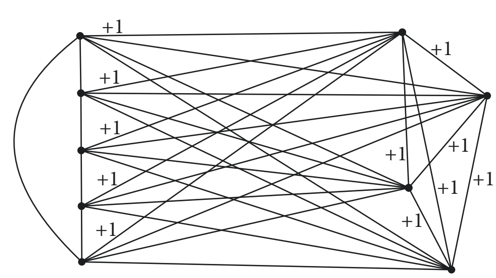

Figure 4: $K_4 \vee C_5$.

In this note, we mainly correct two wrong conclusions in [10, 11] and determine exactly the signed clique domination numbers of $P_m \vee \overline{K_n}$ and $C_m \vee K_n$.

2. Main Results

Theorem 1. *For any two positive integers $m \geq 2$ and $n \geq 1$, then*

$$\gamma_{\mathrm{scl}}'(P_m \vee \overline{K_n}) = \begin{cases} 1, & \text{when } m = 2 \text{ or } n = 1, \\ m - n - 1, & \text{when } m \geq 3 \text{ and } m \text{ is odd}, \\ m - 3, & \text{when } m \geq 4 \text{ and } m \text{ is even}. \end{cases} \tag{9}$$

Proof. Let $G = P_m \vee \overline{K_n}$ be a joint graph, $V(G) = V(P_m) \cup V(\overline{K_n})$, we define $A = V(P_m) = \{u_1, u_2, \ldots, u_m\}$, $E(P_m) = \{u_i u_{i+1} | 1 \leq i \leq m-1\}$, $B = V(\overline{K_n}) = \{v_1, v_2, \ldots, v_n\}$.

Case 1. When $m = 2$.

Let f be a minimum SCDF of G, that is, $\gamma_{\mathrm{scl}}'(G) = f(E(G))$. Obviously, there are n cliques $K^{(j)} = K_3 (1 \leq j \leq n)$ in G, among $K^{(j)}$ is a clique with three vertices u_1, u_2, v_j. The n cliques contain a common edge $u_1 u_2$. According to Definition 1, we have $f(E(K^{(j)})) \geq 1$ held for every clique $K^{(j)}$. Then, we obtain $\sum_{j=1}^{n} f(E(K^{(j)})) \geq n$. In this inequality, note that the function value $f(e)$ of each edge e in $E(P_2)$ is counted exactly n times, and the function value $f(e)$ of each edge e in $E(A, B)$ is counted one time. Then,

$$nf(E(P_2)) + f(E(A, B)) \geq n. \tag{10}$$

For every vertex $v_j \in B$, $E(v_j)$ is the set of edges in G incident to v_j. It is obvious that $f(E(v_j)) \geq 0$ (otherwise, if there exists a vertex v_s such that $f(E(v_s)) < 0$). It implies the contradiction that $f(E(K^{(s)})) < 0$). Thus, we derive $f(E(A, B)) = \sum_{j=1}^{n} f(E(v_j)) \geq 0$. Combining with (10), we have

$$\begin{aligned} \gamma_{\mathrm{scl}}'(G) &= f(E(G)), \\ &= f(E(P_2)) + f(E(A, B)) \geq 1 + \frac{n-1}{n} f(E(A, B)) \geq 1. \end{aligned} \tag{11}$$

Meanwhile, we define such a function f' of G as follows:

$$f'(e) = \begin{cases} -1, & e = u_1 v_j (1 \leq j \leq n), \\ +1, & \text{otherwise}. \end{cases} \tag{12}$$

It is routine to check that f' is an SCDF of G. Hence,

$$\gamma_{\mathrm{scl}}'(G) \leq f'(E(G)) = f'(E(P_2)) + f'(E(A, B)) = 1. \tag{13}$$

Combining with $\gamma_{\mathrm{scl}}'(G) \geq 1$, we finally derive $\gamma_{\mathrm{scl}}'(G) = 1$.

Case 2. When $n = 1$.

We know that $G = P_m \vee K_1$ is connected graph of order $m+1$, $|E(G)| = 2m - 1$, and $w(G) \leq 3$. By Lemma 2, we have $\gamma_{\mathrm{scl}}'(G) \geq 2(m+1) - (2m-1) - 2 = 1$. Meanwhile, we define such a function f of G as follows:

$$f(e) = \begin{cases} +1, & e \in E(A, B), \\ -1, & e \in E(P_m). \end{cases} \tag{14}$$

It is routine to check that f is an SCDF of G. Then,

$$\gamma_{\mathrm{scl}}'(G) \leq f(E(G)) = f(E(P_m)) + f(E(A, B)) = 1. \tag{15}$$

Combining with $\gamma_{\mathrm{scl}}'(G) \geq 1$, we finally have $\gamma_{\mathrm{scl}}'(G) = 1$.

Case 3. When $m \geq 3$ and m is odd.

Let f be a minimum SCDF of G, that is, $\gamma_{\mathrm{scl}}'(G) = f(E(G))$. Clearly, there are $n(m-1)$ cliques $K_{i,j} = K_3 (1 \leq i \leq m-1, 1 \leq j \leq n)$ in G, among $K_{i,j}$ is a clique with three vertices u_i, u_{i+1}, v_j. According to Definition 1, we obtain $f(E(K_{i,j})) \geq 1$holds for every clique $K_{i,j}$. Therefore, we have $\sum_{i=1}^{m-1} \sum_{j=1}^{n} f(E(K_{i,j})) \geq n(m-1)$.

In the above inequality, we know that the function value $f(e)$ of each edge e in $E(P_m)$ is counted exactly n times. Then, let E_1 be the edge set in $E(A, B)$ where the function value $f(e)$ of each edge e is counted one time, and E_2 is the edge set in $E(A, B)$ where the function value $f(e)$ of each edge e is counted exactly 2 times, where

$$\begin{cases} E_1 = \{u_i v_j | i = 1, m, 1 \leq j \leq n\}, \\ E_2 = \{u_i v_j | 2 \leq i \leq m-1, 1 \leq j \leq n\}. \end{cases} \tag{16}$$

Then,

$$\sum_{i=1}^{m-1} \sum_{j=1}^{n} f(E(K_{i,j})) = nf(E(P_m)) + f(E_1) + 2f(E_2) \geq n(m-1). \tag{17}$$

For every vertex $v_j \in B$, it is obvious that $f(E(v_j)) \geq -1$ (otherwise, there must exist a clique $K_{s,t} (1 \leq s \leq m-1, 1 \leq t \leq n)$ such that $f(E(K_{s,t})) < 1$, a contradiction). Then, we have $f(E_1) + (E_2) = \sum_{j=1}^{n} f(E(v_j)) \geq -n$. We assume that there are k vertices $v_j (1 \leq j \leq n)$ such that $\sum_{i=2}^{m-1} f(u_i v_j) \geq 1$, then

$$\begin{cases} 2n - 4k \leq f(E_1) \leq 2n, \\ 2k - n \leq f(E_2) \leq k(m-2) - (n-k), \end{cases} \tag{18}$$

where $0 \leq k \leq n$.

According to (17), we obtain

$$\begin{aligned} \gamma_{\mathrm{scl}}'(G) &= f(E(P_m)) + f(E_1) + f(E_2) \\ &\geq (m-1) + \frac{n-1}{n} f(E_1) + \frac{n-2}{n} f(E_2) \\ &\geq (m-1) + \frac{n-1}{n}(2n - 4k) + \frac{n-2}{n}(2k - n) \\ &= m - 1 + n - 2k \geq m - n - 1. \end{aligned} \tag{19}$$

Meanwhile, we define such a function f' of G as follows:

$$f'(e) = \begin{cases} (-1)^i, & e = u_i v_j (1 \leq i \leq m, \ 1 \leq j \leq n), \\ +1, & e \in E(P_m). \end{cases} \tag{20}$$

It is clear that f' is an SCDF of G. Therefore,

$$\begin{aligned}\gamma_{\mathrm{scl}}'(G) &\le f'(E(G))\\ &= f'(E(P_m)) + f'(E_1) + f'(E_2) = m-n-1.\end{aligned}\tag{21}$$

Together with $\gamma_{\mathrm{scl}}'(G) \ge m-n-1$, we have $\gamma_{\mathrm{scl}}'(G) = m-n-1$.

Case 4. When $m \ge 4$ and m is even.

Let f be a minimum SCDF of G, i.e., $\gamma_{\mathrm{scl}}'(G) = f(E(G))$. The same as Case 3, we have $f(E(v_j)) \ge 0$, then $f(E_1) + f(E_2) = \sum_{j=1}^{n} f(E(v_j)) \ge 0$. We assume that there are k vertices $v_j\,(1 \le j \le n)$ such that $\sum_{i=2}^{m-1} f(u_i v_j) \ge 2$, then

$$\begin{cases} -2k \le f(E_1) \le 2n,\\ 2k \le f(E_2) \le k(m-2),\end{cases}\tag{22}$$

where $0 \le k \le n$.

According to (17), we have

$$\begin{aligned}\gamma_{\mathrm{scl}}'(G) &= f(E(P_m)) + f(E_1) + f(E_2)\\ &\ge (m-1) + \frac{n-1}{n} f(E_1) + \frac{n-2}{n} f(E_2)\\ &\ge (m-1) + \frac{n-1}{n}(-2k) + \frac{n-2}{n}(2k)\\ &= m - 1 - \frac{2k}{n} \ge m-3.\end{aligned}\tag{23}$$

In addition, we define such a function f' of G as follows:

$$f'(e) = \begin{cases} -1, & e \in u_1 v_j \cup u_2 u_3\,(1 \le j \le n),\\ (-1)^{i+1}, & e = u_i v_j\,(3 \le i \le m, 1 \le j \le n),\\ +1, & \text{otherwise}.\end{cases}\tag{24}$$

It is routine to check that f' is an SCDF of G. Therefore,

$$\gamma_{\mathrm{scl}}'(G) \le f'(E(G)) = f'(E(P_m)) + f'(E_1) + f'(E_2) = m-3.\tag{25}$$

Combining with $\gamma_{\mathrm{scl}}'(G) \ge m-3$, we have $\gamma_{\mathrm{scl}}'(G) = m-3$. This completes the proof of Theorem 1.

Theorem 2. *For any two positive integers $m \ge 3$ and $n \ge 3$, then*

$$\gamma_{\mathrm{scl}}'(C_m \vee K_n) = \begin{cases} 3 - 2m + 2\left\lceil \dfrac{m}{2} \right\rceil, & \text{when } n = 3;\\ 6 - 3m, & \text{when } n = 4;\\ 10 - 6m + 2\left\lceil \dfrac{m}{2} \right\rceil, & \text{when } n = 5;\\ -(n+1)m + 2n + 2 + \dfrac{(-1)^{\lfloor n/2 \rfloor + 1} + 1}{2}, & \text{when } n \ge 6.\end{cases}\tag{26}$$

Proof. Let $G = C_m \vee K_n$, $A = V(C_m) = \{u_1, u_2, \ldots, u_m\}$, and $E(C_m) = \{u_i u_{i+1} | 1 \le i \le m\}$, among $u_{m+1} = u_1$; $B = V(K_n) = \{v_1, v_2, \ldots, v_n\}$, $|E(G)| = (n(n-1)/2) + m(n+1)$.

Let f be a minimum SCDF of G, that is, $\gamma_{\mathrm{scl}}'(G) = f(E(G))$. Obviously, there are m cliques $K^{(i)} = K_{n+2}\,(1 \le i \le m)$ in G, among $K^{(i)}$ is a clique with $n+2$ vertices u_i, u_{i+1} and the vertex set B.

Case 5. When $n = 3$. Note that $K^{(i)} = K_5\,(1 \le i \le m)$. According to Definition 1, we have $f(E(K^{(i)})) \ge 1$ held for every clique $K^{(i)}$. Since $|E(K_5)| = 10$, by Lemma 1, we have $f(E(K^{(i)})) \ge 2$. Then, we obtain $\sum_{i=1}^{m} f(E(K^{(i)})) \ge 2m$. In this inequality, we know that the function value $f(e)$ of each edge e in $E(K_3)$ is counted exactly m times, the function value $f(e)$ of each edge e in $E(C_m)$ is counted exactly one time, and the function value $f(e)$ of each edge e in $E(A,B)$ is counted 2 times. Then,

$$m f(E(K_3)) + f(E(C_m)) + 2f(E(A,B)) \ge 2m.\tag{27}$$

Since $f(E(K_3)) \le 3$, we have

$$f(E(C_m)) + 2f(E(A,B)) \ge -m.\tag{28}$$

According to (27), we obtain

$$\begin{aligned}\gamma_{\mathrm{scl}}'(G) &= f(E(K_3)) + f(E(C_m))\\ &\quad + f(E(A,B)) \ge 2 + \frac{m-1}{m} f(E(C_m))\\ &\quad + \frac{m-2}{m} f(E(A,B)).\end{aligned}\tag{29}$$

Together with (28), we have

(i) When $f(E(A,B)) \ge 0$,

$$\begin{aligned}\gamma_{\mathrm{scl}}'(G) &\ge 2 + \frac{m-1}{m} f(E(C_m)) + \frac{m-2}{m} f(E(A,B))\\ &\ge 2 + \frac{m-1}{m}\cdot(-m) + 0 = 3 - m.\end{aligned}\tag{30}$$

(ii) When $f(E(A,B)) \le 0$,

$$\begin{aligned}\gamma_{\mathrm{scl}}'(G) &\ge 2 + \frac{m-1}{m} f(E(C_m)) + \frac{m-2}{m} f(E(A,B))\\ &= 2 + \frac{m-1}{m}\left[f(E(C_m)) + 2f(E(A,B))\right] - f(E(A,B))\\ &\ge 2 + \frac{m-1}{m}\cdot(-m) - 0 = 3 - m.\end{aligned}\tag{31}$$

Then, we have $\gamma_{\mathrm{scl}}'(G) \ge 3 - m$. Note that, when m is odd, $|E(G)| = 4m+3$ is also odd. As per Lemma 1, we derive $\gamma_{\mathrm{scl}}'(G) \ge 4 - m$. Thus, $\gamma_{\mathrm{scl}}'(G) \ge 3 - 2m + 2\lceil m/2 \rceil$. In addition, we define such a function f' of G as follows:

$$f'(e)=\begin{cases}+1, & e\in E(K_3)\cup\{u_iv_j\mid 1\le i\le m,\ j=1\}\cup\{u_iv_j\mid i\equiv 1\ (\mathrm{mod}\,2),\ j=2\},\\ -1, & \text{otherwise.}\end{cases}\tag{32}$$

It is not difficult to check that f' is an SCDF of G, then $\gamma_{\mathrm{scl}}'(G)\le f'(E(G))=3-2m+2\lceil m/2\rceil$. In summary, when $n=3$, $\gamma_{\mathrm{scl}}'(G)=3-2m+2\lceil m/2\rceil$.

Case 6. When $n=4$. We know $K^{(i)}=K_6\,(1\le i\le m)$. By the Definition 1, we have $f(E(K^{(i)}))\ge 1$ held for every clique $K^{(i)}$. Then, we obtain $\sum_{i=1}^{m} f(E(K^{(i)}))\ge m$. The same as Case 5, we have

$$mf(E(K_4))+f(E(C_m))+2f(E(A,B))\ge m.\tag{33}$$

Since $f(E(K_4))\le 6$, we derive

$$f(E(C_m))+2f(E(A,B))\ge -5m.\tag{34}$$

According to (33), we have

$$\begin{aligned}\gamma_{\mathrm{scl}}'(G)&=f(E(K_4))+f(E(C_m))\\&\quad+f(E(A,B))\ge 1+\frac{m-1}{m}f(E(C_m))\\&\quad+\frac{m-2}{m}f(E(A,B)).\end{aligned}\tag{35}$$

Combining with (34), we have

(i) When $f(E(A,B))\ge -2m$,

$$\begin{aligned}\gamma_{\mathrm{scl}}'(G)&\ge 1+\frac{m-1}{m}f(E(C_m))+\frac{m-2}{m}f(E(A,B))\\&\ge 1+\frac{m-1}{m}\cdot(-m)+\frac{m-2}{m}\cdot(-2m)=6-3m.\end{aligned}\tag{36}$$

(ii) When $f(E(A,B))\le -2m$,

$$\begin{aligned}\gamma_{\mathrm{scl}}'(G)&\ge 1+\frac{m-1}{m}f(E(C_m))+\frac{m-2}{m}f(E(A,B))\\&=1+\frac{m-1}{m}[f(E(C_m))+2f(E(A,B))]-f(E(A,B))\\&\ge 1+\frac{m-1}{m}\cdot(-5m)+2m=6-3m.\end{aligned}\tag{37}$$

Then, we have $\gamma_{\mathrm{scl}}'(G)\ge 6-3m$. Meanwhile, we define such a function f' of G as follows:

$$f'(e)=\begin{cases}+1, & e\in E(K_4)\cup\{u_iv_j|1\le i\le m,\ j=1\},\\ -1, & \text{otherwise.}\end{cases}\tag{38}$$

Clearly, f' is an SCDF of G, and then $\gamma_{\mathrm{scl}}'(G)\le f'(E(G))=6-3m$. In summary, when $n=4$, whether m is odd or m is even, we finally have $\gamma_{\mathrm{scl}}'(G)=6-3m$.

Case 7. When $n=5$. We know $K^{(i)}=K_7\,(1\le i\le m)$. As per Definition 1, we have $f(E(K^{(i)}))\ge 1$ held for every clique $K^{(i)}$. Then, we obtain $\sum_{i=1}^{m} f(E(K^{(i)}))\ge m$. The same as Case 5, we have

$$mf(E(K_5))+f(E(C_m))+2f(E(A,B))\ge m.\tag{39}$$

Since $f(E(K_5))\le 10$, we derive

$$f(E(C_m))+2f(E(A,B))\ge -9m.\tag{40}$$

According to (39), we have

$$\begin{aligned}\gamma_{\mathrm{scl}}'(G)&=f(E(K_5))+f(E(C_m))+f(E(A,B))\ge 1\\&\quad+\frac{m-1}{m}f(E(C_m))+\frac{m-2}{m}f(E(A,B)).\end{aligned}\tag{41}$$

Combining with (40), we have

(i) When $f(E(A,B))\ge -4m$,

$$\begin{aligned}\gamma_{\mathrm{scl}}'(G)&\ge 1+\frac{m-1}{m}f(E(C_m))+\frac{m-2}{m}f(E(A,B))\\&\ge 1+\frac{m-1}{m}\cdot(-m)+\frac{m-2}{m}\cdot(-4m)=10-5m.\end{aligned}\tag{42}$$

(ii) When $f(E(A,B))\le -4m$,

$$\begin{aligned}\gamma_{\mathrm{scl}}'(G)&\ge 1+\frac{m-1}{m}f(E(C_m))+\frac{m-2}{m}f(E(A,B))\\&=1+\frac{m-1}{m}[f(E(C_m))+2f(E(A,B))]-f(E(A,B))\\&\ge 1+\frac{m-1}{m}\cdot(-9m)+4m=10-5m.\end{aligned}\tag{43}$$

Then, we have $\gamma_{\mathrm{scl}}'(G)\ge 10-5m$. Notice that when m is odd, $|E(G)|=6m+10$ is even. By Lemma 1, we obtain $\gamma_{\mathrm{scl}}'(G)\ge 11-5m$. Thus, $\gamma_{\mathrm{scl}}'(G)\ge 10-6m+2\lceil m/2\rceil$. In addition, we define such a function f' of G as follows:

$$f'(e)=\begin{cases}+1, & e\in E(K_5)\cup\{u_iv_j\mid i\equiv 1\ (\mathrm{mod}\,2),\ j=1\},\\ -1, & \text{otherwise.}\end{cases}\tag{44}$$

It is not difficult to check that f' is an SCDF of G, then $\gamma_{\mathrm{scl}}'(G)\le f'(E(G))=10-6m+2\lceil m/2\rceil$. In summary, when $n=5$, we have $\gamma_{\mathrm{scl}}'(G)=10-6m+2\lceil m/2\rceil$.

Case 8. When $n\ge 6$. Let f be a minimum SCDF of G, that is, $\gamma_{\mathrm{scl}}'(G)=f(E(G))$. Write $s=\{e\in E(G)|f(e)=1\}$, $s_1=\{e\in E(K_{n+2})|f(e)=1\}$.

According to Definition 1, we have $f(E(K_{n+2})) \geq 1$ held for every clique K_{n+2} in G. Thus, we have $s_1 \geq \lfloor (n+2)(n+1)/4 \rfloor + 1$. It implies $s \geq s_1 \geq \lfloor (n+2)(n+1)/4 \rfloor + 1$, and then,

$$\gamma_{\mathrm{scl}}'(G) = 2s - |E(G)| \geq -(n+1)m + 2n + 2 + \frac{(-1)^{\lfloor n/2 \rfloor + 1} + 1}{2}. \quad (45)$$

In addition, since $n \geq 6$, we know $(n(n-1)/2) \geq \lfloor (n+2)(n+1)/4 \rfloor + 1$. Now define a function f', that is, let the number of +1 edges in K_n is $\lfloor (n+2)(n+1)/4 \rfloor + 1$, the other edges are assigned as -1. It is obvious that

$$\gamma_{\mathrm{scl}}'(G) \leq f'(E(G)) = -(n+1)m + 2n + 2 + \frac{(-1)^{\lfloor n/2 \rfloor + 1} + 1}{2}. \quad (46)$$

In summary, when $n \geq 6$, we have $\gamma_{scl}'(G) = -(n+1)m + 2n + 2 + ((-1)^{\lfloor n/2 \rfloor + 1} + 1/2)$. We complete the proof of Theorem 2.

Data Availability

All the results and data in this paper are obtained through theoretical analysis and logical reasoning.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Grant nos. 11961026, 11861032). The authors would like to thank the referees for their helpful comments.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan, London, UK, 1977.

[2] J. E. Dunbar, S. T. Hedetniemi, M. A. Henning, and P. J. Slater, *Signed domination in graphs. Graph Theory, Combinatorics, Algorithms and Applications*, Wiley, New York, NY, USA, 1995.

[3] M. Talanda-Fisher and P. Zhang, "Total dominating functions of graphs: antiregularity versus regularity," *Contributions to Mathematics*, vol. 2, pp. 61–70, 2020.

[4] B. Xu, "On signed edge domination numbers of graphs," *Discrete Mathematics*, vol. 239, no. 1–3, pp. 179–189, 2001.

[5] M. A. Henning, "Signed total domination in graphs," *Discrete Mathematics*, vol. 278, no. 1–3, pp. 109–125, 2004.

[6] E. J. Cockayne, P. A. Dreyer, S. M. Hedetniemi, and S. T. Hedetniemi, "Roman domination in graphs," *Discrete Mathematics*, vol. 278, no. 1–3, pp. 11–22, 2004.

[7] B. Xu, "On signed cycle domination in graphs," *Discrete Mathematics*, vol. 309, no. 4, pp. 1007–1012, 2009.

[8] B. Xu, S. Wang, and S. Zhou, "On signed clique domination numbers of graphs," *Modern Technologies System Analysis*, vol. 2, pp. 123–129, 2006.

[9] B. Xu, T. Lan, J. Zhang, and G. Li, "On signed clique domination numbers of graphs," *Journal of Jiangxi Normal University*, vol. 45, pp. 331–334, 2021.

[10] G. Ao, X. Hong, and Y. Ren, "Jirimutu, Signed edge domination numbers of graphs," *Utilitas Mathematica*, vol. 93, pp. 323–331, 2014.

[11] G. Ao, Jirimutu, and L. Zhao, "Signed clique edge domination numbers of graphs," *Journal of Mathematics*, vol. 35, pp. 1109–1114, 2015.