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Research Article

The Fractional Series Solutions for the Conformable Time-Fractional Swift-Hohenberg Equation through the Conformable Shehu Daftardar-Jafari Approach with Comparative Analysis

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The major objective of this study is to derive fractional series solutions of the time-fractional Swift-Hohenberg equations (TFSHEs) in the sense of conformable derivative using the conformable Shehu transform (CST) and the Daftardar-Jafari approach (DJA). We call it the conformable Shehu Daftardar-Jafari approach (CSDJA). One of the universal equations used in the description of pattern formation in spatially extended dissipative systems is the Swift-Hohenberg equation. To assess the effectiveness and consistency of the suggested approach, the numerical results are compared with those obtained by the Elzaki decomposition method (EDM) in the sense of relative and absolute error functions, proving that the CSDJA is an effective substitute for techniques that use He's or Adomian polynomials to solve TFSHEs. The transition from the imprecise solution to the precise solution at various values of fractional-order derivatives is shown using the recurrence error function. Furthermore, the exact and approximative solutions are compared using 2D and 3D graphics and also numerically in the form of relative and absolute error functions. The results show that the procedure is quick, precise, and easy to implement, and it yields outstanding results. The recommended approach's strength, which gives it an advantage over the Adomian decomposition and homotopy perturbation methods, is its algorithm for dealing with nonlinear problems without the use of Adomian polynomials or He's polynomials. The advantage of this method is that it does not make any assumptions about physical parameters. As a result, it can be used to solve both weakly and strongly nonlinear problems and circumvent some of the drawbacks of perturbation techniques.

1. Introduction

A more than the 300-year-old extension of traditional calculus is fractional calculus (FC). Leibniz and L'Hospital started the FC as a result of a correspondence that lasted for several months in 1695. Leibniz addressed a letter to L'Hospital in that year, posing the following query [1].

Is it possible to generalize the definition of derivatives with integer orders to derivatives with noninteger orders? The preceding query piqued L'Hopital's interest, so it asked Leibniz another straightforward query in response: "What if the order is 1/2?" In a letter dated September 30, 1695, Leibniz said, "It will lead to a paradox, from which one-day valuable conclusions will be deduced."

That date is regarded as the actual birthday of the FC. The fractional-order differential equations (FODEs), fractional dynamics, and other practical disciplines have all benefited greatly from the rapid development of the theory of FC since the 19th century. These days, FC is used in a wide variety of applications. It is safe to argue that nearly no field of contemporary engineering or research is unaffected by the methods and instruments of FC. For instance, mechanical engineering, electrical engineering, control theory, viscoelasticity, optics, rheology, chemistry, physics,

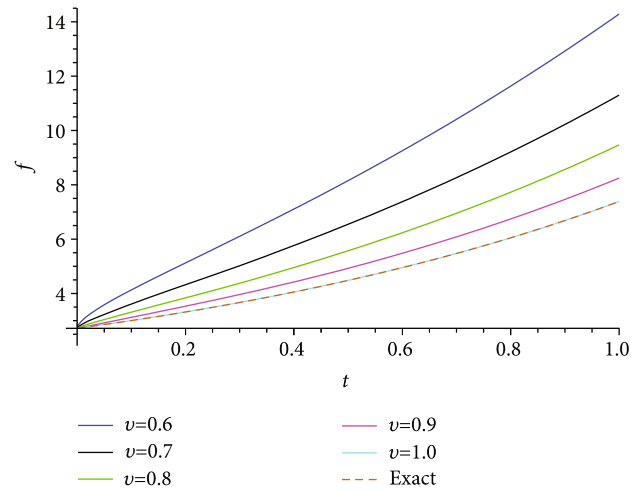

Figure 1: The 2D plot of the 5th-step approximate solution and exact solution of Example 3 for various quantities of υ in the interval $t \in [0, 1.0]$ when $x = 1.0$ is shown.

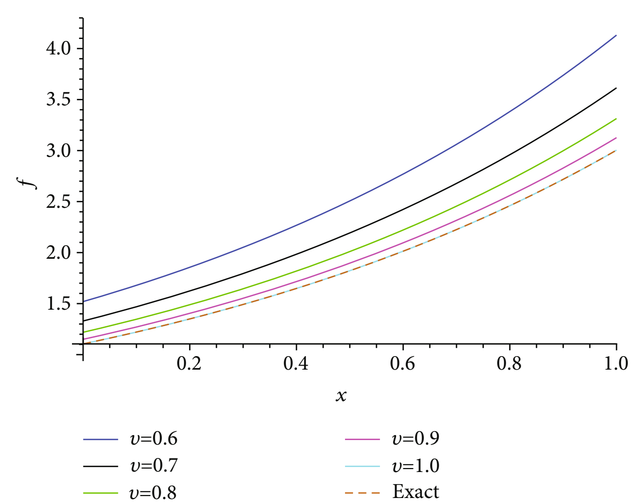

Figure 2: The 5th iteration's approximate and exact solutions for Example 3 are shown in a 2D graphic for various values of υ in the interval $x \in [0, 1.0]$ with $t = 0.1$.

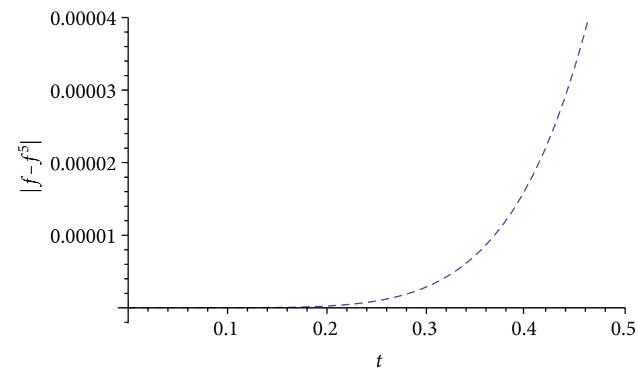

Figure 3: The 2D plot of the absolute error graph of $f(x,t)$ for the 5th iteration approximate solution in the interval $t \in [0,0.5]$ when $\upsilon = 1.0$ and $x = 1.0$ in Example 3.

statistics, robotics, and bioengineering are just a few fields with numerous and fruitful applications [2–5]. In fact, it may be argued that fractional-order systems in general characterize real-world processes. The success of FC applications is mostly due to the fact that these new fractional-order models are frequently more accurate than integer-order ones since they have more degrees of freedom than the corresponding classical ones. The fact that fractional derivatives are not a local number is one of the subject's interesting aspects. The nonlocal and distributed effects frequently observed in technical and natural phenomena can be modeled by all fractional operators since they consider the whole history of the activity under consideration. In order to describe the memory and hereditary characteristics of various materials and processes, FC is a great collection of tools.

Examples of fractional derivative definitions used in a range of natural phenomena and applications include Hadamard, Grunwald, Riemann-Liouville, Letnikov, Riesz, and Caputo derivatives [6–10]. In contrast to past formulations, the conformable derivative (CD), which Khalil et al. proposed [11], is a new formulation of the fractional operators that is far simpler to compute. The main benefits of CD are as follows [12–17]: in contrast to the previous fractional formulations, it satisfies all the concepts and guidelines of an ordinary derivative, including the chain, product, and quotient criteria. It is easily and quickly adaptable to solve exact and numerical FODEs and systems. It simplifies the well-known transforms, like the Sumudu and Laplace transforms, which are used as tools to explain some FODEs. Class conformable fractional operators, modified conformable fractional operators, fuzzy generalized conformable fractional operators, deformable fractional operators, M-conformable fractional operators, and Katugampola fractional operators are just a few examples of the new definitions that can be created and expanded upon. In a variety of applications, it produces new comparisons between CD and the earlier fractional definitions. Researchers have shown a great deal of interest in CD due to the numerous applications and phenomena that it may represent and the necessity to address them.

Various integral transforms are used with other analytical, numerical, or homotopy-based methods to handle FODEs. The Laplace transform (LT) [18], the Elzaki transform (ET) [19], the traveling wave transform (TWT) [20], the Yang transform (YT) [21], the Aboodh transform (AT) [22], the fractional complex transform (FCT) [23], and the natural transform (NT) [24] are all transformations that can be used to solve FODEs. The Shehu transform (ST), which has been used by numerous academics for the solutions of FODEs, has recently piqued the curiosity of many mathematicians [25–27].

The significant advantages of ST include the following:

(i) The ET, SIT, and NT are all more challenging to understand than the ST

(ii) When variable $n = 1$ is used, the ST becomes LT, and when variable $m = 1$ is used, it becomes YT

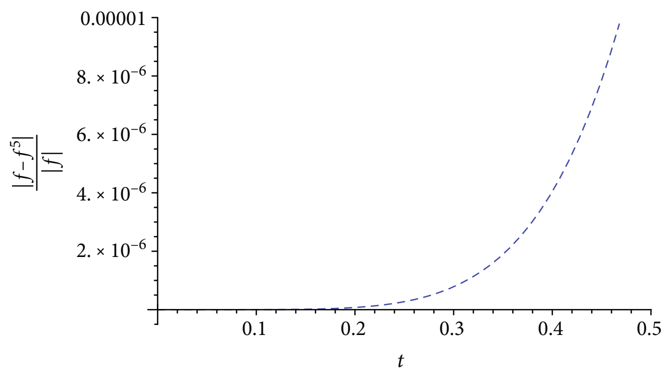

Figure 4: The 2D plot of the relative error graph of $f(x,t)$ for the 5th iteration approximate solution in the interval $t \in [0, 0.5]$ when $\upsilon = 1.0$ and $x = 1.0$ in Example 3.

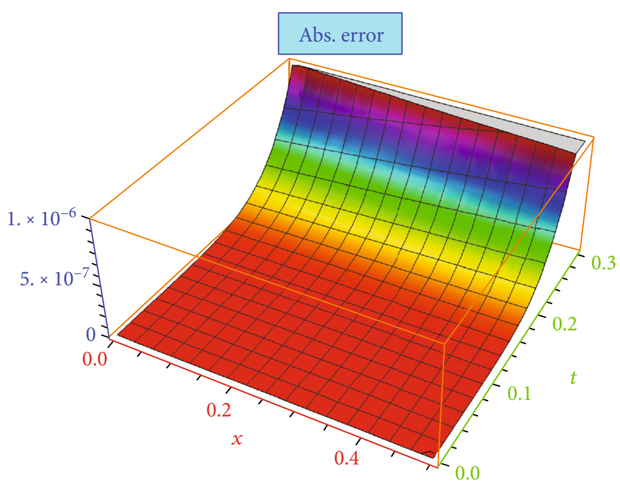

Figure 5: The 3D plot of the absolute error graph of $f = (x,t)$ for the 5th iteration approximate solution when $\upsilon = 1.0$ with $x \in [0, 0.5]$ and $t \in [0, 0.3]$ in Example 3.

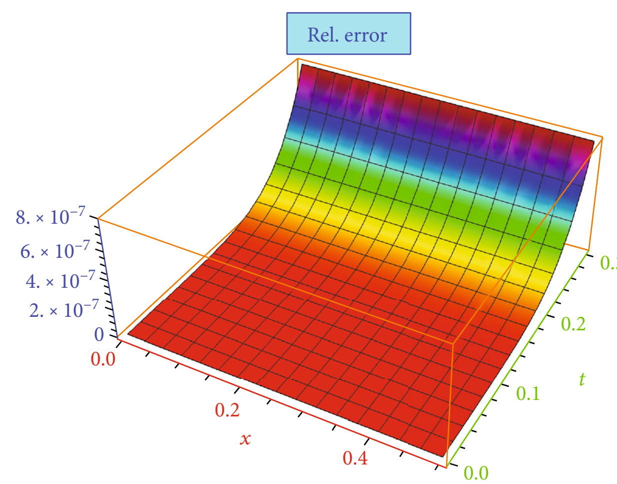

Figure 6: The 3D plot of the relative error graph of $f = (x,t)$ for the 5th iteration approximate solution when $\upsilon = 1.0$ and $x \in [0, 0.5]$ and $t \in [0, 0.3]$ in Example 3.

(iii) It is an extension of the LT and SIT

(iv) It can be used efficiently to find exact and numerical solutions to FODEs

We understand that the majority of engineering issues are nonlinear and that addressing them analytically is challenging. Obtaining closed form or approximate solutions to nonlinear FODEs remains a fundamental topic in physics and mathematics, requiring innovative ways to locate exact or approximate solutions. As a result of the foregoing, researchers have devised a variety of numerical strategies for solving nonlinear FODEs. A few examples include the Elzaki residual power series method [28], the Haar Wavelet method [29], the operational Matrix Technique [30], the reduced differential transform method [31], the spectral Tau approach [32], the reproducing kernel technique [33], and the fractional power series technique [34].

Finding the solutions to TFSHEs is an interesting and important field for researchers [35–39]. Each of these techniques has distinct restrictions and flaws. These techniques have long run times and enormous computational demands. In this study, we used CNDJM to acquire approximate and closed-form results of TFSHEs in the sense of CD. The recurrence, relative, and absolute error analyses among the exact solution and approximate solution of five linear-nonlinear problems have been used to demonstrate the accuracy and efficacy of the proposed method. The results obtained using the recommended method show excellent agreement with EDM [36], proving that the CSDJA is an acceptable substitute tool for the He's or Adomian polynomial-based methods used to solve FODEs. The effectiveness of the CSDJA has been demonstrated by results in both graphs and numerically. The approximate solutions achieved using CSDJA are in perfect agreement with the corresponding precise solutions, as can be seen from the graphs and tables. The numerical evidence for the convergence of the approximative solution to the exact solution is presented in the tables. The absolute, relative, and recurrence error analyses have demonstrated a more accurate and faster convergence.

Table 1: Absolute and relative error in the 4th-step approximate solution of $f(x,t)$ when $v = 1.0$, $Y = 3$, and $\Theta = 2$ for Example 3.

(x,t)	$f^4(x,t)$	$f(x,t)$	Abs.error $= \lvert f - f^4 \rvert$	Re l.error $= \lvert f - f^4 \rvert / \lvert f \rvert$
(0.02, 0.02)	1.040810774165092	1.0408107741923882	$2.729616532803902 \times 10^{-11}$	$2.622586737653570 \times 10^{-11}$
(0.12, 0.12)	1.271248911766325	1.2712491503214047	$2.385550796901725 \times 10^{-7}$	$1.8765407208324160 \times 10^{-7}$
(0.22, 0.22)	1.552701664430785	1.5527072185113360	0.0000055408055080697	0.00000357703016034920
(0.32, 0.32)	1.896440220522181	1.8964808793049515	0.00004065878276993118	0.00002143906812539674
(0.42, 0.42)	2.316188883508124	2.3163669767810915	0.00017809327296758326	0.00007688474009203334
(0.52, 0.52)	2.828634231524732	2.8292170143515600	0.00058278282682744380	0.00020598731870733296
(0.62, 0.62)	3.454033558387693	3.4556134647626755	0.00157990637498217620	0.00045719997074113760
(0.72, 0.72)	4.216940784284310	4.2206958169965520	0.00375503271224175700	0.00088967148428948830
(0.82, 0.82)	5.303822828551244	5.1551695122346800	0.00809833753451094700	0.00157091585743035090
(0.92, 0.92)	6.280329876306269	6.2965382610266570	0.01620838472038777000	0.00257417394264272700

Table 2: Absolute and relative error in the 5th iteration approximate solution of $f(x,t)$ when $v = 1.0$, $Y = 3$, and $\Theta = 2$ for Example 3.

(x,t)	$f^5(x,t)$	$f(x,t)$	Abs.error $= \lvert f - f^5 \rvert$	Re l.error $= \lvert f - f^5 \rvert / \lvert f \rvert$
(0.02, 0.02)	1.0408107741922974	1.0408107741923882	$9.08162434143378 \times 10^{-14}$	$8.725528757598248 \times 10^{-14}$
(0.12, 0.12)	1.2712491455640722	1.2712491503214047	$4.757332527915992 \times 10^{-9}$	$3.742250310816896 \times 10^{-9}$
(0.22, 0.22)	1.5527070159482126	1.5527072185113360	$2.025631233859570 \times 10^{-7}$	$1.304580290289145 \times 10^{-7}$
(0.32, 0.32)	1.8592962781683804	1.8964808793049515	0.00000215149950033577	0.00000113446938685945
(0.42, 0.42)	2.3163546393670910	2.3163669767810915	0.00001233741400064047	0.00000532619145597775
(0.52, 0.52)	2.8291671596033580	2.8292170143515600	0.00004985474820173863	0.00001762139416978059
(0.62, 0.62)	3.4554527458972446	3.4556134647626755	0.00016071886543089775	0.00004650950318077187
(0.72, 0.72)	4.2202534168201780	4.2206958169965520	0.00044240017637431350	0.00010481688222894147
(0.82, 0.82)	5.1540858809814400	5.1551695122346800	0.00108363125324029140	0.00021020283633128398
(0.92, 0.92)	6.2941117674091895	6.2965382610266570	0.00242649361746760660	0.00038536947079107625

Table 3: The recurrence error for $f(x,t)$ at various values of v for Example 3.

(x,t)	$v = 0.7$	$v = 0.8$	$v = 0.9$	$v = 1.0$
(0.03, 0.03)	$2.389361050347296 \times 10^{-7}$	$2.122670930232615 \times 10^{-8}$	$2.04023732018427 \times 10^{-9}$	$2.086670431255871 \times 10^{-10}$
(0.13, 0.13)	0.00004472893622536946	0.00000827180924889488	0.00000165504411671953	$3.523658391081241 \times 10^{-7}$
(0.23, 0.23)	0.00036413607844953396	0.00008957109715237295	0.00002383796620121089	0.000006750651136424606
(0.33, 0.33)	0.00142378535825781070	0.00041950945889177800	0.00013373223882468582	0.000045363403631587120
(0.43, 0.43)	0.00397388024822836750	0.00133656248951324060	0.00048636339807192120	0.000188324862522390530
(0.53, 0.53)	0.00912998728886662200	0.00340916776057963170	0.00137728461232044700	0.000592071528646969300
(0.63, 0.63)	0.01847676778128128800	0.00752205562040674600	0.00331317368190818600	0.001552841126987777000
(0.73, 0.73)	0.03419743526328207000	0.01498632791930036700	0.00710549027016691900	0.003584825550795332800
(0.83, 0.83)	0.05923335025813710600	0.02767869489816617700	0.01399336113675867500	0.007527896437711811000
(0.93, 0.93)	0.09747937634589607000	0.04821636669617927000	0.02580318662074363600	0.014693571306273750000

Many applications of the TFSHEs are found in engineering and science, including physics, biology, laser studies, fluids, and hydrodynamics. In fluid layers restricted between horizontal well-conducting barriers, the Swift-Hohenberg (SH) equations play a significant role in pattern creation theory. There are numerous uses for this equation in the modeling of pattern generation and its many difficulties, such as pattern selection, noise effects on bifurcations, defect

Table 4: The absolute and relative error in various approaches for Example 3 at $\upsilon = 1.0$.

(x, t)	Abs.error (CSDJA)	Abs.error(EDM) [36]	Re l.error (CSDJA)	Re l.error(EDM) [36]
(0.06, 0.06)	$6.940115149234316 \times 10^{-11}$	$6.940115149234316 \times 10^{-11}$	$6.15532995902626 \times 10^{-11}$	$6.15532995902626 \times 10^{-11}$
(0.16, 0.16)	$2.798253562197317 \times 10^{-8}$	$2.798253562197317 \times 10^{-8}$	$2.031949129677607 \times 10^{-8}$	$2.031949129677607 \times 10^{-8}$
(0.26, 0.26)	$5.778071316964173 \times 10^{-7}$	$5.778071316964173 \times 10^{-7}$	$3.435182125572402 \times 10^{-7}$	$3.435182125572402 \times 10^{-7}$
(0.36, 0.36)	0.0000045667026689066	0.0000045667026689066	0.0000022228528263887	0.0000022228528263887
(0.46, 0.46)	0.0000222971843411023	0.0000222971843411023	0.0000088858525225007	0.0000088858525225007
(0.56, 0.56)	0.0000814362943066804	0.0000814362943066804	0.0000265710173812450	0.0000265710173812450
(0.66, 0.66)	0.0002449136020046261	0.0002449136020046261	0.0000654250690270498	0.0000654250690270498
(0.76, 0.76)	0.0006408613764907756	0.0006408613764907756	0.0001401640009270912	0.0001401640009270912
(0.86, 0.86)	0.0015103634891575624	0.0015103634891575624	0.0002704549719496070	0.0002704549719496070
(0.96, 0.96)	0.0032809722748394776	0.0032809722748394776	0.0004810133780481200	0.0004810133780481200

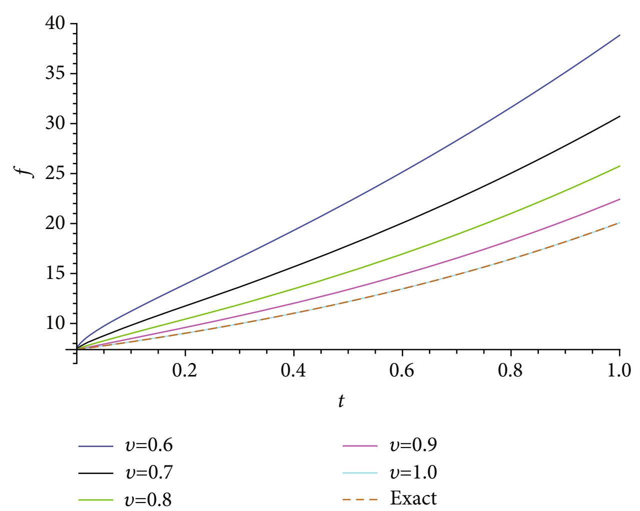

Figure 7: The 2D plot of the 5th step exact solution and approximate solution of Example 4 for various values of υ in the interval $t \in [0, 1.0]$, when $x = 2.0$ is shown.

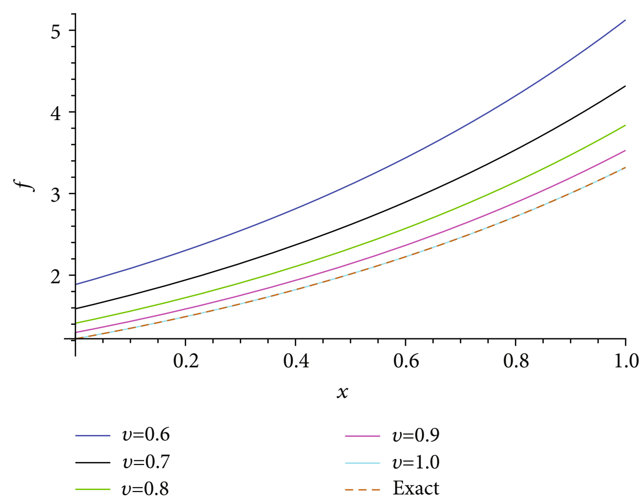

Figure 8: The 2D plot of the 5th iteration approximate and exact solutions of Example 4 for different values of υ in the interval $x \in [0, 1.0]$, with $t = 0.2$.

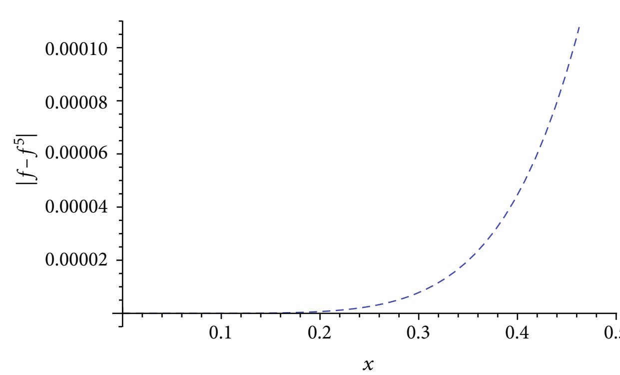

Figure 9: The 2D plot of the absolute error graph of $f(x, t)$ for the 5th iteration approximate solution in the interval $t \in [0,0.5]$ when $\upsilon = 1.0$ and $x = 2.0$ in Example 4.

dynamics, and spatiotemporal chaos. The general form of the S-H equation is given by [40]

$$\frac{\partial}{\partial t} f(x, t) = w f(x, t) - \left(1 + \nabla^2\right)^2 f(x, t) - f^3(x, t), \qquad (1)$$

where $x \in R, t > 0, w$ is bifurcation parameter, and $f(x, t)$ is a scaler function of x and t defined on the line or the plane.

The following is how this study is structured: first, in Section 2, we use crucial terminology and findings from FC theory. Next, in Section 3, we described how to solve TFSHEs using the CSDJA. In Section 4, to assess the efficiency of the CSDJA, we solved five numerical problems with a conclusion remark. Finally, we summarize our results in the conclusion.

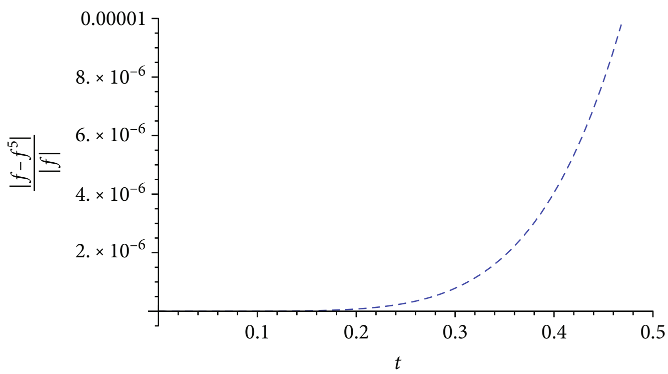

FIGURE 10: The 2D plot of the relative error graph of $f(x,t)$ for the 5th iteration approximate solutions in the interval $t \in [0, 0.5]$ when $\upsilon = 1.0$ and $x = 2.0$ in Example 4.

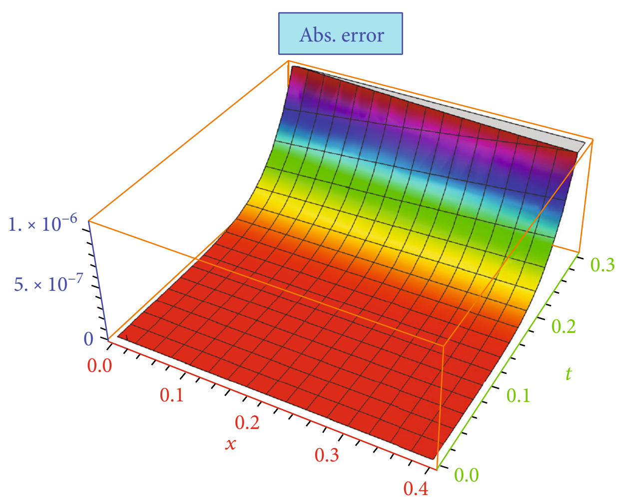

FIGURE 11: The 3D plot of the absolute error graph of $f = (x, t)$ for the 5th iteration approximate solutions when $\upsilon = 1.0$ with $x \in [0, 0.4]$ and $t \in [0, 0.3]$ in Example 4.

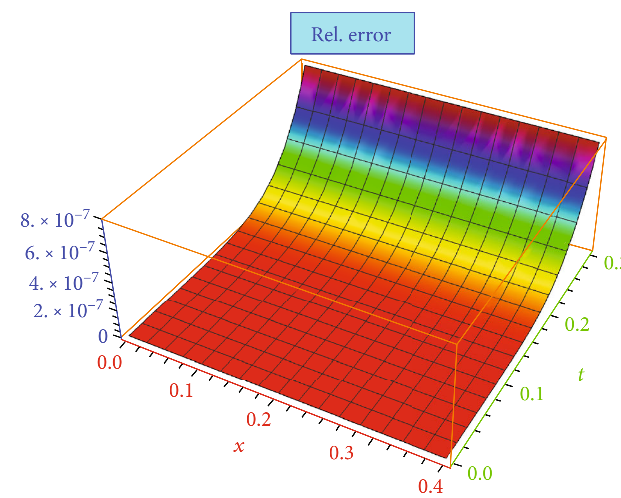

FIGURE 12: The 3D plot of the relative error graph of $f = (x, t)$ for the 5th iteration approximate solutions when $\upsilon = 1.0$ and $x \in [0, 0.4]$ and $t \in [0, 0.3]$ in Example 4.

2. Preliminaries

The definitions, theorems, and mathematical foundations of CD and CST that will be used in this work are reviewed in this part.

Definition 1. Assumed a function $f : [0,\infty) \longrightarrow R$ as well the CD of f order υ is defined as follows [41]:

$$T_t^{\upsilon} f(t) = \lim_{\epsilon \longrightarrow 0} \frac{f\left(t + \epsilon t^{1-\upsilon}\right) - f(t)}{\epsilon}, \tag{2}$$

for $t > 0$ and $\upsilon \in (0, 1]$. If f is υ – differentiable in some (0, P),$P > 0$, and $\lim_{t \longrightarrow 0^+} (T_{\upsilon} f)(t)$, happen afterward, it is drawn as

$$(T_{\upsilon} f)(0) = \lim_{t \longrightarrow 0^+} (T_{\upsilon} f)(t). \tag{3}$$

Theorem 2. *Let f_1 and f_2 be υ – differentiable at a point $t > 0$. Then, for $\upsilon \in (0, 1]$, we take the following [42]:*

(i) $T_t^{\upsilon}(e_1 f_1 + e_2 f_2) = e_1 T_t^{\upsilon}(f_1) + e_2 T_t^{\upsilon}(f_2), \forall e_1, e_2 \in R$

(ii) $T_t^{\upsilon}(t^e) = e t^{e-\upsilon}, \forall e \in R \; T_t^{\upsilon}(e) = 0,$ *where* $e \in R$

(iii) $T_t^{\upsilon}(f_1 f_2) = f_1 T_t^{\upsilon}(f_2) + f_2 T_t^{\upsilon}(f_1)$

(iv) $T_t^{\upsilon}(f_1 / f_2) = (f_2 T_t^{\upsilon}(f_1) - f_1 T_t^{\upsilon}(f_2)) / (f_2)^2$

Definition 3. Let $0 < \upsilon \le 1$ and $f : [0,\infty) \longrightarrow R$ be a real value function. Then, the CST of order υ is defined by [43]:

$$\mathfrak{I}_{\upsilon}[f(t)] = R_{\upsilon}(m, n) = \int_0^{\infty} e^{-(m t^{\upsilon} / n \upsilon)} f(t) t^{\upsilon - 1} dt, \tag{4}$$

assuming the integral is present.

Theorem 4. *Let $[0,\infty) \longrightarrow R$ be differentiable function and $0 < \upsilon \le 1$. Then, we have the following [44]:*

$$\mathfrak{I}_{\upsilon}[T_t^{\upsilon} f(x,t)] = \frac{m}{n} \mathfrak{I}_{\upsilon}[f(x,t)] - f(x,0). \tag{5}$$

Table 5: Relative and absolute error in the 4th-step approximate solution of $f(x,t)$ when $\upsilon = 1.0$, $Y = 3$, and $\Theta = 2$ for Example 4.

(x,t)	$f^4(x,t)$	$f(x,t)$	Re l.error $= \lvert f-f^4\rvert/\lvert f\rvert$	Abs.error $= \lvert f-f^4\rvert$
(0.03, 0.03)	1.0618365463356450	1.0618365465453596	$1.975018572683782 \times 10^{-10}$	$2.097146900581492 \times 10^{-10}$
(0.13, 0.13)	1.2969297265212163	1.2969300866657718	$2.776900306454735 \times 10^{-7}$	$3.601445555112548 \times 10^{-7}$
(0.23, 0.23)	1.5683052371426760	1.5840739849944818	0.00000443046196873528	0.00000701817954618100
(0.33, 0.33)	1.9347443533545630	1.9347923344020317	0.00002479906841452054	0.05280880783415465000
(0.43, 0.43)	2.3629579963630247	2.3631606937057947	0.00008577382964683130	0.00020269734277000850
(0.53, 0.53)	2.8857223796091587	2.8863709892679585	0.00022471458492741108	0.00064860965879987020
(0.63, 0.63)	3.5236896816699144	3.5254214873653824	0.00049123365863474670	0.00173180569546804720
(0.73, 0.73)	4.3018885509799105	4.3059595283452060	0.00094542861782547080	0.00407097736529582500
(0.83, 0.83)	5.2506040155785140	5.2593108444468980	0.00165550756095314270	0.00870682886838469700
(0.93, 0.93)	6.4064239349551840	6.4237367714291350	0.00269513479303095400	0.01731283647395098800

Table 6: Relative and absolute error in the 5th-step approximate solution of $f(x,t)$ when $\upsilon = 1.0$, $Y = 3$, and $\Theta = 2$ for Example 4.

(x,t)	$f^5(x,t)$	$f(x,t)$	Re l.error $= \lvert f-f^5\rvert/\lvert f\rvert$	Abs.error $= \lvert f-f^5\rvert$
(0.03, 0.03)	1.0618365465443120	1.0618365465453596	$9.865985960312264 \times 10^{-13}$	$1.047606446036297 \times 10^{-12}$
(0.13, 0.13)	1.2969300788870555	1.2969300866657718	$5.997791567719124 \times 10^{-9}$	$7.778716337725200 \times 10^{-9}$
(0.23, 0.23)	1.5840737174660720	1.5840739849944818	$1.688863098591869 \times 10^{-7}$	$2.675284098696551 \times 10^{-7}$
(0.33, 0.33)	1.9347897167581944	1.9347923344020317	0.00000135293271050344	0.00000261764383724383
(0.43, 0.43)	2.3631463212255475	2.3631606937057947	0.00000608188866950595	0.00001437248024727111
(0.53, 0.53)	2.8863144511378060	2.8863709892679585	0.00001958796369660575	0.00005653813015271680
(0.63, 0.63)	3.5252425227969020	3.5254214873653824	0.00005076402045020574	0.00017896456848021103
(0.73, 0.73)	4.3054733765307050	4.3059595283452060	0.00011290208635281496	0.00048615181450095690
(0.83, 0.83)	5.2581319120162260	5.2593108444468980	0.00022416100997668992	0.00117893243067257460
(0.93, 0.93)	6.4211175062614570	6.4237367714291350	0.00040774789828359850	0.00261926516767729820

Theorem 5. *Let $e, a \in R$ and $0 < \upsilon \leq 1$. After that, we have the following [45]:*

(i) $\mathfrak{J}_\upsilon[e] = ne/m, m > 0$

(ii) $\mathfrak{J}_\upsilon[e^{a(t^\upsilon/\upsilon)}] = 1/(n - am), m > an$

(iii) $\mathfrak{J}_\upsilon[\sin(a(t^\upsilon/\upsilon))] = an/(m^2 + a^2n^2), m > 0$

(iv) $\mathfrak{J}_\upsilon[\cos(a(t^\upsilon/\upsilon))] = m/(m^2 + a^2n^2), m > 0$

(v) $\mathfrak{J}_\upsilon[\sinh(a(t^\upsilon/\upsilon))] = an/(m^2 - a^2n^2), m > |an|$

(vi) $\mathfrak{J}_\upsilon[\cosh(a(t^\upsilon/\upsilon))] = m/(m^2 - a^2n^2), m > |an|$

(vii) $\mathfrak{J}_\upsilon[t^e] = \upsilon^{(e/\upsilon)}\Gamma((e/\upsilon) + 1)(n/m)^{(e/\upsilon)+1}, m > 0$

In the succeeding part, we generate the foremost suggestion of the CSDJA to obtain the results for linear and nonlinear TFSHEs.

3. The Methodology of the CSDJA for the TFSHEs

To show a detailed understanding of the CSDJA for the TFSHEs, we consider the following problem in common operator systems with the preliminary condition [35, 40]:

$$\begin{cases} T_t^\upsilon f(x,t) + L(f(x,t)) + N(f(x,t)) = X(x,t), & 0 < \upsilon \leq 1, \\ f_0(x,t) = f(x,0) = M(x), \end{cases} \tag{6}$$

where T_t^υ is the CD, $L(f(x,t))$ is a linear operator, $N(f(x,t))$ is nonlinear operator, $X(x,t)$ is a source operator, and $M(x)$ is a function of x.

Considering CST over both sides of Equation (6), we get the following:

$$\mathfrak{J}_\upsilon[T_t^\upsilon f(x,t) + L(f(x,t)) + N(f(x,t))] = \mathfrak{J}_\upsilon[X(x,t)]. \tag{7}$$

Table 7: The recurrence error for $f(x,t)$ at various values of υ for Example 4.

(x,t)	$\upsilon = 0.7$	$\upsilon = 0.8$	$\upsilon = 0.9$	$\upsilon = 1.0$
(0.04, 0.04)	$6.605570054968454 \times 10^{-7}$	$6.776111811148356 \times 10^{-8}$	$7.520521323907584 \times 10^{-9}$	$8.88158527310838 \times 10^{-10}$
(0.14, 0.14)	0.00005855687695655334	0.000011237823043034677	0.000002333369918901773	$5.155373796638246 \times 10^{-7}$
(0.24, 0.24)	0.00042687289116174175	0.000107261647058368440	0.000029159996674309360	0.000008435399161940688
(0.34, 0.34)	0.00159648516526208390	0.000477468374476977100	0.000154497500358521600	0.000053195324562529430
(0.44, 0.44)	0.00435013258370755200	0.001480025155353154600	0.000544794636100933600	0.000213388842878693700
(0.54, 0.54)	0.00984522698715773800	0.003710760102727795500	0.001513202834688249000	0.000656608610049396100
(0.64, 0.64)	0.01972000093709039600	0.008091651751701122000	0.003592233495567985300	0.001696942365438352200
(0.74, 0.74)	0.03622575866606469000	0.015983563638428833000	0.007630040833314809000	0.003875745658449868000
(0.84, 0.84)	0.06238977506975238000	0.029328734766364786000	0.014916617684664520000	0.008072769438923630000
(0.94, 0.94)	0.10221458141443295000	0.050829641468808535000	0.027347545964240397000	0.015656505501509240000

Table 8: The absolute and relative error in different methods for Example 4 at $\upsilon = 1.0$.

(x,t)	Abs.errors (CSDJA)	Abs.errors(EDM) [36]	Re l.errors (CSDJA)	Re l.errors(EDM) [36]
(0.07, 0.07)	$1.770170676707039 \times 10^{-10}$	$1.770170676707039 \times 10^{-10}$	$1.538912455856741 \times 10^{-10}$	$1.538912455856741 \times 10^{-10}$
(0.17, 0.17)	$4.072256909459781 \times 10^{-8}$	$4.072256909459781 \times 10^{-8}$	$2.898511614818456 \times 10^{-8}$	$2.898511614818456 \times 10^{-8}$
(0.27, 0.27)	$7.330046871700802 \times 10^{-7}$	$7.330046871700802 \times 10^{-7}$	$4.271572004303072 \times 10^{-7}$	$4.271572004303072 \times 10^{-7}$
(0.37, 0.37)	0.0000054448562791797	0.0000054448562791797	0.0000025978166988087	0.0000025978166988087
(0.47, 0.47)	0.0000256617119771540	0.0000256617119771540	0.0000100241790012295	0.0000100241790012295
(0.57, 0.57)	0.0000916097361671752	0.0000916097361671752	0.0000292985362098356	0.0000292985362098356
(0.67, 0.67)	0.0002711490539653027	0.0002711490539653027	0.0000709992053204676	0.0000709992053204676
(0.77, 0.77)	0.0007012039032332495	0.0007012039032332495	0.0001503248651000401	0.0001503248651000401
(0.87, 0.87)	0.0016376884224653665	0.0016376884224653665	0.0002874477279969388	0.0002874477279969388
(0.97, 0.97)	0.0035321427112258164	0.0035321427112258164	0.0005075828587816742	0.0005075828587816742

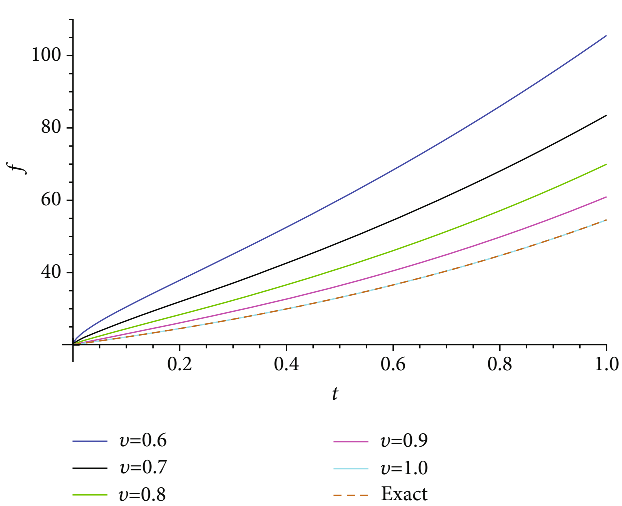

Figure 13: The 2D plot of the 5th iteration approximate and exact solutions of Example 5 for different values of υ in the interval $t \in [0, 1.0]$, when $x = 3.0$ is shown.

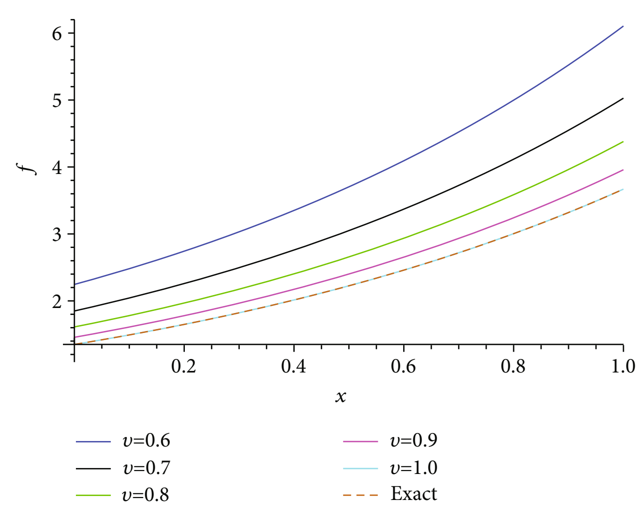

Figure 14: The 2D plot of the 5th iteration approximate and exact solutions of Example 5 for different values of υ in the interval $x \in [0, 1.0]$, with $t = 0.3$ is shown.

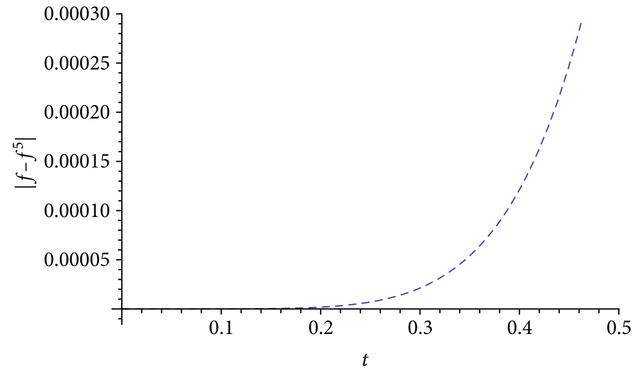

FIGURE 15: The 2D plot of the absolute error graph of $f(x,t)$ for the 5th iteration approximate solution in the intervals $t \in [0{,}0.5]$ when $\upsilon = 1.0$ and $x = 3.0$ in Example 5.

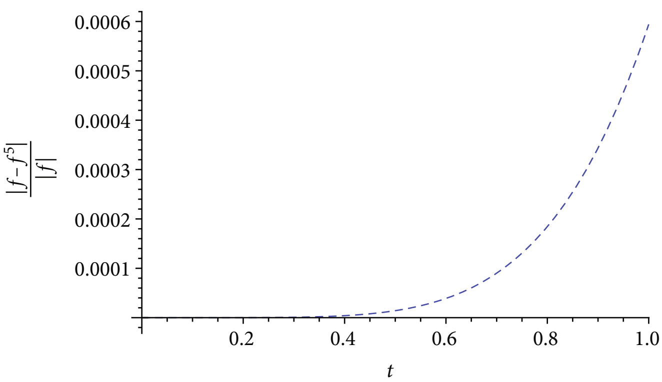

FIGURE 16: The 2D plot of the relative error graph of $f(x,t)$ for the 5th iteration approximate solution in the interval $t \in [0, 0.5]$ when $\upsilon = 1.0$ and $x = 3.0$ in Example 5.

We acquire the following by simplifying Equation (7).

$$\mathfrak{I}_\upsilon[f(x,t)] = \frac{n}{m}f(x,0) + \frac{n}{m}\mathfrak{I}_\upsilon[X(x,t)] - \frac{n}{m}\mathfrak{I}_\upsilon[L(f(x,t))] - \frac{n}{m}\mathfrak{I}_\upsilon[N(f(x,t))]. \tag{8}$$

We obtain the following when we use the inverse CST on both sides of Equation (8).

$$f(x,t) = \mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}f(x,0) + \frac{n}{m}\mathfrak{I}_\upsilon[X(x,t)]\right\} - \mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}\mathfrak{I}_\upsilon[L(f(x,t))]\right\} - \mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}\mathfrak{I}_\upsilon[Nf((x,t))]\right\}. \tag{9}$$

Next, assume the following:

$$\begin{aligned} A(f(x,t)) &= \mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}f(x,0) + \frac{n}{m}\mathfrak{I}_\upsilon[X(x,t)]\right\}. \\ B(f(x,t)) &= \mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}\mathfrak{I}_\upsilon[L(f(x,t))]\right\}. \\ C(f(x,t)) &= \mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}\mathfrak{I}_\upsilon[N(f(x,t))]\right\}. \end{aligned} \tag{10}$$

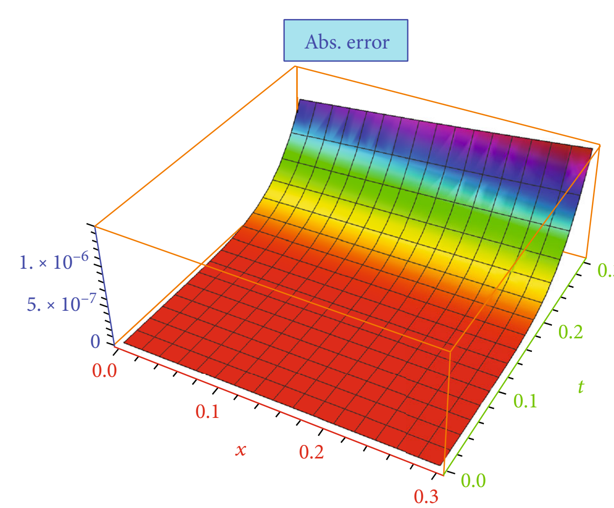

FIGURE 17: The 3D plot of the absolute error graph of $f = (x,t)$ for the 5th iteration approximate solution when $\upsilon = 1.0$ with $x \in [0, 0.3]$ and $t \in [0, 0.3]$ in Example 5.

As a result, Equation (9) can be written as

$$f(x,t) = A(f(x,t)) + B(f(x,t)) + C(f(x,t)), \tag{11}$$

where B and C are given as linear and nonlinear operators of

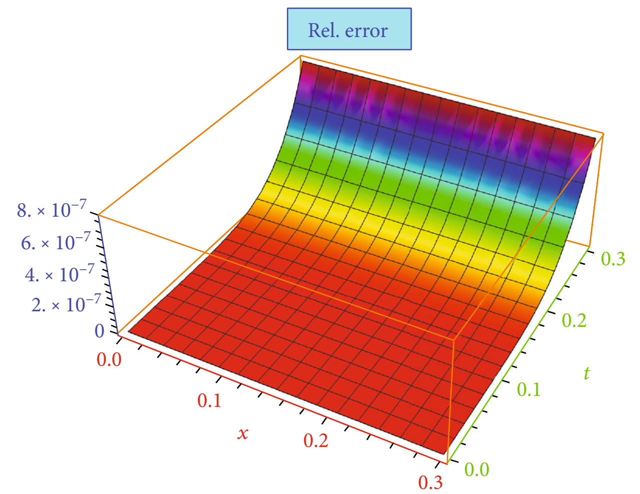

Figure 18: The 3D plot of the relative error graph of $f = (x, t)$ for the 5th iteration approximate solution when $\upsilon = 1.0$ and $x \in [0, 0.3]$ and $t \in [0, 0.3]$ in Example 5.

$f(x, t)$, respectively, where $A(f(x, t))$ is a known function. The solution to Equation (6) can be expressed in the following expansion form:

$$f(x,t) = \sum_{i=0}^{\infty} f_i(x,t). \tag{12}$$

Assume the following:

$$B\left(\sum_{i=0}^{\infty} f_i(x,t)\right) = \sum_{i=0}^{\infty} B(f_i(x,t)). \tag{13}$$

Decompose the nonlinear operator as [45]

$$C\left(\sum_{i=0}^{\infty} f_i(x,t)\right) = C(f_0(x,t)) + \sum_{i=0}^{\infty}\left\{C\left(\sum_{\kappa=0}^{i} f_\kappa(x,t)\right) - C\left(\sum_{\kappa=0}^{i-1} f_\kappa(x,t)\right)\right\}. \tag{14}$$

As a result, Equation (6) can be considered the following arrangement.

$$\sum_{i=0}^{\infty} f_i(x,t) = A(f(x,t)) + \sum_{i=0}^{\infty} B(f_i(x,t)) + C(f_0(x,t)) + \sum_{i=0}^{\infty}\left\{CC\left(\sum_{\kappa=0}^{i-1} f_\kappa(x,t)\right)\right\}. \tag{15}$$

Use Equation (15) to define the recurrence relation.

$$\begin{aligned}
f_0(x,t) &= A(f(x,t)),\\
f_1(x,t) &= B(f_0(x,t)) + C(f_0(x,t)),\\
f_{j+1}(x,t) &= B\Big(f_k(x,t) + C\Big(f_0(x,t) + f_1(x,t) + \cdots + f_j(x,t)\Big)\\
&\quad - C\Big(f_0(x,t) + f_1(x,t) + \cdots + f_{j-1}(x,t)\Big).
\end{aligned} \tag{16}$$

As a result, we have the following:

$$\begin{aligned}
&\Big(f_1(x,t) + f_2(x,t) + \cdots + f_{j+1}(x,t)\Big)\\
&\quad = B\Big(f_0(x,t) + f_1(x,t) + \cdots + f_j(x,t)\Big)\\
&\qquad + C\Big(f_0(x,t) + f_1(x,t) + \cdots + f_j(x,t)\Big).
\end{aligned} \tag{17}$$

Particularly,

$$\sum_{i=0}^{\infty} f_i(x,t) = A\left(\sum_{i=0}^{\infty} f_i(x,t)\right) + B\left(\sum_{i=0}^{\infty} f_i(x,t)\right) + C\left(\sum_{i=0}^{\infty} f_i(x,t)\right). \tag{18}$$

Equation (6) has jth approximate solution, which is given by

$$f_j(x,t) = f_0(x,t) + f_1(x,t) ++ f_{j-1}(x,t). \tag{19}$$

In the next section, we will assess the suitability of the suggested approach by solving numerical problems.

4. Numerical Examples and Concluding Remarks

In this section, five TFSHEs in the CD sense are solved in order to assess the effectiveness and suitability of the suggested approach.

Example 1. Take the following SH equation, which is linear and time-fractional [36]:

$$T_t^{\upsilon} f(x,t) + (1-\Upsilon) f(x,t) + 2\frac{\partial^2 f(x,t)}{\partial x^2} + \frac{\partial^4 f(x,t)}{\partial x^4} = 0, \quad 0 < \upsilon \le 1, \tag{20}$$

with the initial condition:

$$f(x,0) = \sin x. \tag{21}$$

The results of applying CST to both sides of Equation (20) are as follows:

$$\mathfrak{T}_\upsilon\left[T_t^{\upsilon} f(x,t) + (1-\Upsilon) f(x,t) + 2\frac{\partial^2 f(x,t)}{\partial x^2} + \frac{\partial^4 f(x,t)}{\partial x^4}\right] = 0. \tag{22}$$

By using the method described in Section 3, we achieve the following outcome:

$$\mathfrak{T}_\upsilon[f(x,t)] = \frac{n}{m}\sin x - \frac{n}{m}\mathfrak{T}_\upsilon[(1-\Upsilon) f(x,t)] - \frac{n}{m}\mathfrak{T}_\upsilon\left[2\frac{\partial^2 f(x,t)}{\partial x^2}\right] - \frac{n}{m}\mathfrak{T}_\upsilon\left[\frac{\partial^4 f(x,t)}{\partial x^4}\right]. \tag{23}$$

Table 9: Absolute and relative error in the 4th-step approximate solution of $f(x,t)$ when $v = 1.0$ for Example 5.

(x,t)	$f^4(x,t)$	$f(x,t)$	Re l.error $= \|f - f^4\|/\|f\|$	Abs.error $= \|f - f^4\|$
(0.05, 0.05)	1.1051709153149847	1.1051709180756477	$2.497951209498041 \times 10^{-9}$	$2.760663031509125 \times 10^{-9}$
(0.15, 0.15)	1.3498580535708110	1.3498588075760032	$5.585807849316814 \times 10^{-7}$	$7.540051922827473 \times 10^{-7}$
(0.25, 0.25)	1.6487103698322902	1.6487212707001282	0.0000661171056119114	0.00001090086783794852
(0.35, 0.35)	2.0136867849099350	2.0137527074704766	0.00003273617475320743	0.00006592256054149814
(0.45, 0.45)	2.4593426274569340	2.4596031111569500	0.00010590476928346425	0.00026048370001596766
(0.55, 0.55)	3.0033668369534470	3.0041660239464334	0.00026602624043272600	0.00079918699298620060
(0.65, 0.65)	3.6672236017846527	3.6692966676192444	0.00056497634897884670	0.00207306583459176960
(0.75, 0.75)	4.4769175156003390	4.4816890703380645	0.00106467777278576500	0.00477155473772583600
(0.85, 0.85)	5.4639046334569470	5.4739473917272000	0.00183464647201958890	0.01004275827025313800
(0.95, 0.95)	6.6661756508830850	6.6858944422792685	0.00294931240186632400	0.01971879139618337700

Table 10: Absolute and relative error in the 5th-step approximate solution of $f(x,t)$ when $v = 1.0$ for Example 5.

(x,t)	$f^5(x,t)$	$f(x,t)$	Rel.error $= \|f - f^5\|/\|f\|$	Abs.error $= \|f - f^5\|$
(0.05, 0.05)	1.1051709180526696	1.1051709180756477	$2.079141200717798 \times 10^{-11}$	$2.297806389606194 \times 10^{-11}$
(0.15, 0.15)	1.3498587887940425	1.3498588075760032	$1.391401867217576 \times 10^{-8}$	$1.878196065341342 \times 10^{-8}$
(0.25, 0.25)	1.6487208192578820	1.6487212707001282	$2.738135634414047 \times 10^{-7}$	$4.514422462520429 \times 10^{-7}$
(0.35, 0.35)	2.0137488949836050	2.0137527074704766	0.00000189322495130197	0.00000381248687153501
(0.45, 0.45)	2.4595837922127695	2.4596031111569500	0.00000785449656196916	0.00001931894418039093
(0.55, 0.55)	3.0040937695882923	3.0041660239464334	0.00002405138649632585	0.00007225435814106618
(0.65, 0.65)	3.6690757551727438	3.6692966676192444	0.00006020566514836269	0.00022091244650068730
(0.75, 0.75)	4.4811039658285040	4.4816890703380645	0.00013055446292173030	0.00058510450956017480
(0.85, 0.85)	5.4725555812700595	5.4739473917272000	0.00025426083912386135	0.00139181045714043000
(0.95, 0.95)	6.6828487579188190	6.6858944422792685	0.00045553880438038890	0.00304568436044938550

Table 11: The recurrence error for $f(x,t)$ at various values of v for Example 5.

(x,t)	$v = 0.7$	$v = 0.8$	$v = 0.9$	$v = 1.0$
(0.05, 0.05)	0.000001456928681921672	$1.6709504069900890 \times 10^{-7}$	$2.073413635883548 \times 10^{-8}$	$2.7376851468125640 \times 10^{-9}$
(0.15, 0.15)	0.000075299507010213980	0.000014958154942352114	0.000003214852447088009	$7.3522323172649170 \times 10^{-7}$
(0.25, 0.25)	0.000497384451313588700	0.000127556464740409500	0.000035392387988058570	0.000010449425591534353
(0.35, 0.35)	0.001784718714941603100	0.000541556864447976400	0.000177793334429172840	0.000062110073669809540
(0.45, 0.45)	0.004753405550388819000	0.001635503172715662700	0.000608828586232166100	0.000241164755835960970
(0.55, 0.55)	0.010603761039668216000	0.004033495177363797000	0.001659970134744652700	0.000726932634845213500
(0.65, 0.65)	0.021028907727128978000	0.008695882418547608000	0.003890520650822312500	0.001852153388091293700
(0.75, 0.75)	0.038349864797284756000	0.017034709587258750000	0.008186584586292290000	0.004186450228164713000
(0.85, 0.85)	0.065681808688387200000	0.031059526557876346000	0.015890650727901650000	0.008650947813112190000
(0.95, 0.95)	0.107137357433544080000	0.053560299636793150000	0.028969579299671387000	0.016673107035735170000

Table 12: The absolute and relative error in various approaches for Example 5 at $\upsilon = 1.0$.

(x, t)	Abs.errors (CSDJA)	Abs.errors(EDM) [36]	Re l.errors (CSDJA)	Re l.errors(EDM) [36]
(0.08, 0.08)	$3.989657493264076 \times 10^{-10}$	$3.989657493264076 \times 10^{-10}$	$3.399761852987486 \times 10^{-10}$	$3.399761852987486 \times 10^{-10}$
(0.18, 0.18)	$5.804326486114064 \times 10^{-8}$	$5.804326486114064 \times 10^{-8}$	$4.049541178148838 \times 10^{-8}$	$4.049541178148838 \times 10^{-8}$
(0.28, 0.28)	$9.222621923932195 \times 10^{-7}$	$9.222621923932195 \times 10^{-7}$	$5.268045235400865 \times 10^{-7}$	$5.268045235400865 \times 10^{-7}$
(0.38, 0.38)	0.0000064634844112454	0.0000064634844112454	0.0000030227546606414	0.0000030227546606414
(0.48, 0.48)	0.0000294538494971519	0.0000294538494971519	0.0000112776694370410	0.0000112776694370410
(0.58, 0.58)	0.0001028641032836397	0.0001028641032836397	0.0000322464748883020	0.0000322464748883020
(0.68, 0.68)	0.0002997943734013297	0.0002997943734013297	0.0000769454568034897	0.0000769454568034897
(0.78, 0.78)	0.0007664534267357581	0.0007664534267357581	0.0001610595118526153	0.0001610595118526153
(0.88, 0.88)	0.0017743428907426306	0.0017743428907426306	0.0003052665810132136	0.0003052665810132136
(0.98, 0.98)	0.0038001237563074497	0.0038001237563074497	0.0005352794316180174	0.0005352794316180174

Apply the inverse CST on both sides of Equation (23).

$$f(x,t) = \mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}\sin x\right\} - \mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}\mathfrak{I}_\upsilon[(1-\Upsilon)f(x,t)]\right\} - \mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}\mathfrak{I}_\upsilon\left[2\frac{\partial^2 f(x,t)}{\partial x^2}\right]\right\} - \mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}\mathfrak{I}_\upsilon\left[\frac{\partial^4 f(x,t)}{\partial x^4}\right]\right\}. \tag{24}$$

The procedure described in Section 3 leads to the following outcome:

$$\begin{aligned} B(f(x,t)) = &-\mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}\mathfrak{I}_\upsilon[(1-\Upsilon)f(x,t)]\right\} \\ &- \mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}\mathfrak{I}_\upsilon\left[2\frac{\partial^2 f(x,t)}{\partial x^2}\right]\right\} \\ &- \mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}\mathfrak{I}_\upsilon\left[\frac{\partial^4 f(x,t)}{\partial x^4}\right]\right\}, \end{aligned} \tag{25}$$

$$\begin{aligned} \sum_{i=0}^{\infty} f_i(x,t) = &\mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}\sin x\right\} - \mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}\mathfrak{I}_\upsilon\left[(1-\Upsilon)\sum_{i=0}^{\infty} f_i(x,t)\right]\right\} \\ &- \mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}\mathfrak{I}_\upsilon\left[2\frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty} f_i(x,t)\right]\right\} \\ &- \mathfrak{I}_\upsilon^{-1}\left\{\frac{n}{m}\mathfrak{I}_\upsilon\left[\frac{\partial^4}{\partial x^4}\sum_{i=0}^{\infty} f_i(x,t)\right]\right\}. \end{aligned} \tag{26}$$

Here, we will discuss the following two cases.

Case 1. $\Upsilon \neq 0$.

Using the iteration process outlined in Section 3, the results from Equation (26) are as follows:

$$\begin{aligned} f_0(x,t) &= \sin x, \\ f_1(x,t) &= \sin x((\Upsilon-1)+1)\frac{t^\upsilon}{1!\upsilon^1}, \\ f_2(x,t) &= \sin x\left((\Upsilon-1)^2+(\Upsilon-1)+1\right)\frac{t^{2\upsilon}}{2!\upsilon^2}, \\ f_3(x,t) &= \sin x\left((\Upsilon-1)^3+(\Upsilon-1)^2+(\Upsilon-1)+1\right)\frac{t^{3\upsilon}}{3!\upsilon^3}, \\ f_4(x,t) &= \sin x\left((\Upsilon-1)^4+(\Upsilon-1)^3+(\Upsilon-1)^2 + (\Upsilon-1)+1\right)\frac{t^{4\upsilon}}{4!\upsilon^4}, \\ f_5(x,t) &= \sin x\left((\Upsilon-1)^5+(\Upsilon-1)^4+(\Upsilon-1)^3+(\Upsilon-1)^2 + (\Upsilon-1)+1\right)\frac{t^{5\upsilon}}{5!\upsilon^5}. \end{aligned} \tag{27}$$

As a result, we get the 5th step approximate solution to Equations (20) and (21) obtained from the 5th iteration as follows:

$$\begin{aligned} f^{(5)}(x,t) = &\sin x + \sin x((\Upsilon-1)+1)\frac{t^\upsilon}{1!\upsilon^1} + \sin x\left((\Upsilon-1)^2 + (\Upsilon-1)+1\right)\frac{t^{2\upsilon}}{2!\upsilon^2} + \sin x\left((\Upsilon-1)^3+(\Upsilon-1)^2 \right.\\ &\left.+ (\Upsilon-1)+1\right)\frac{t^{3\upsilon}}{3!\upsilon^3} + \sin x\left((\Upsilon-1)^4+(\Upsilon-1)^3 + \left((\Upsilon-1)^2+(\Upsilon-1)+1\right)\right)\frac{t^{4\upsilon}}{4!\upsilon^4} + \sin x\left((\Upsilon-1)^5 \right.\\ &\left.+ (\Upsilon-1)^4+(\Upsilon-1)^3+(\Upsilon-1)^2+(\Upsilon-1)+1\right)\frac{t^{5\upsilon}}{5!\upsilon^5}. \end{aligned} \tag{28}$$

We get the following exact solution to Equations (20) and (21):

$$f(x,t) = -\frac{1}{\Upsilon}\sin x \lim_{j\longrightarrow\infty}\sum_{i=0}^{j}\left(1-(\Upsilon-1)^{i+1}\right)\frac{t^{iv}}{i!v^{i}}. \tag{29}$$

Case 2. $\Upsilon = 0$.

Using the iteration process outlined in Section 3, the results from Equation (26) are as follows.

$$\begin{aligned} &f_0(x,t) = \sin x,\\ &f_1(x,t) = 0,\\ &f_2(x,t) = 0,\\ &f_i(x,t) = 0. i = 3, 4, \cdots. \end{aligned} \tag{30}$$

Hence, the solution to Equations (20) and (21) is

$$f(x,t) = \sin x. \tag{31}$$

As a consequence, the solution of Equations (20) and (21) is

$$f(x,t) = \begin{cases} -\dfrac{1}{\Upsilon}\sin x \lim\limits_{j\longrightarrow\infty}\sum\limits_{i=0}^{j}\left(1-(\Upsilon-1)^{i+1}\right)\dfrac{t^{iv}}{i!v^{i}}, & \Upsilon \neq 0,\\ \sin x & \Upsilon = 0. \end{cases} \tag{32}$$

Example 2. Take the following SH equation, which is linear and time-fractional [37]:

$$T_t^v f(x,t) + (1-\Upsilon)f(x,t) + 2\frac{\partial^2 f(x,t)}{\partial x^2} + \frac{\partial^4 f(x,t)}{\partial x^4} = 0, \quad 0 < v \leq 1, \tag{33}$$

subject to the initial condition:

$$f(x,0) = \cos x. \tag{34}$$

When CST is applied to both sides of an equation (33), we get the following:

$$\mathfrak{I}_v\left[T_t^v f(x,t) + (1-\Upsilon)f(x,t) + 2\frac{\partial^2 f(x,t)}{\partial x^2} + \frac{\partial^4 f(x,t)}{\partial x^4}\right] = 0. \tag{35}$$

By using the method described in Section 3, we achieve the following outcome:

$$\begin{aligned} \mathfrak{I}_v[f(x,t)] = \frac{n}{m}\cos x - \frac{n}{m}\mathfrak{I}_v[(1-\Upsilon)f(x,t)] \\ - \frac{n}{m}\mathfrak{I}_v\left[2\frac{\partial^2 f(x,t)}{\partial x^2}\right] - \frac{n}{m}\mathfrak{I}_v\left[\frac{\partial^4 f(x,t)}{\partial x^4}\right]. \end{aligned} \tag{36}$$

Consider the inverse of CST on both sides of Equation (33).

$$\begin{aligned} f(x,t) = \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\cos x\right\} - \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v[(1-\Upsilon)f(x,t)]\right\} \\ - \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[2\frac{\partial^2 f(x,t)}{\partial x^2}\right]\right\} - \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[\frac{\partial^4 f(x,t)}{\partial x^4}\right]\right\}. \end{aligned} \tag{37}$$

We obtain the following outcome by following the procedure described in Section 3:

$$\begin{aligned} B(f(x,t)) = -\mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v[(1-\Upsilon)f(x,t)]\right\} \\ - \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[2\frac{\partial^2 f(x,t)}{\partial x^2}\right]\right\} \\ - \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[\frac{\partial^4 f(x,t)}{\partial x^4}\right]\right\}. \end{aligned} \tag{38}$$

$$\begin{aligned} \sum_{i=0}^{\infty} f_i(x,t) = \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\sin x\right\} - \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[(1-\Upsilon)\sum_{i=0}^{\infty} f_i(x,t)\right]\right\} \\ - \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[2\frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty} f_i(x,t)\right]\right\} \\ - \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[\frac{\partial^4}{\partial x^4}\sum_{i=0}^{\infty} f_i(x,t)\right]\right\}. \end{aligned} \tag{39}$$

Here, we will deal with the following two situations:

Case 1. $\Upsilon \neq 0$.

The outcomes of Equation (39) are as follows when applying the iteration procedure described in Section 3:

$$\begin{aligned} &f_0(x,t) = \cos x,\\ &f_1(x,t) = \cos x((\Upsilon-1)+1)\frac{t^{v}}{1!v^{1}},\\ &f_2(x,t) = \cos x\left((\Upsilon-1)^2+(\Upsilon-1)+1\right)\frac{t^{2v}}{2!v^{2}},\\ &f_3(x,t) = \cos x\left((\Upsilon-1)^3+(\Upsilon-1)^2+(\Upsilon-1)+1\right)\frac{t^{3v}}{3!v^{3}},\\ &f_4(x,t) = \cos x\left((\Upsilon-1)^4+(\Upsilon-1)^3+(\Upsilon-1)^2+(\Upsilon-1)+1\right)\frac{t^{4v}}{4!v^{4}}, \end{aligned}$$

$$f_5(x,t) = \cos x\left((\Upsilon-1)^5 + (\Upsilon-1)^4 + (\Upsilon-1)^3 + (\Upsilon-1)^2 + (\Upsilon-1) + 1\right)\frac{t^{5v}}{5!v^5}. \tag{40}$$

As a result, we get the 5th step approximate solution to Equations (33) and (34) obtained from the 5th iteration as follows:

$$\begin{aligned} f^{(5)}(x,t) &= \cos x + \cos x((\Upsilon-1)+1)\frac{t^{v}}{1!v^1} + \cos x\left((\Upsilon-1)^2 + (\Upsilon-1)+1\right)\frac{t^{2v}}{2!v^2} + \cos x\left((\Upsilon-1)^3 + (\Upsilon-1)^2 + (\Upsilon-1)+1\right)\frac{t^{3v}}{3!v^3} \\ &\quad + \cos x\left((\Upsilon-1)^4 + (\Upsilon-1)^3 + (\Upsilon-1)^2 + (\Upsilon-1)+1\right)\frac{t^{4v}}{4!v^4} + \cos x\left((\Upsilon-1)^5 + (\Upsilon-1)^4 + (\Upsilon-1)^3 + (\Upsilon-1)^2 + (\Upsilon-1)+1\right)\frac{t^{5v}}{5!v^5}. \end{aligned} \tag{41}$$

We get the following exact solution to Equations (33) and (34).

$$f(x,t) = -\frac{1}{\Upsilon}\cos x \lim_{j\longrightarrow\infty}\sum_{i=0}^{j}\left(1-(\Upsilon-1)^{i+1}\right)\frac{t^{iv}}{i!v^i}. \tag{42}$$

Case 2. $\Upsilon = 0$.

Using the iteration process outlined in Section 3, the results from Equation (39) are as follows.

$$\begin{aligned} f_0(x,t) &= \cos x \\ f_1(x,t) &= 0, \\ f_2(x,t) &= 0, \\ f_i(x,t) &= 0. i = 3, 4, \cdots. \end{aligned} \tag{43}$$

Hence, the solution to Equations (33) and (34) is

$$f(x,t) = \cos x. \tag{44}$$

As a consequence, the solution of Equations (33) and (34) is

$$f(x,t) = \begin{cases} -\dfrac{1}{\Upsilon}\cos x \lim\limits_{j\longrightarrow\infty}\sum\limits_{i=0}^{j}\left(1-(\Upsilon-1)^{i+1}\right)\dfrac{t^{iv}}{i!v^i}, & \Upsilon \neq 0, \\ \cos x, & \Upsilon = 0. \end{cases} \tag{45}$$

Example 3. Consider the following SH equation, which is linear and time-fractional [36]:

$$T_t^{v}f(x,t) + (1-\Upsilon)f(x,t) + 2\frac{\partial^2 f(x,t)}{\partial x^2} + \frac{\partial^4 f(x,t)}{\partial x^4} - \Theta\frac{\partial^3 f(x,t)}{\partial x^3} = 0, \quad 0 < v \le 1, \tag{46}$$

with the initial condition:

$$f(x,0) = e^x. \tag{47}$$

Using CST on both sides of Equation (46),

$$\mathfrak{I}_v\left[T_t^{v}f(x,t) + (1-\Upsilon)f(x,t) + 2\frac{\partial^2 f(x,t)}{\partial x^2} + \frac{\partial^4 f(x,t)}{\partial x^4} - \Theta\frac{\partial^3 f(x,t)}{\partial x^3}\right] = 0. \tag{48}$$

We obtain the following result from Equation (48) by carrying out the processes outlined in Section 3.

$$\mathfrak{I}_v[f(x,t)] = \frac{n}{m}e^x - \frac{n}{m}\mathfrak{I}_v[(1-\Upsilon)f(x,t)] - \frac{n}{m}\mathfrak{I}_v\left[2\frac{\partial^2 f(x,t)}{\partial x^2}\right] - \frac{n}{m}\mathfrak{I}_v\left[\frac{\partial^4 f(x,t)}{\partial x^4}\right] + \frac{n}{m}\mathfrak{I}_v\left[\Theta\frac{\partial^3 f(x,t)}{\partial x^3}\right]. \tag{49}$$

Use inverse CST on both sides of Equation (49).

$$\begin{aligned} f(x,t) &= \mathfrak{I}_v^{-1}\left\{\frac{n}{m}e^x\right\} - \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v[(1-\Upsilon)f(x,t)]\right\} - \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[2\frac{\partial^2 f(x,t)}{\partial x^2}\right]\right\} - \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[\frac{\partial^4 f(x,t)}{\partial x^4}\right]\right\} \\ &\quad + \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[\Theta\frac{\partial^3 f(x,t)}{\partial x^3}\right]\right\}. \end{aligned} \tag{50}$$

By following the steps indicated in Section 3, we obtain the following outcome:

$$A(f(x,t)) = \mathfrak{I}_v^{-1}\left\{\frac{n}{m}e^x\right\}, \tag{51}$$

$$\begin{aligned} B(f(x,t)) &= -\mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v[(1-\Upsilon)f(x,t)]\right\} - \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[2\frac{\partial^2 f(x,t)}{\partial x^2}\right]\right\} \\ &\quad - \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[\frac{\partial^4 f(x,t)}{\partial x^4}\right]\right\} + \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[\Theta\frac{\partial^3 f(x,t)}{\partial x^3}\right]\right\}, \end{aligned} \tag{52}$$

$$\begin{aligned} \sum_{i=0}^{\infty} f_i(x,t) &= \mathfrak{I}_v^{-1}\left\{\frac{n}{m}e^x\right\} - \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[(1-\Upsilon)\sum_{i=0}^{\infty} f_i(x,t)\right]\right\} \\ &\quad - \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[2\frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty} f_i(x,t)\right]\right\} \\ &\quad + \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[\Theta\frac{\partial^3}{\partial x^3}\sum_{i=0}^{\infty} f_i(x,t)\right]\right\} \\ &\quad - \mathfrak{I}_v^{-1}\left\{\frac{n}{m}\mathfrak{I}_v\left[\frac{\partial^4}{\partial x^4}\sum_{i=0}^{\infty} f_i(x,t)\right]\right\}. \end{aligned} \tag{53}$$

The findings from Equation (53) are as follows using the iteration method described in Section 3:

$$\begin{aligned} f_0(x,t) &= e^x,\\ f_1(x,t) &= e^x(\Upsilon - 4 + \Theta)^1 \frac{t^{1\upsilon}}{1!\upsilon^1}.\\ f_2(x,t) &= e^x(\Upsilon - 4 + \Theta)^2 \frac{t^{2\upsilon}}{2!\upsilon^2}.\\ f_3(x,t) &= e^x(\Upsilon - 4 + \Theta)^3 \frac{t^{3\upsilon}}{3!\upsilon^3}.\\ f_4(x,t) &= e^x(\Upsilon - 4 + \Theta)^4 \frac{t^{4\upsilon}}{4!\upsilon^4}.\\ f_5(x,t) &= e^x(\Upsilon - 4 + \Theta)^5 \frac{t^{5\upsilon}}{5!\upsilon^5}. \end{aligned} \tag{54}$$

As a result, we get the approximate solution to Equations (46) and (47) obtained from the 5th iteration as follows:

$$\begin{aligned} f^{(5)}(x,t) = e^x &+ e^x(\Upsilon - 4 + \Theta)^1 \frac{t^{1\upsilon}}{1!\upsilon^1} + e^x(\Upsilon - 4 + \Theta)^2 \frac{t^{2\upsilon}}{2!\upsilon^2}\\ &+ e^x(\Upsilon - 4 + \Theta)^3 \frac{t^{3\upsilon}}{3!\upsilon^3} + e^x(\Upsilon - 4 + \Theta)^4 \frac{t^{4\upsilon}}{4!\upsilon^4}\\ &+ e^x(\Upsilon - 4 + \Theta)^5 \frac{t^{5\upsilon}}{5!\upsilon^5}. \end{aligned} \tag{55}$$

When $\upsilon = 1$ is employed in Equation (55), we get the following exact solution to Equations (46) and (47):

$$f(x,t) = e^{x+(\Theta+\Upsilon-4)t}. \tag{56}$$

We compare the numerical and graphical results of the exact and approximative solutions to the models presented in Examples 3–5. Error functions can be used to evaluate the numerical method's accuracy and capabilities. It is important to provide the error of the analytical approximate solution that CSDJA offers in terms of an infinite fractional power series. We employed the recurrence, absolute, and relative error functions to show the accuracy and efficiency of CSDJA.

The 2D graph of the comparison between the exact solution and the approximation is shown in Figures 1 and 2 obtained by the proposed method in Example 3. Figure 1 represents the 2D graph of the exact solution and the 5th iteration approximate solution attained by CSDJA at$\upsilon = 0.6, 0.7, 0.8, 0.9,$ and$\upsilon = 1.0$with$x = 1.0, Y = 3$, and $\Theta =$ 2in the interval$t \in [0, 1.0]$for Example 3. Figure 2 represents the 2D graph of the exact solution and 5th iteration approximate solution attained by CSDJA at $\upsilon = 0.6, 0.7, 0.8, 0.9,$ and $\upsilon = 1.0$ with $t = 0.1$, $Y = 3$, and $\Theta = 2$ in the interval $x \in [0,1.0]$ for Example 3. These graphs show how effective the approximative solution provided by the CSDJA is when $\upsilon \longrightarrow 1.0$. The approximate solution corresponds with the precise solution at $\upsilon = 1.0$, which proves the suggested method's effectiveness and precision.

Figures 3 and 4 show the 2D graph of the comparative study of the approximate solution obtained from the five iterations and the exact solution obtained by the proposed method in the form of absolute and relative error with $\upsilon =$ 1.0 at $x = 1.0$, $Y = 3$, and $\Theta = 2$ in the interval $t \in [0, 0.5]$ to Example 3, respectively. The comparison has shown that the suggested method's fifth-step approximation and precise solution are extremely close. The relative and absolute error comparison of the approximate and exact solutions on a graph demonstrates the superior accuracy of CSDJA.

Figures 5 and 6 display the 3D graphs of the comparison research in terms of absolute and relative error for the approximate solution produced by the suggested method in the fifth stage and the exact solution at $\upsilon = 1.0$, $Y = 3$, and $\Theta = 2$ in the intervals $x \in [0, 0.5]$ and $t \in [0, 0.3]$ to Example 3, respectively. The comparison has proven that the suggested method's fifth-step approximation solution is quite near to the precise solution, and this proves the effectiveness and precision of the suggested method.

Tables 1 and 2 show that the approximate solution for the fourth phase has very low relative and absolute error to Example 3. The relative and absolute error will be even smaller if we consider the fifth-step approximation. The relative and absolute error processes demonstrate the precision of our suggested strategy, and as a result, the approximation is quickly approaching the exact solution. We anticipate that our approach will be a key step in the management of several FODEs with engineering and applied mathematics fields of physical interest.

Recurrence error has been used to quantitatively demonstrate how the approximate solution for Example 3 converges to the exact solution for suitably specified grid locations in the intervals $x \in [0, 1.0]$ and $t \in [0, 1.0]$ when Y $= 3$ and $\Theta = 2$ as in Table 3. From Table 3, we see that the approximate solution obtained by the proposed method in the fifth step quickly approaches the exact solution as the order of the fractional-order derivative increases. The increased degree of accuracy and convergence rates have been proven by the recurrence error analysis. We conclude that the proposed method is a viable and efficient algorithm for solving particular classes of FODEs with minimal calculations and iterative phases.

Table 4 also compares the absolute and relative error of the approximation from the fifth iterations obtained by the CSDJA of Example 3 at plausible short-listed grid points in the range $x \in [0,1.0]$ and $t \in [0,1.0]$ when $Y = 3$ and $\Theta = 2$ with the absolute and relative error of the 5th-step approximation obtained by EDM [36]. The comparison has proven that the suggested procedure and EDM yield the same results. Because of the comparison, we concluded that CSDJA can be used as a substitute for method EDM to solve TFSHEs.

Example 4. Consider the following SH equation, which is nonlinear and time-fractional [36]:

$$\begin{aligned} T_t^{\upsilon} f(x,t) + (1-\Upsilon) f(x,t) + 2\frac{\partial^2 f(x,t)}{\partial x^2} + \frac{\partial^4 f(x,t)}{\partial x^4} - \Theta \frac{\partial^3 f(x,t)}{\partial x^3}\\ - f^2(x,t) + \left(\frac{\partial f(x,t)}{\partial x}\right)^2 = 0, \quad 0 < \upsilon \leq 1, \end{aligned} \tag{57}$$

subject to the following initial conditions:

$$f(x,0) = e^{x}. \tag{58}$$

Using CST on both sides of Equation (57),

$$\begin{aligned}\mathfrak{I}_{\upsilon}\Bigg[&T_t^{\upsilon}f(x,t) + (1-\Upsilon)f(x,t) + 2\frac{\partial^2 f(x,t)}{\partial x^2} + \frac{\partial^4 f(x,t)}{\partial x^4}\\ &-\Theta\frac{\partial^3 f(x,t)}{\partial x^3} - f^2(x,t) + \left(\frac{\partial f(x,t)}{\partial x}\right)^2\Bigg] = 0.\end{aligned} \tag{59}$$

We obtain the following result from Equation (59) by carrying out the processes outlined in Section 3.

$$\begin{aligned}\mathfrak{I}_{\upsilon}[f(x,t)] = {}&\frac{n}{m}e^{x} - \frac{n}{m}\mathfrak{I}_{\upsilon}[(1-\Upsilon)f(x,t)] - \frac{n}{m}\mathfrak{I}_{\upsilon}\left[2\frac{\partial^2 f(x,t)}{\partial x^2}\right]\\ &-\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\frac{\partial^4 f(x,t)}{\partial x^4}\right] + \frac{n}{m}\mathfrak{I}_{\upsilon}\left[\Theta\frac{\partial^3 f(x,t)}{\partial x^3}\right]\\ &+\frac{n}{m}\mathfrak{I}_{\upsilon}\left[f^2(x,t)\right] - \frac{n}{m}\mathfrak{I}_{\upsilon}\left[\left(\frac{\partial f(x,t)}{\partial x}\right)^2\right].\end{aligned} \tag{60}$$

Use inverse CST on both sides of Equation (60).

$$\begin{aligned}f(x,t) = {}&\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}e^{x}\right\} - \mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}[(1-\Upsilon)f(x,t)]\right\}\\ &-\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[2\frac{\partial^2 f(x,t)}{\partial x^2}\right]\right\} - \mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\frac{\partial^4 f(x,t)}{\partial x^4}\right]\right\}\\ &+\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\Theta\frac{\partial^3 f(x,t)}{\partial x^3}\right]\right\} + \mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[f^2(x,t)\right]\right\}\\ &-\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\left(\frac{\partial f(x,t)}{\partial x}\right)^2\right]\right\}.\end{aligned} \tag{61}$$

By following the steps indicated in Section 3, we obtain the following outcome:

$$\begin{aligned}B(f(x,t)) = {}&-\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}[(1-\Upsilon)f(x,t)]\right\} - \mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[2\frac{\partial^2 f(x,t)}{\partial x^2}\right]\right\}\\ &-\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\frac{\partial^4 f(x,t)}{\partial x^4}\right]\right\} + \mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\Theta\frac{\partial^3 f(x,t)}{\partial x^3}\right]\right\},\end{aligned} \tag{62}$$

$$C(f(x,t)) = \mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[f^2(x,t)\right]\right\} - \mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\left(\frac{\partial f(x,t)}{\partial x}\right)^2\right]\right\}, \tag{63}$$

$$\begin{aligned}\sum_{i=0}^{\infty}f_i(x,t) = {}&\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}e^{x}\right\} - \mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[(1-\Upsilon)\sum_{i=0}^{\infty}f_i(x,t)\right]\right\}\\ &-\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[2\frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty}f_i(x,t)\right]\right\}\\ &+\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\Theta\frac{\partial^3}{\partial x^3}\sum_{i=0}^{\infty}f_i(x,t)\right]\right\}\\ &+\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\frac{\partial^4}{\partial x^4}\sum_{i=0}^{\infty}f_i(x,t)\right]\right\}\\ &+\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\sum_{i=0}^{\infty}f_i(x,t)\right]^2\right\}\\ &-\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\left(\frac{\partial}{\partial x}\sum_{i=0}^{\infty}f_i(x,t)\right)^2\right]\right\}.\end{aligned} \tag{64}$$

The findings from Equation (64) are as follows using the iteration method described in Section 3:

$$\begin{aligned}f_0(x,t) &= e^{x},\\ f_1(x,t) &= e^{x}(\Upsilon-4+\Theta)^1\frac{t^{1\upsilon}}{1!\upsilon^1},\\ f_2(x,t) &= e^{x}(\Upsilon-4+\Theta)^2\frac{t^{2\upsilon}}{2!\upsilon^2},\\ f_3(x,t) &= e^{x}(\Upsilon-4+\Theta)^3\frac{t^{3\upsilon}}{3!\upsilon^3},\\ f_4(x,t) &= e^{x}(\Upsilon-4+\Theta)^4\frac{t^{4\upsilon}}{4!\upsilon^4},\\ f_5(x,t) &= e^{x}(\Upsilon-4+\Theta)^5\frac{t^{5\upsilon}}{5!\upsilon^5}.\end{aligned} \tag{65}$$

As a result, we get the approximate solution to Equations (46) and (47) obtained from the 5th iteration as follows:

$$\begin{aligned}f^{(5)}(x,t) = {}&e^{x} + e^{x}(\Upsilon-4+\Theta)^1\frac{t^{1\upsilon}}{1!\upsilon^1} + e^{x}(\Upsilon-4+\Theta)^2\frac{t^{2\upsilon}}{2!\upsilon^2}\\ &+e^{x}(\Upsilon-4+\Theta)^3\frac{t^{3\upsilon}}{3!\upsilon^3} + e^{x}(\Upsilon-4+\Theta)^4\frac{t^{4\upsilon}}{4!\upsilon^4}\\ &+e^{x}(\Upsilon-4+\Theta)^5\frac{t^{5\upsilon}}{5!\upsilon^5}.\end{aligned} \tag{66}$$

When $\upsilon = 1$ is employed in Equation (66), we get the following exact solution to Equations (57) and (58):

$$f(x,t) = e^{x+(\Theta+\Upsilon-4)t}. \tag{67}$$

Figure 7 represents the 2D graph of the exact solution and 5th iteration approximate solution attained by CSDJA at $\upsilon = 0.6, 0.7, 0.8, 0.9$, and $\upsilon = 1.0$ with $x = 2.0$ in the interval $t \in [0,1.0]$ for Example 4. Figure 8 represents the 2D graph of the 5th-step approximate and the exact solutions obtained by CSDJA at $\upsilon = 0.6, 0.7, 0.8, 0.9$, and $\upsilon = 1.0$ with $t = 0.2$,

$\Upsilon=3$, and $\Theta=2$ in the interval $x\in[0,1.0]$ for Example 4. These figures display that the approximate solution attained through CSDJA approach the exact solution when $\upsilon\longrightarrow 1.0$. The approximate solution corresponds with the precise solution at $\upsilon=1.0$, demonstrating the precision and effectiveness of the suggested approach.

Figures 9 and 10 show the 2D graph of the comparative study of the approximate solution obtained from the five iterations and exact solution obtained by the proposed method in the form of absolute and relative error with υ =1.0 at $x=2.0, \Upsilon=3$, and $\Theta=2$ in the interval $t\in[0,0.5]$ for Example 4, respectively. The comparison has proven that the suggested method's fifth-step approximation is quite near to the exact solution. Lastly, we draw the conclusion from the graphical results that the suggested method provides a solution in the form of a fractional series with high accuracy and few calculations.

Figures 11 and 12 display a 3D graph of the comparison research that compares the approximate solution and exact solution achieved in the fifth stage using the suggested method at $\upsilon=1.0$, $\Upsilon=3$, and $\Theta=2$ in the intervals $x\in[0,0.4]$ and $t\in[0,0.3]$ to Example 4, respectively. The comparison has proven that the suggested method's fifth-step approximations are quite near to the precise solutions. As a result, the suggested approach is a methodical, potent, and useful tool for analytic approximations and precise solutions to FODEs.

Tables 5 and 6 demonstrate that Example 4's absolute and relative error for the fourth phase is quite small. If we consider the fifth-step approximation, the absolute and relative error will be even lower. The approximation is rapidly getting closer to the exact solution as a result of the accuracy of our suggested method being shown by the absolute and relative error processes. We anticipate that our approach will be a key step in obtaining solutions to numerous FODEs with a physical interest in the domains of applied mathematics and engineering.

Recurrence error has been used to numerically demonstrate the convergence of the approximate solution to the exact solution for Example 4 for appropriately set grid locations in the interval $x\in[0,1.0]$ and $t\in[0,1.0]$ when $\Upsilon=3$, and $\Theta=2$ as in Table 7. Table 6 demonstrates how the approximate solution found by the suggested method in the fifth step progressively approaches the exact solution as the order of the fractional-order derivative increases. Higher degrees of accuracy and convergence rates have been shown for the recurrent error analysis. We conclude that the suggested approach is a practical and efficient procedure for resolving specific classes of FODEs with a minimum number of calculations and iterations.

Table 8 also compares the absolute and relative error of the approximations from the fifth iterations obtained by the CSDJA of Example 4 at plausible short-listed grid points in the range $x\in[0,1.0]$ and $t\in[0,1.0]$ when $\Upsilon=3$ and $\Theta=2$ with the absolute and relative error of the 5th-step approximations obtained by EDM [36]. The comparison has shown that the suggested method and EDM have the same outcomes. The comparison led us to the conclusion that CSDJA can be used in place of EDM to solve TFSHEs.

Example 5. Consider the following SH equation, which is nonlinear and time-fractional [36]:

$$T_t^{\upsilon}f(x,t)+(1-\Upsilon)f(x,t)+2\frac{\partial^2 f(x,t)}{\partial x^2}+\frac{\partial^4 f(x,t)}{\partial x^4}-f^2(x,t)+\left(\frac{\partial f(x,t)}{\partial x}\right)^2=0,\quad 0<\upsilon\le 1, \tag{68}$$

with the initial condition:

$$f(x,0)=e^x. \tag{69}$$

Using CST on both sides of Equation (68),

$$\mathfrak{I}_{\upsilon}\left[T_t^{\upsilon}f(x,t)+(1-\Upsilon)f(x,t)+2\frac{\partial^2 f(x,t)}{\partial x^2}+\frac{\partial^4 f(x,t)}{\partial x^4}-f^2(x,t)+\left(\frac{\partial f(x,t)}{\partial x}\right)^2\right]=0. \tag{70}$$

We get the following result from the Equation (70) by following the processes outlined in Section 3.

$$\mathfrak{I}_{\upsilon}[f(x,t)]=\frac{n}{m}e^x-\frac{n}{m}\mathfrak{I}_{\upsilon}[(1-\Upsilon)f(x,t)]-\frac{n}{m}\mathfrak{I}_{\upsilon}\left[2\frac{\partial^2 f(x,t)}{\partial x^2}\right]-\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\frac{\partial^4 f(x,t)}{\partial x^4}\right]+\frac{n}{m}\mathfrak{I}_{\upsilon}\left[f^2(x,t)\right]-\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\left(\frac{\partial f(x,t)}{\partial x}\right)^2\right]. \tag{71}$$

Applying inverse CST to Equation (71),

$$\begin{aligned}f(x,t)={}&\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}e^x\right\}-\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}[(1-\Upsilon)f(x,t)]\right\}\\&-\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[2\frac{\partial^2 f(x,t)}{\partial x^2}\right]\right\}-\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\frac{\partial^4 f(x,t)}{\partial x^4}\right]\right\}\\&+\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[f^2(x,t)\right]\right\}-\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\left(\frac{\partial f(x,t)}{\partial x}\right)^2\right]\right\}.\end{aligned} \tag{72}$$

Following the procedure given in Section 3 yields the following result:

$$\begin{aligned}B(f(x,t))={}&-\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}[(1-\Upsilon)f(x,t)]\right\}\\&-\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[2\frac{\partial^2 f(x,t)}{\partial x^2}\right]\right\}\\&-\mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\frac{\partial^4 f(x,t)}{\partial x^4}\right]\right\},\end{aligned} \tag{73}$$

$$C(f(x,t)) = \mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[f^2(x,t)\right]\right\} - \mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\left(\frac{\partial f(x,t)}{\partial x}\right)^2\right]\right\}, \tag{74}$$

$$\begin{aligned}\sum_{i=0}^{\infty} f_i(x,t) = {} & \mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}e^x\right\} - \mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[(1-\Upsilon)\sum_{i=0}^{\infty} f_i(x,t)\right]\right\} \\ & - \mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[2\frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty} f_i(x,t)\right]\right\} \\ & - \mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\frac{\partial^4}{\partial x^4}\sum_{i=0}^{\infty} f_i(x,t)\right]\right\} \\ & + \mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\sum_{i=0}^{\infty} f_i(x,t)\right]^2\right\} \\ & - \mathfrak{I}_{\upsilon}^{-1}\left\{\frac{n}{m}\mathfrak{I}_{\upsilon}\left[\left(\frac{\partial}{\partial x}\sum_{i=0}^{\infty} f_i(x,t)\right)^2\right]\right\}.\end{aligned} \tag{75}$$

Using the iteration process given in Section 3, the following results are obtained from Equation (75):

$$\begin{aligned} f_0(x,t) &= e^x, \\ f_1(x,t) &= e^x(\Upsilon-4)^1\frac{t^{1\upsilon}}{1!\upsilon^1}, \\ f_2(x,t) &= e^x(\Upsilon-4)^2\frac{t^{2\upsilon}}{2!\upsilon^2}, \\ f_3(x,t) &= e^x(\Upsilon-4)^3\frac{t^{3\upsilon}}{3!\upsilon^3}, \\ f_4(x,t) &= e^x(\Upsilon-4)^4\frac{t^{4\upsilon}}{4!\upsilon^4}, \\ f_5(x,t) &= e^x(\Upsilon-4)^5\frac{t^{5\upsilon}}{5!\upsilon^5}. \end{aligned} \tag{76}$$

As a result, we get the approximate solution to Equations (68) and (69) obtained from the 5th iteration as follows:

$$\begin{aligned} f^{(5)}(x,t) = {} & e^x + e^x(\Upsilon-4)^1\frac{t^{\upsilon}}{1!\upsilon^1} + e^x(\Upsilon-4)^2\frac{t^{2\upsilon}}{2!\upsilon^2} \\ & + e^x(\Upsilon-4)^3\frac{t^{3\upsilon}}{3!\upsilon^3} + e^x(\Upsilon-4)^4\frac{t^{4\upsilon}}{4!\upsilon^4} \\ & + e^x(\Upsilon-4)^5\frac{t^{5\upsilon}}{5!\upsilon^5}. \end{aligned} \tag{77}$$

When $\upsilon = 1$ is employed in Equation (77), we get the following precise solution to Equations (68) and (69):

$$f(x,t) = e^{x+(\Upsilon-4)t}. \tag{78}$$

Figure 13 represents the 2D graph of the 5th-step approximate and the exact solutions obtained by CSDJA at $\upsilon = 0.6, 0.7, 0.8, 0.9$, and $\upsilon = 1.0$ with $x = 3.0$ in the interval $t \in [0,1.0]$ for Example 5. Figure 14 represents the 2D graph of the 5th-step approximate and the exact solutions obtained by CSDJA at $\upsilon = 0.6, 0.7, 0.8, 0.9$, and $\upsilon = 1.0$ with $Y = 5$ and $t = 0.3$ in the interval $x \in [0, 1.0]$ for Example 5. One can notice the identical conclusions described for Examples 3 and 4.

Figures 15 and 16 show the 2D graph of the comparative study of the approximate solution obtained from the five iterations and exact solution obtained by the proposed method in the form of absolute and relative error with $\upsilon = 1.0$ at $x = 3.0$ and $Y = 5$ in the interval $t \in [0, 0.5]$ to Example 5, respectively. One can notice the identical conclusions described for Examples 3 and 4.

The accuracy of the fifth step approximation obtained using the suggested method and the precise solution is shown in a 3D graph in Figures 17 and 18 together with the absolute and relative error at $\upsilon = 1.0$ and $Y = 5$ in the intervals $x \in [0, 0.3]$ and $t \in [0, 0.3]$, to Example 5, respectively. From Figures 17 and 18, we yield the same outcomes as in Examples 3 and 4.

From Tables 9 and 10, one can observe the corresponding findings portrayed for Examples 3 and 4.

Recurrence error has been used to quantitatively demonstrate how the approximate solution for Example 5 converges to the exact solution for suitably specified grid locations in the interval $x \in [0, 1.0]$ and $t \in [0, 1.0]$ as in Table 11. One can perceive the equivalent verdicts depicted for Examples 3 and 4.

Table 12 also compares the absolute and relative error of the approximations from the fifth iterations obtained by the CSDJA of Example 5 at plausible short-listed grid points in the range $x \in [0,1.0]$ and $t \in [0,1.0]$ with the absolute and relative error of the 5th-step approximations obtained by EDM [36]. One can notice the identical conclusions described for Examples 3 and 4.

Finally, the primary advantages of the CSDJA are as follows, as shown by the numerical and graphical results: the suggested approach is a methodical, potent, and useful tool for both precise and approximate analytical solutions to FODEs. The strength of the suggested method is its modest size of computation, which allows it to be more efficient than existing numerical methods with fewer computations. In the form of absolute and relative error, the numerical results acquired by CSDJA are also contrasted with the other results obtained by EDM. Because of the comparison's great agreement with this method, CSDJA can be used as a substitute for the method EDM to solve TFSHEs. The higher degrees of accuracy and convergence rates were confirmed by the error analysis, demonstrating the suggested method's efficacy and reliability.

5. Conclusion

In this study, we solved TFSHEs in the sense of CD using the CSDJA. We were able to successfully solve both linear and nonlinear TFSHEs using the aforementioned approach. Results in graphs and numerical information have been used to prove the CSDJA's efficacy. According to Table 2, the magnitude of absolute and relative error ranges from $9.08162434143378 \times 10^{-14}$ to 0.00242649361746760660 and from $8.72552875759824810^{-14}$ to 0.00038536947079107625, respectively, in Example 3. According to Table 6, the magnitude of absolute and relative error ranges from $1.047606446036297 \times 10^{-12}$ to 0.00261926516767729820 and from $9.865985960312264 \times 10^{-13}$ to 0.00040774789828 359850, respectively, in Example 4. According to Table 10, the magnitude of absolute and relative error ranges from $2.297806389606194 \times 10^{-11}$ to 0.00304568436044938550 and from $2.079141200717798 \times 10^{-11}$ to 0.00045553880438 038890, respectively, in Example 5.

The results of the CSDJA are also contrasted with the EDM in the sense of absolute and relative error. Moreover, the numerical evidence for the convergence of the approximative solution to the exact solution is presented numerically.

The following are the main advantages of the CSDJA, as illustrated by the results obtained using the proposed technique, which demonstrates excellent agreement with EDM. The CSDJA is a suitable replacement tool for the He or Adomian polynomial-based methods used to solve FODEs. The pattern between the coefficients of the series solution made it simple to find accurate solution to numerical problems, and we did so to achieve the exact solutions as indicated in the five applications. The greater degree of accuracy and convergence rates have been proven by the absolute, relative, and recurrence error analysis. Making any large or small physical parametric assumptions about the problem is not necessary. So, it applies to both weak and strongly nonlinear systems, overcoming some of the inherent limits of standard perturbation techniques. The CSDJA can solve nonlinear problems without the aid of He's or Adomian polynomials. The number of calculations needed to solve nonlinear TFSHEs is extremely low. As a result, it performs far better than homotopy analysis and Adomian decomposition methods. As a solution to problems, the CSDJA provides a quick and simple procedure for figuring out the values of the fractional power series. Finally, the CSDJA can generate expansion solutions for linear and nonlinear TFSHEs without the need for perturbation, linearization, or discretization, in contrast to traditional analytic approximation methods. The results led us to the conclusion that our technique is easy to use, accurate, flexible, and effective. We intend to use the CSDJA to address other FODE systems that develop in different contexts in the future.

Data Availability

The numerical data used to support the findings of this study is included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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