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Research Article

Three-Dimensional Expansion and Graphical Concept of Generalized Triangular Fuzzy Set

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We have studied the extended algebraic operations between two fuzzy numbers and calculated Zadeh's max-min composition operator for two generalized triangular fuzzy sets in $\mathbb{R}^2$. And we generalized the triangular fuzzy numbers from $\mathbb{R}^2$ to $\mathbb{R}^3$. We prove that the result of the three-dimensional case is an extension of two-dimensional case and presented it in a graph. The extension is proved by showing that the result obtained by restricting the three-dimensional result to two-dimensional result is consistent with the existing two-dimensional result.

1. Introduction

Fuzzy theory has been increasingly applied to humanities including logics and sociology as well as natural sciences from engineering to medicine. In mathematics, triangular fuzzy sets have been extensively studied, which resulted in numerous fuzzy theories. In applications of the fuzzy set theories, many operators between two fuzzy sets have been defined and calculated. In particular, Zadeh's operators have been widely applied and developed [1–3]. Recently, the application expands to fuzzy control theory [4, 5] and fuzzy logic [6–8]. The theories of triangular fuzzy numbers have been extended to generalized triangular fuzzy sets that do not have the maximum value of 1. And as the number of ambiguous fuzzy variables increases, the theories have been extended to the studies of two-dimensional and three-dimensional fuzzy sets. In that respect, the study that extended Zadeh's operator theory to two or three dimensions is meaningful. We have studied the extended algebraic operations between two fuzzy numbers [9–12] and calculated Zadeh's max-min composition operator for two generalized triangular fuzzy sets in $\mathbb{R}^2$ [13–15]. In [16], we generalized the triangular fuzzy numbers from $\mathbb{R}^2$ to $\mathbb{R}^3$. By defining a parametric operator between two α-cuts with ellipsoidal values containing the interior, we defined a parametric operator for the two triangular fuzzy numbers defined in $\mathbb{R}^3$. We proved that the results for the parametric operator are the generalization of Zadeh's extended algebraic operator on $\mathbb{R}$ [9]. In addition, we calculated the parametric operators for two generalized three-dimensional triangular fuzzy sets and presented the calculation in three-dimensional graphs [17].

In this paper, we prove that the result of the three-dimensional case is an extension of two dimensions and presented it in a graph. The extension is proved by showing that the result obtained by restricting the three-dimensional result to two dimensions is consistent with the existing two-dimensional result. The graph of the fuzzy set defined in three dimensions expresses the function value by color density. When the graph is cut with a vertical plane passing through the vertex of a generalized three-dimensional triangular fuzzy set, the function value is shown through color density on the cross section of the graph. The value of the membership function defined on the cross section can be expressed in a graph of the function defined in two dimensions. We show that this graph is consistent with the three-dimensional representation of the results in two dimensions.

2. Zadeh's Max-Min Composition Operations for Generalized Triangular Fuzzy Sets on $\mathbb{R}^2$

We define α-cut and α-set of the fuzzy set A on $\mathbb{R}$ with the membership function $\mu_A(x)$.

Definition 1. An α-cut of the fuzzy number A is defined by $A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geq \alpha\}$ if $\alpha \in (0,1]$ and $A_0 = \mathrm{cl}\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$, where $\mathrm{cl}(B)$ is the closure of $B \subset \mathbb{R}$. For $\alpha \in (0,1)$, the set $A^\alpha = \{x \in X \mid \mu_A(x) = \alpha\}$ is said to be the α-set of the fuzzy set A, A^0 is the boundary of $\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$, and $A^1 = A_1$.

We define the generalized two-dimensional triangular fuzzy numbers on $\mathbb{R}^2$ as a generalization of generalized triangular fuzzy sets on $\mathbb{R}$ and the parametric operations between two generalized two-dimensional triangular fuzzy sets. For that, we have to calculate operations between α-cuts in $\mathbb{R}$. The α-cuts are intervals in $\mathbb{R}$, but in $\mathbb{R}^2$, the α-cuts are regions, which makes the existing method of calculations between α-cuts unusable. We interpret the existing method from a different perspective and apply the method to the region valued α-cuts on $\mathbb{R}^2$.

Definition 2. A fuzzy set A with a membership function:

$$\mu_A(x,y) = \begin{cases} h - \sqrt{\dfrac{(x-x_1)^2}{a^2} + \dfrac{(y-y_1)^2}{b^2}}, & b^2(x-x_1)^2 + a^2(y-y_1)^2 \leq a^2b^2h^2, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where $a, b > 0$ and $0 < h < 1$ is called the generalized two-dimensional triangular fuzzy set and denoted by $(a, x_1, h, b, y_1)^2$.

The intersections of $\mu_A(x,y)$ and the vertical planes $y - y_1 = k(x - x_1)$ $(k \in \mathbb{R})$ are symmetric triangular fuzzy numbers in those planes. If $a = b$, ellipses become circles. The α-cut A_α of a generalized two-dimensional triangular fuzzy number $A = (a, x_1, h, b, y_1)^2$ is an interior of ellipse in an xy-plane including the boundary

$$A_\alpha = \left\{(x,y) \in \mathbb{R}^2 \middle| \left(\frac{x-x_1}{a(h-\alpha)}\right)^2 + \left(\frac{y-y_1}{b(h-\alpha)}\right)^2 \leq 1\right\}. \tag{2}$$

Definition 3. A two-dimensional fuzzy number A defined on $\mathbb{R}^2$ is called convex fuzzy number if for all $\alpha \in (0,1)$, the α-cuts,

$$A_\alpha = \{(x,y) \in \mathbb{R}^2 | \mu_A(x,y) \geq \alpha\}, \tag{3}$$

are convex subsets in $\mathbb{R}^2$.

Theorem 1 (see [9]). *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha = \{(x,y) \in \mathbb{R}^2 | \mu_A(x,y) = \alpha\}$ be the α-set of A. Then, for all $\alpha \in (0,1)$, there exist continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0, 2\pi]$ such that*

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\}. \tag{4}$$

Definition 5. Let A and B be convex fuzzy sets defined on $\mathbb{R}^2$ and

$$\begin{aligned} A^\alpha &= \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\}, \\ B^\alpha &= \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\}, \end{aligned} \tag{5}$$

be the α-sets of A and B, respectively. For $\alpha \in (0,1)$, the parametric addition, parametric subtraction, parametric multiplication, and parametric division are fuzzy sets that have their α-sets as follows.

2.1. Parametric Addition $A(+)_pB$.

The parametric addition is given by the following:

$$(A(+)_pB)^\alpha = \{(f_1^\alpha(t) + g_1^\alpha(t), f_2^\alpha(t) + g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\} \tag{6}$$

2.2. Parametric Subtraction $A(-)_pB$.

The parametric subtraction is given by the following:

$$(A(-)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\}, \tag{7}$$

where

$$\begin{aligned} x_\alpha(t) &= \begin{cases} f_1^\alpha(t) - g_1^\alpha(t+\pi), & \text{if } 0 \leq t \leq \pi, \\ f_1^\alpha(t) - g_1^\alpha(t-\pi), & \text{if } \pi \leq t \leq 2\pi, \end{cases} \\ y_\alpha(t) &= \begin{cases} f_2^\alpha(t) - g_2^\alpha(t+\pi), & \text{if } 0 \leq t \leq \pi, \\ f_2^\alpha(t) - g_2^\alpha(t-\pi), & \text{if } \pi \leq t \leq 2\pi. \end{cases} \end{aligned} \tag{8}$$

2.3. Parametric Multiplication $A(\cdot)_pB$.

The parametric multiplication is given by the following:

$$(A(\cdot)_pB)^\alpha = \{(f_1^\alpha(t) \cdot g_1^\alpha(t), f_2^\alpha(t) \cdot g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\}. \tag{9}$$

2.4. Parametric Division $A(/)_pB$. The parametric division is given by the following:

$$\left(A(/)_pB\right)^\alpha = \left\{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\right\}, \quad (10)$$

where

$$\begin{aligned} x_\alpha(t) &= \frac{f_1^\alpha(t)}{g_1^\alpha(t+\pi)}, \quad (0 \le t \le \pi),\\ x_\alpha(t) &= \frac{f_1^\alpha(t)}{g_1^\alpha(t-\pi)}, \quad (\pi \le t \le 2\pi),\\ y_\alpha(t) &= \frac{f_2^\alpha(t)}{g_2^\alpha(t+\pi)}, \quad (0 \le t \le \pi),\\ y_\alpha(t) &= \frac{f_2^\alpha(t)}{g_2^\alpha(t-\pi)}, \quad (\pi \le t \le 2\pi). \end{aligned} \quad (11)$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_pB)^0 = \lim_{\alpha \longrightarrow 0^+}(A(*)_pB)^\alpha$ and $(A(*)_pB)^1 = \lim_{\alpha \longrightarrow 1^-}(A(*)_pB)^\alpha$, where $* = +, -, \cdot, /$.

Theorem 2 (see [10]). *Let $A = (a_1, x_1, h_1, b_1, y_1)^2$ and $B = (a_2, x_2, h_2, b_2, y_2)^2$ be two generalized two-dimensional triangular fuzzy sets. If $0 < h_1 < h_2 < 1$, then we have the following:*

(1) For $0 < \alpha < h_1$, the α-set of $A(+)_pB$ is

$$\left(A(+)_pB\right)^\alpha = \left\{(x,y) \in \mathbb{R}^2 \Big| \left(\frac{x - x_1 - x_2}{a_1(h_1-\alpha) + a_2(h_2-\alpha)}\right) + \left(\frac{y - y_1 - y_2}{b_1(h_1-\alpha) + b_2(h_2-\alpha)}\right)^2 = 1\right\}. \quad (12)$$

(2) For $0 < \alpha < h_1$, the α-set of $A(-)_pB$ is

$$\left(A(-)_pB\right)^\alpha = \left\{(x,y) \in \mathbb{R}^2 \Big| \left(\frac{x - x_1 + x_2}{a_1(h_1-\alpha) + a_2(h_2-\alpha)}\right)^2 + \left(\frac{y - y_1 + y_2}{b_1(h_1-\alpha) + b_2(h_2-\alpha)}\right)^2 = 1\right\}. \quad (13)$$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) | 0 \le t \le 2\pi\}$, where

$$\begin{aligned} x_\alpha(t) &= x_1x_2 + (x_1a_2(h_2-\alpha) + x_2a_1(h_1-\alpha))\cos t + a_1a_2(h_1-\alpha)(h_2-\alpha)\cos^2 t, \quad 0 < \alpha < h_1,\\ y_\alpha(t) &= y_1y_2 + (y_1b_2(h_2-\alpha) + y_2b_1(h_1-\alpha))\sin t + b_1b_2(h_1-\alpha)(h_2-\alpha)\sin^2 t, \quad 0 < \alpha < h_1. \end{aligned} \quad (14)$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) | 0 \le t \le 2\pi\}$, where

$$\begin{aligned} x_\alpha(t) &= \frac{x_1 + a_1(h_1-\alpha)\cos t}{x_2 - a_2(h_2-\alpha)\cos t},\\ y_\alpha(t) &= \frac{y_1 + b_1(h_1-\alpha)\sin t}{y_2 - b_2(h_2-\alpha)\sin t},\\ &0 < \alpha < h_1. \end{aligned} \quad (15)$$

Furthermore, we have

$$\begin{aligned} \left(A(*)_pB\right)^0 &= \lim_{\alpha \longrightarrow 0^+}\left(A(*)_pB\right)^\alpha, \quad * = +, -, \cdot, /,\\ \left(A(*)_pB\right)^{h_1} &= \lim_{\alpha \longrightarrow h_1^-}\left(A(*)_pB\right)^\alpha, \quad * = +, -, \cdot, /. \end{aligned} \quad (16)$$

If $h_1 < \alpha \le h_2$, by the Zadeh's max-min principle operations, we obtain

$$\left(A(*)_pB\right)^\alpha = \varnothing, \quad * = +, -, \cdot, /. \quad (17)$$

Example 1. (see [10]). Let $A = (6, 3, (1/2), 8, 5)^2$ and $B = (4, 2, (2/3), 5, 3)^2$. Then, by Theorem 2, we have the following:

(1) For $0 < \alpha < (1/2)$, the α-set of $A(+)_pB$ is

$$\left(A(+)_pB\right)^\alpha = \left\{(x,y) \in \mathbb{R}^2 \Big| \left(\frac{3x-15}{17-30\alpha}\right)^2 + \left(\frac{3y-24}{22-39\alpha}\right)^2 = 1\right\}. \quad (18)$$

(2) For $0 < \alpha < (1/2)$, the α-set of $A(-)_pB$ is

$$\left(A(-)_pB\right)^\alpha = \left\{(x,y) \in \mathbb{R}^2 \Big| \left(\frac{3x-3}{17-30\alpha}\right)^2 + \left(\frac{3y-6}{22-39\alpha}\right)^2 = 1\right\}. \quad (19)$$

(3) $(A(\cdot)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) | 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = 6 + (14 - 24\alpha)\cos t + 4(1 - 2\alpha)(2 - 3\alpha)\cos^2 t, \quad 0 < \alpha < \frac{1}{2},$$
$$y_\alpha(t) = 15 + \left(\frac{86}{3} - 49\alpha\right)\sin t + 20(1 - 2\alpha)\left(\frac{2}{3} - \alpha\right)\sin^2 t, \quad 0 < \alpha < \frac{1}{2}. \tag{20}$$

(4) $(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) | 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{9 + 9(1 - 2\alpha)\cos t}{6 - 4(2 - 3\alpha)\cos t},$$
$$y_\alpha(t) = \frac{15 + 12(1 - 2\alpha)\sin t}{9 - 15(2 - 3\alpha)\sin t}, \tag{21}$$
$$0 < \alpha < \frac{1}{2}.$$

3. Parametric Operations for Generalized Three-Dimensional Triangular Fuzzy Sets on $\mathbb{R}^3$

We define the generalized three-dimensional triangular fuzzy sets on $\mathbb{R}^3$ as a generalization of generalized triangular fuzzy sets on $\mathbb{R}^2$. Then, we define the parametric operations between two generalized three-dimensional triangular fuzzy sets. For that, we have to calculate operations between α-sets in $\mathbb{R}^3$. The α-sets are regions in $\mathbb{R}^2$, but in $\mathbb{R}^3$, the α-sets are ellipsoids including interior, which makes the existing method of calculations between α-sets unusable. We interpret the existing method from a different perspective and apply the method to the ellipsoids including interior-valued α-sets on $\mathbb{R}^3$.

Definition 6. A fuzzy set A with a membership function $\mu_A(x, y, z)$ such that

$$\begin{cases} h - \sqrt{\dfrac{(x - x_1)^2}{a^2} + \dfrac{(y - y_1)^2}{b^2} + \dfrac{(z - z_1)^2}{c^2}}, & \text{if } b^2c^2(x - x_1)^2 + c^2a^2(y - y_1)^2 + a^2b^2(z - z_1)^2 \le a^2b^2c^2h^2, \\ 0, & \text{otherwise}, \end{cases} \tag{22}$$

where $a, b, c > 0$ and $0 < h < 1$ is called the generalized three-dimensional triangular fuzzy set and denoted by $(h, a, x_1, b, y_1, c, z_1)^3$.

Note that $\mu_A(x, y)$ is a cone in $\mathbb{R}^2$, but we cannot know the shape of $\mu_A(x, y, z)$ in $\mathbb{R}^3$. The α-cut A_α of a generalized three-dimensional triangular fuzzy number $A = (h, a, x_1, b, y_1, c, z_1)^3$ is the following set:

$$A_\alpha = \left\{(x, y, z) \in \mathbb{R}^3 \middle| \left(\frac{x - x_1}{a(h - \alpha)}\right)^2 + \left(\frac{y - y_1}{b(h - \alpha)}\right)^2 + \left(\frac{z - z_1}{c(h - \alpha)}\right)^2 \le 1\right\}. \tag{23}$$

Definition 7. A three-dimensional fuzzy number A defined on $\mathbb{R}^3$ is called convex fuzzy number if for all $\alpha \in (0, 1)$, the α-cuts,

$$A_\alpha = \{(x, y, z) \in \mathbb{R}^3 | \mu_A(x, y, z) \ge \alpha\}, \tag{24}$$

are convex subsets in $\mathbb{R}^3$.

Theorem 3 (see [17]). *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^3$ and $A^\alpha = \{(x, y, z) \in \mathbb{R}^3 | \mu_A(x, y, z) = \alpha\}$ be the α-set of A. Then, for all $\alpha \in (0, 1)$, there exist continuous functions $f_1^\alpha(s), f_2^\alpha(s, t)$, and $f_3^\alpha(s, t)$ $(0 \le s \le 2\pi, -(\pi/2) \le t \le (\pi/2))$ such that*

$$A^\alpha = \left\{(f_1^\alpha(s), f_2^\alpha(s, t), f_3^\alpha(s, t)) \in \mathbb{R}^3 \middle| 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\right\}. \tag{25}$$

Definition 8 (see [17]). Let A and B are two continuous convex fuzzy sets defined on $\mathbb{R}^3$ and

$$A^{\alpha} = \left\{ \left(f_1^{\alpha}(s), f_2^{\alpha}(s,t), f_3^{\alpha}(s,t)\right) \in \mathbb{R}^3 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2} \right\},$$
$$B^{\alpha} = \left\{ \left(g_1^{\alpha}(s), g_2^{\alpha}(s,t), g_3^{\alpha}(s,t)\right) \in \mathbb{R}^3 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2} \right\}, \tag{26}$$

be the α-set of A and B, respectively. For $\alpha \in (0,1)$, we define that the parametric addition, parametric subtraction, parametric multiplication, and parametric division of two fuzzy sets A and B are fuzzy numbers that have their α-sets as follows:

(1) Parametric addition $A(+)_p B$:

$$\left(A(+)_p B\right)^{\alpha} = \left\{ \left(f_1^{\alpha}(s) + g_1^{\alpha}(s), f_2^{\alpha}(s,t) + g_2^{\alpha}(s,t), f_3^{\alpha}(s,t) + g_3^{\alpha}(s,t)\right) \in \mathbb{R}^3 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2} \right\}. \tag{27}$$

(2) Parametric subtraction $A(-)_p B$:

$$\left(A(-)_p B\right)^{\alpha} = \left\{ \left(f_1^{\alpha}(s) - g_1^{\alpha}(s+\pi),\ f_2^{\alpha}(s,t) - g_2^{\alpha}(s+\pi,t), f_3^{\alpha}(s,t) - g_3^{\alpha}(s+\pi,t)\right) \in \mathbb{R}^3 | 0 \le s \le \pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}, \right.$$
$$\left. \left(A(-)_p B\right)^{\alpha} = \left\{ \left(f_1^{\alpha}(s) - g_1^{\alpha}(s-\pi) f_2^{\alpha}(s,t) - g_2^{\alpha}(s-\pi,t), f_3^{\alpha}(s,t) - g_3^{\alpha}(s-\pi,t)\right) \in \mathbb{R}^3 | \pi \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2} \right\}. \tag{28}$$

(3) Parametric multiplication $A(\cdot)_p B$:

$$\left(A(\cdot)_p B\right)^{\alpha} = \left\{ \left(f_1^{\alpha}(s) \cdot g_1^{\alpha}(s), f_2^{\alpha}(s,t) \cdot g_2^{\alpha}(s,t), f_3^{\alpha}(s,t) \cdot g_3^{\alpha}(s,t)\right) \in \mathbb{R}^3 | 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2} \right\}. \tag{29}$$

(4) Parametric division $A(/)_p B$:

$$\left(A(/)_p B\right)^{\alpha} = \left\{ \left(\frac{f_1^{\alpha}(s)}{g_1^{\alpha}(s+\pi)}, \frac{f_2^{\alpha}(s,t)}{g_2^{\alpha}(s+\pi,t)}, \frac{f_3^{\alpha}(s,t)}{g_3^{\alpha}(s+\pi,t)} \right) \in \mathbb{R}^3 | 0 \le s \le \pi, \frac{\pi}{2} \le t \le \frac{\pi}{2} \right\},$$
$$\left(A(/)_p B\right)^{\alpha} = \left\{ \left(\frac{f_1^{\alpha}(s)}{g_1^{\alpha}(s-\pi)}, \frac{f_2^{\alpha}(s,t)}{g_2^{\alpha}(s-\pi,t)}, \frac{f_3^{\alpha}(s,t)}{g_3^{\alpha}(s-\pi,t)} \right) \in \mathbb{R}^3 | \pi \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2} \right\}. \tag{30}$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_p B)^0 = \lim_{\alpha \longrightarrow 0^+} (A(*)_p B)^{\alpha}$ and $(A(*)_p B)^1 = \lim_{\alpha \longrightarrow 1^-} (A(*)_p B)^{\alpha}$, where $* = +, -, \cdot, /$.

Theorem 4 (see [17]). *Let $A = (h_1, a_1, x_1, b_1, y_1, c_1, z_1)^3$ and $B = (h_2, a_2, x_2, b_2, y_2, c_2, z_2)^3$ be two generalized three-dimensional triangular fuzzy sets. If $0 < h_1 < h_2 < 1$, then we have the following:*

(1) For $0 < \alpha < h_1$, the α-set of $A(+)_p B$ is

$$\left(A(+)_p B\right)^{\alpha} = \left\{ (x,y,z) \in \mathbb{R}^3 | \left(\frac{x - x_1 - x_2}{a_1(h_1 - \alpha) + a_2(h_2 - \alpha)} \right)^2 + \left(\frac{y - y_1 - y_2}{b_1(h_1 - \alpha) + b_2(h_2 - \alpha)} \right)^2 + \left(\frac{z - z_1 - z_2}{c_1(h_1 - \alpha) + c_2(h_2 - \alpha)} \right)^2 = 1 \right\}. \tag{31}$$

(2) For $0 < \alpha < h_1$, the α-set of $A(-)_p B$ is

$$\left(A(-)_p B\right)^\alpha = \left\{(x, y, z) \in \mathbb{R}^3 \middle| \left(\frac{x - x_1 + x_2}{a_1(h_1 - \alpha) + a_2(h_2 - \alpha)}\right)^2 + \left(\frac{y - y_1 + y_2}{b_1(h_1 - \alpha) + b_2(h_2 - \alpha)}\right)^2 + \left(\frac{z - z_1 + z_2}{c_1(h_1 - \alpha) + c_2(h_2 - \alpha)}\right)^2 = 1\right\}. \tag{32}$$

(3) For $0 < \alpha < h_1$, $(A(\cdot)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) | 0 \le s \le 2\pi, -(\pi/2) \le t \le (\pi/2)\}$, where

$$\begin{aligned}
x_\alpha(s) &= x_1 x_2 + (x_1 a_2(h_2 - \alpha) + x_2 a_1(h_1 - \alpha))\cos s + a_1 a_2 (h_1 - \alpha)(h_2 - \alpha)\cos^2 s,\\
y_\alpha(s,t) &= y_1 y_2 + (y_1 b_2(h_2 - \alpha) + y_2 b_1(h_1 - \alpha))\sin s\cos t + b_1 b_2 (h_1 - \alpha)(h_2 - \alpha)\sin^2 s\cos^2 t,\\
z_\alpha(s,t) &= z_1 z_2 + (z_1 c_2(h_2 - \alpha) + z_2 c_1(h_1 - \alpha))\sin s\sin t + c_1 c_2 (h_1 - \alpha)(h_2 - \alpha)\sin^2 s\sin^2 t.
\end{aligned} \tag{33}$$

(4) For $0 < \alpha < h_1$, $(A(/)_p B)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) | 0 \le s \le 2\pi, -(\pi/2) \le t \le (\pi/2)\}$, where

$$\begin{aligned}
x_\alpha(s) &= \frac{x_1 + a_1(h_1 - \alpha)\cos s}{x_2 - a_2(h_2 - \alpha)\cos s},\\
y_\alpha(s.t) &= \frac{y_1 + b_1(h_1 - \alpha)\sin s\cos t}{y_2 - b_2(h_2 - \alpha)\sin s\cos t},\\
z_\alpha(s.t) &= \frac{z_1 + c_1(h_1 - \alpha)\sin s\sin t}{z_2 - c_2(h_2 - \alpha)\sin s\sin t}.
\end{aligned} \tag{34}$$

Furthermore, we have

$$\begin{aligned}
\left(A(*)_p B\right)^0 &= \lim_{\alpha \longrightarrow 0^+} \left(A(*)_p B\right)^\alpha, \quad * = +, -, \cdot, /,\\
\left(A(*)_p B\right)^{h_1} &= \lim_{\alpha \longrightarrow h_1^-} \left(A(*)_p B\right)^\alpha, \quad * = +, -, \cdot, /,
\end{aligned} \tag{35}$$

If $h_1 < \alpha \le h_2$, by the Zadeh's max-min principle operations, we obtain

$$\left(A(*)_p B\right)^\alpha = \varnothing, \quad * = +, -, \cdot, /, \tag{36}$$

4. A Generalization from $\mathbb{R}^2$ to $\mathbb{R}^3$ of Generalized Triangular Fuzzy Sets

In this section, we show that the parametric operations for two generalized triangular fuzzy sets defined on $\mathbb{R}^3$ are a generalization of parametric operations for two generalized triangular fuzzy sets defined on $\mathbb{R}^2$. For that, we have to prove that the intersections of the results on $\mathbb{R}^3$ and $z = 0$ are the same as those on $\mathbb{R}^2$.

Theorem 5. *For $* = +, -, \cdot, /$, let $\mu_{A(*)B}(x, y, z)$ and $\mu_{A(*)B}(x, y)$ are the results in Theorem 3 and Theorem 2, respectively. Then, we have $\mu_{A(*)B}(x, y, 0) = \mu_{A(*)B}(x, y)$.*

Proof. Consider $A = (h_1, a_1, x_1, b_1, y_1, 0, 0)^3$ and $B = (h_2, a_2, x_2, b_2, y_2, 0, 0)^3$, where $0 < h_1 < h_2 < 1$.

(1) For $0 < \alpha < h_1$, the α-set of $\mu_{A(+)B}(x, y, 0)$ is

$$\left\{(x, y) \in \mathbb{R}^2 \middle| \left(\frac{x - x_1 - x_2}{a_1(h_1 - \alpha) + a_2(h_2 - \alpha)}\right)^2 + \left(\frac{y - y_1 - y_2}{b_1(h_1 - \alpha) + b_2(h_2 - \alpha)}\right)^2 = 1\right\}. \tag{37}$$

Similarly, we can prove that the 0–set and h_1–set of $\mu_{A(+)B}(x, y, 0)$ are the same as those of $\mu_{A(+)B}(x, y)$.

(2) For $0 < \alpha < h_1$, the α-set of $\mu_{A(-)B}(x, y, 0)$ is

$$\left\{(x, y) \in \mathbb{R}^2 \middle| \left(\frac{x - x_1 + x_2}{a_1(h_1 - \alpha) + a_2(h_2 - \alpha)}\right)^2 + \left(\frac{y - y_1 + y_2}{b_1(h_1 - \alpha) + b_2(h_2 - \alpha)}\right)^2 = 1\right\}. \tag{38}$$

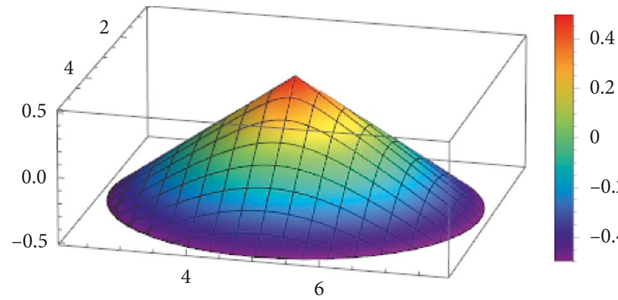

Figure 1: *A2D*.

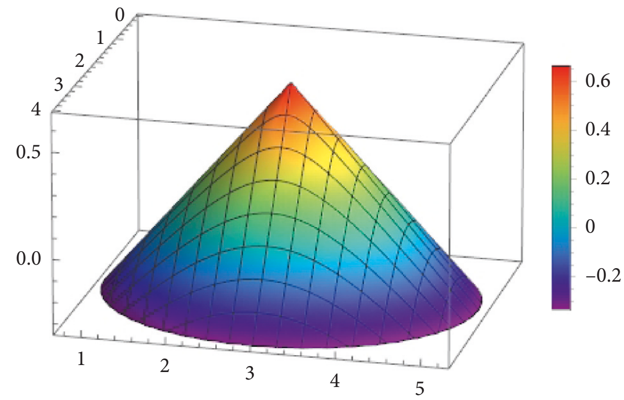

Figure 2: *B2D*.

Similarly, we can prove that the 0−set and h_1−set of $\mu_{A(-)B}(x,y,0)$ are the same as those of $\mu_{A(-)B}(x,y)$.

(3) For $0<\alpha<h_1$, the α-set of $\mu_{A(\cdot)B}(x,y,0)$ is

$$S_1=\left\{(x_\alpha(s),y_\alpha(s,t))\in\mathbb{R}^2|0\le s\le 2\pi,-\frac{\pi}{2}\le t\le\frac{\pi}{2}\right\}, \tag{39}$$

where

$$\begin{aligned}x_\alpha(s)&=x_1x_2+(x_1a_2(h_2-\alpha)+x_2a_1(h_1-\alpha))\cos s+a_1a_2(h_1-\alpha)(h_2-\alpha)\cos^2 s,\\ y_\alpha(s,t)&=y_1y_2+(y_1b_2(h_2-\alpha)+y_2b_1(h_1-\alpha))\sin s\cos t+b_1b_2(h_1-\alpha)(h_2-\alpha)\sin^2 s\cos^2 t.\end{aligned} \tag{40}$$

In Theorem 2, the α-set of $\mu_{A(\cdot)B}(x,y)$ is

$$S_2=\left\{(x_\alpha(t),y_\alpha(t))\in\mathbb{R}^2|0\le t\le 2\pi\right\}, \tag{41}$$

where

$$\begin{aligned}x_\alpha(t)&=x_1x_2+(x_1a_2(h_2-\alpha)+x_2a_1(h_1-\alpha))\cos t+a_1a_2(h_1-\alpha)(h_2-\alpha)\cos^2 t,\\ y_\alpha(t)&=y_1y_2+(y_1b_2(h_2-\alpha)+y_2b_1(h_1-\alpha))\sin t+b_1b_2(h_1-\alpha)(h_2-\alpha)\sin^2 t.\end{aligned} \tag{42}$$

In three-dimensional case, the α−set becomes a convex set in $\mathbb{R}^2$. The boundary of S_1 is S_2. Clearly, $x_0(s)=x_0(t)$, $x_{h_1}(s)=y_{h_1}(t)$, and we can prove that

$$\begin{aligned}y_0(s,t)&=y_0(t),\\ y_{h_1}(s,t)&=y_{h_1}(t).\end{aligned} \tag{43}$$

(4) For $0<\alpha<h_1$, the α-set of $\mu_{A(/)B}(x,y,0)$ is

$$S_3=\left\{(x_\alpha(s),y_\alpha(s,t))\in\mathbb{R}^2|0\le s\le 2\pi,-\frac{\pi}{2}\le t\le\frac{\pi}{2}\right\}, \tag{44}$$

where

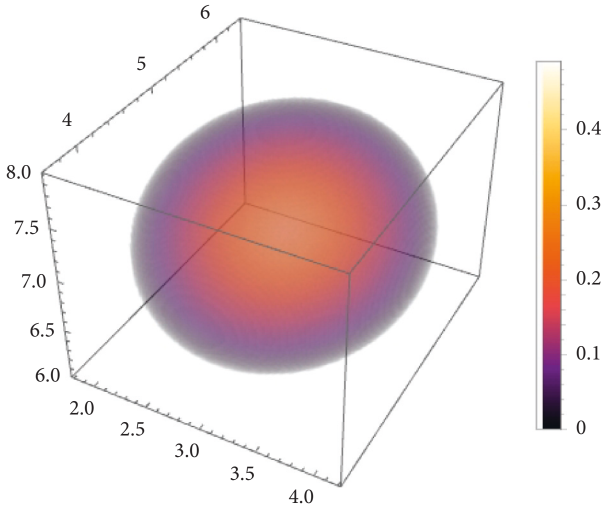

Figure 3: $A3D$.

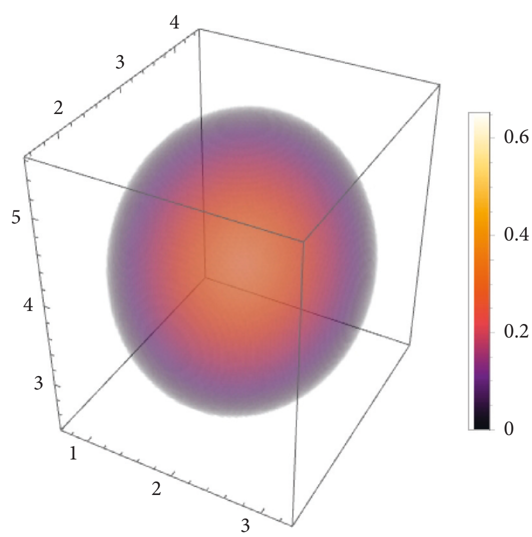

Figure 4: $B3D$.

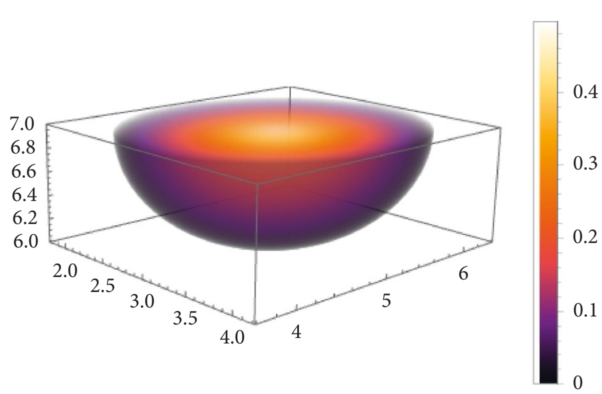

Figure 5: $A\,3D\,half-cut$.

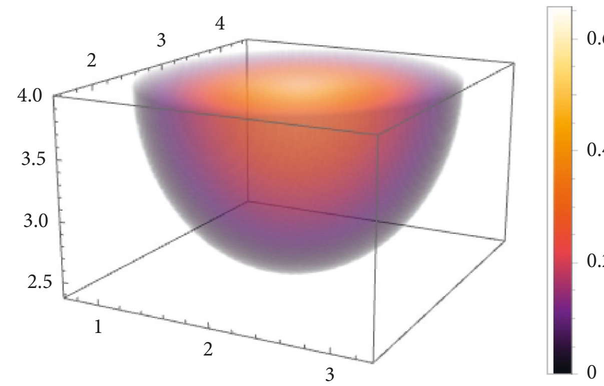

Figure 6: $B\,3D\,half-cut$.

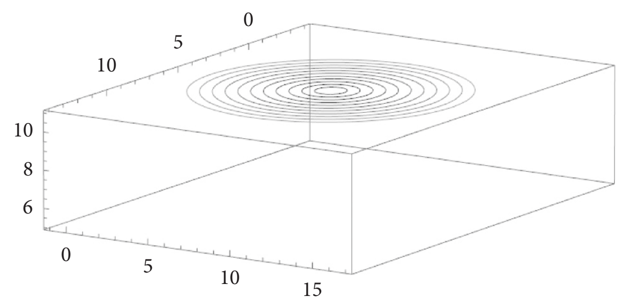

Figure 7: $A+B\,2D\,a-cut$.

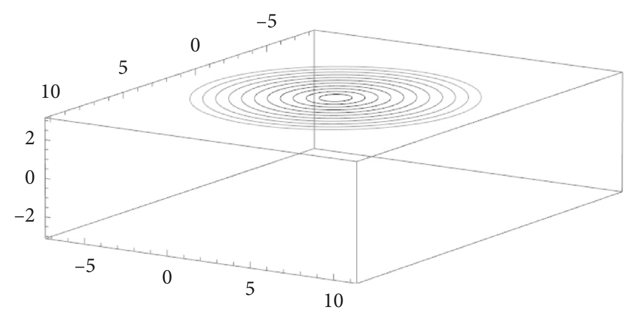

Figure 8: $A-B\,2D\,a-cut$.

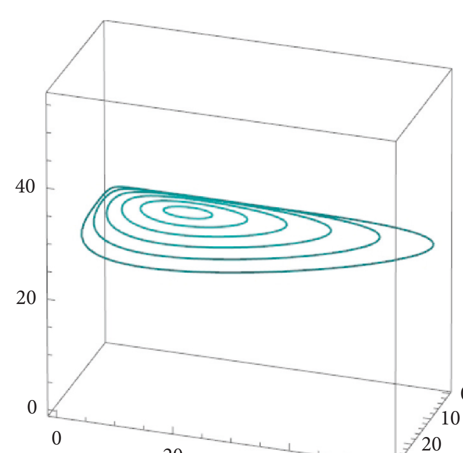

Figure 9: $AB\,2D\,a-cut$.

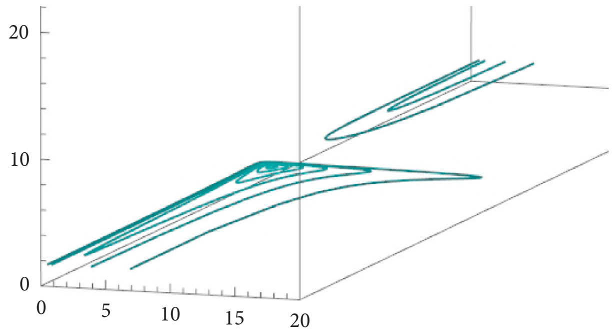

Figure 10: $AdB\, 2D\, a - cut$.

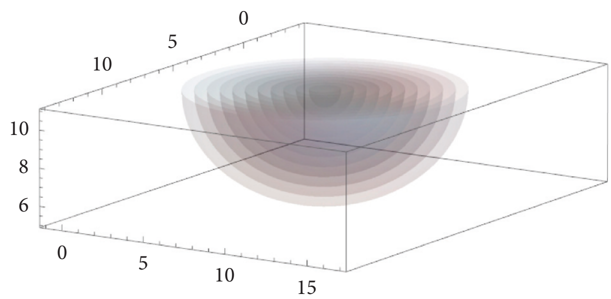

Figure 11: $A + B\, 3D\, half - cut$.

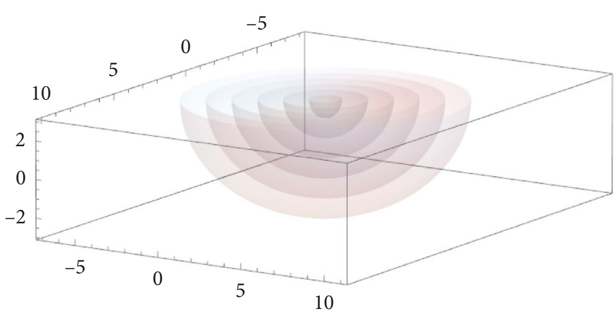

Figure 12: $A - B\, 3D\, half - cut$.

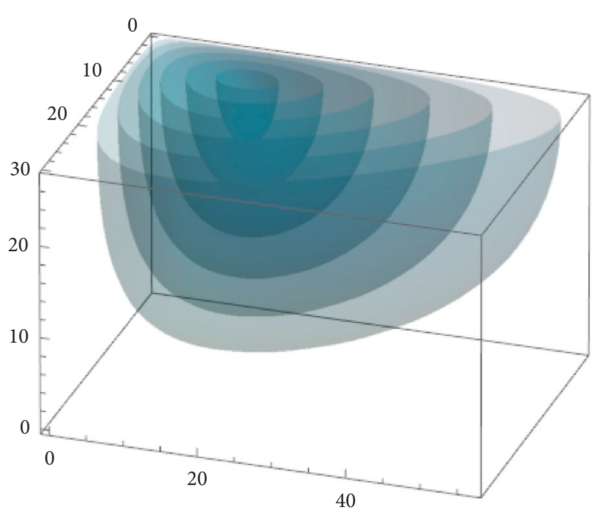

Figure 13: $AB\, 3D\, half - cut$.

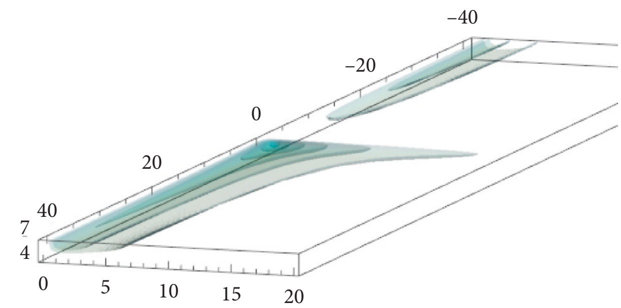

Figure 14: $AdB\, 3D\, half - cut$.

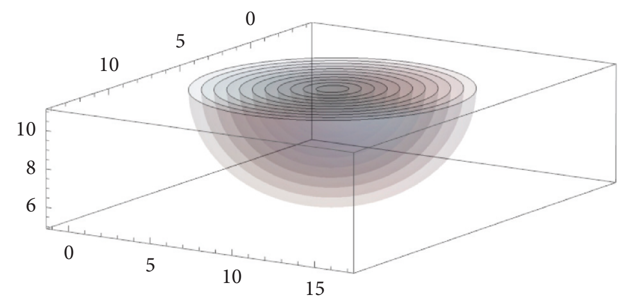

Figure 15: $A + B\, 2D = 3D$.

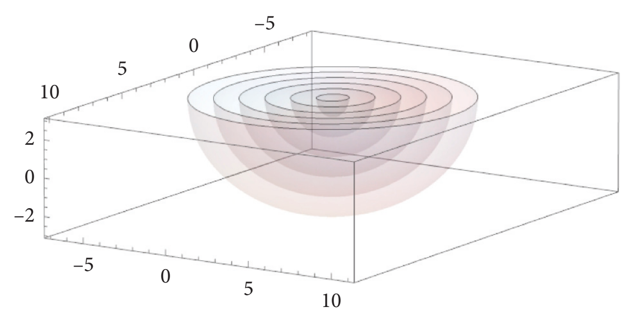

Figure 16: $A - B\, 2D = 3D$.

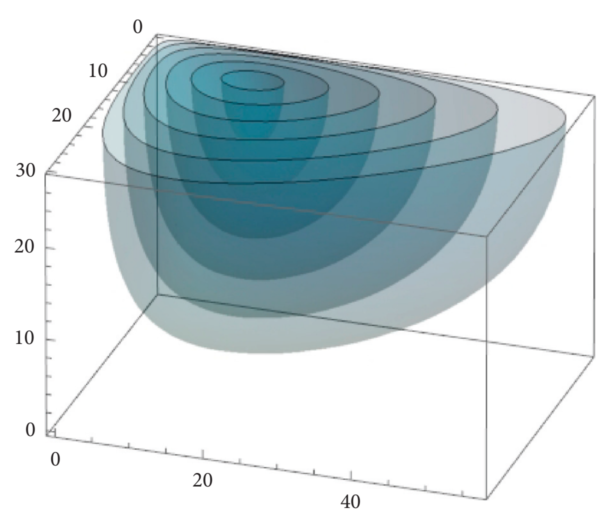

Figure 17: $AB\, 2D = 3D$.

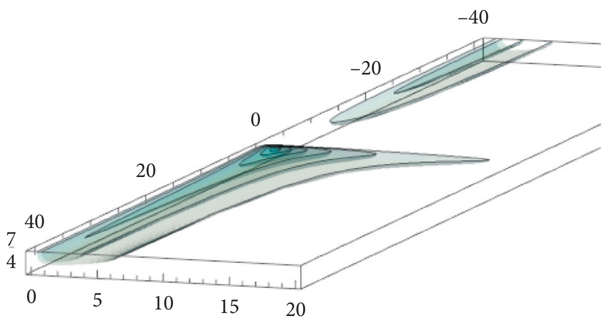

Figure 18: $AdB\,2D = 3D$.

$$x_\alpha(s) = \frac{x_1 + a_1(h_1 - \alpha)\cos s}{x_2 - a_2(h_2 - \alpha)\cos s},$$
$$y_\alpha(s.t) = \frac{y_1 + b_1(h_1 - \alpha)\sin s\cos t}{y_2 - b_2(h_2 - \alpha)\sin s\cos t}. \tag{45}$$

In Theorem 2, the α-set of $\mu_{A(/)B}(x, y)$ is

$$S_4 = \left\{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\right\}, \tag{46}$$

where

$$x_\alpha(t) = \frac{x_1 + a_1(h_1 - \alpha)\cos t}{x_2 - a_2(h_2 - \alpha)\cos t},$$
$$y_\alpha(t) = \frac{y_1 + b_1(h_1 - \alpha)\sin t}{y_2 - b_2(h_2 - \alpha)\sin t}. \tag{47}$$

In a three-dimensional case, the α-set becomes a convex set in $\mathbb{R}^2$. The boundary of S_3 is S_4. Clearly, $x_0(s) = x_0(t), x_{h_1}(s) = y_{h_1}(t)$, and we can prove that

$$y_0(s,t) = y_0(t),$$
$$y_{h_1}(s,t) = y_{h_1}(t). \tag{48}$$

Thus, $\mu_{A(*)B}(x, y, 0) = \mu_{A(*)B}(x, y)$. □

5. Conclusion

Extensive use of fuzzy theory in many different fields has facilitated active research on operators between fuzzy sets [18–20]. Operators of various concepts have been defined and studied, but Zadeh's operator concept is commonly studied and utilized. A correct understanding of the generalized triangular fuzzy set will be helpful in interpreting Zadeh's operators [21–24].

The conclusion of a general triangular fuzzy set in two-dimensional space is a function defined on a plane and can be expressed in a graph in three-dimensional space (Figures 1–6). Therefore, the graph in the form of an elliptical cone in two dimensions, the result of Section 2, is not difficult to understand (Figures 7–10).

However, if it is expanded to three dimensions, the domain becomes a three-dimensional set, making visual expression difficult. To help easier understanding, graphs were expressed in color density (Figures 11–14).

Each point on the cross section has a different color density, noting that each has a specific function value. In Section 4, we proved that the three-dimensional result is an extended concept of the two-dimensional result, which is indicated in graphs in Figures 15–18. When the three-dimensional result is cut with a vertical plane passing through the vertex, the cross section of the graph is two-dimensional. Function values in two-dimensional were represented in graphs with the z-axis value. Visualization of the results will lead to more application and utilization.

In Section 1, we discussed the theoretical flow and application of fuzzy sets. The study that extended Zadeh's operator theory to two or three dimensions is important in application. We have studied the extended algebraic operations between two fuzzy numbers and calculated Zadeh's max-min composition operator for two generalized triangular fuzzy sets in $\mathbb{R}^2$ and generalized the triangular fuzzy numbers from $\mathbb{R}^2$ to $\mathbb{R}^3$. The purpose of the paper is also presented.

In Section 2, we defined the generalized two-dimensional triangular fuzzy numbers on $\mathbb{R}^2$ as a generalization of generalized triangular fuzzy sets on $\mathbb{R}$ and the parametric operations between two generalized two-dimensional triangular fuzzy sets. In Theorem 2, we calculated the parametric operations between two generalized two-dimensional triangular fuzzy sets and gave an example.

In Section 3, we defined the generalized three-dimensional triangular fuzzy sets on $\mathbb{R}^3$ as a generalization of generalized triangular fuzzy sets on $\mathbb{R}^2$. Then, we defined the parametric operations between two generalized three-dimensional triangular fuzzy sets. We calculated the parametric operations between two generalized three-dimensional triangular fuzzy sets in Theorem 3.

In Section 4, we showed that the parametric operations for two generalized triangular fuzzy sets defined on $\mathbb{R}^3$ are a generalization of parametric operations for two generalized triangular fuzzy sets defined on $\mathbb{R}^2$. What has been proven is presented as an example. And the examples are expressed in various types of graphs for easier understanding.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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