Hindawi
Journal of Mathematics
Volume 2022, Article ID 3375246, 3 pages
https://doi.org/10.1155/2022/3375246

Research Article

On a Conjecture about the Saturation Number of Corona Product of Graphs

Mostafa Tavakoli

Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran

Correspondence should be addressed to Mostafa Tavakoli; m_tavakoli@ferdowsi.um.ac.ir

Received 7 June 2022; Accepted 20 August 2022; Published 28 September 2022

Academic Editor: Shaofang Hong

Let $G = (V_G, E_G)$ be a simple and connected graph. A set $M \subseteq E_G$ is called a matching if no two edges of M have a common endpoint. A matching M is maximal if it cannot be extended to a larger matching in G. The smallest size of a maximal matching is called the saturation number of G. In this paper, we confirm a conjecture of Alikhani and Soltani about the saturation number of corona product of graphs. We also present the exact value of $s(G \circ H)$ where H is a randomly matchable graph.

1. Introduction

All graphs considered in this paper are connected and simple; that is, they do not have loops and multiple edges [1–3]. For notation and graph theory terminology, we ingeneral follow [11, 12, 15].

Let $G = (V_G, E_G)$ be a graph. A collection of edges $M_G \subseteq E_G$ is called a **matching** of G if no two edges of M_G are adjacent. The vertices incident to the edges of a matching M_G are said to be **saturated** by M_G (or M_G-saturated); the others are said to be **unsaturated** (or M_G-unsaturated). A matching whose edges meet all vertices of G is called a **perfect matching** of G. If there does not exist a matching M_G'in G such that $|M_G| < |M_G'|$, then M_G is called a **maximum matching** of G. A matching M_G is **maximal** if it cannot be extended to a larger matching in G. The cardinality of any maximum matching, $\nu(G)$, and the cardinality of any smallest maximal matching in G, $s(G)$, are called the **matching number** and the **saturation number** of G, respectively.

If any maximal matching in G is also perfect (i.e., if $s(G) = |V_G|/2$), then G is called **randomly matchable**.

Smallest maximal matchings have a wide range of applications in real-world problems. For example, application of smallest maximal matchings related to a telephone switching network was presented in [4]. Finding a smallest maximal matching is NP-hard even for especial family of graphs (such as planar graphs), see [4–6]. Also, one can find some bounds for this invariant in [7–10]. See [10, 12,13] for more details on this topic. See [11–13] Recently, Alikhani and Soltani presented the following conjecture about the saturation number of corona product of graphs.

Conjecture. [14] Let G and H be two graphs and $|V_G| = n$. Then,

$$ns(H) \leq s(G \circ H) \leq ns(H) + \nu(G) + l, \tag{1}$$

where $\nu(G)$ is the size of a maximum matching M_G of the graph G and l is the number of M-unsaturated vertices of G.

In this paper, we confirm this conjecture. We also present some more efficient results on the saturation number of corona product of graphs.

For two graphs, $G = (V_G, E_G)$ and $H = (V_H, E_H)$. The **corona product** of G and H, denoted by $G \circ H$, is obtained from one copy of G and $|V_G|$ copies of H by joining each vertex of the i^{th} copy of H, $i \in \{1, \ldots, |V_G|\}$, to the i^{th} vertex of G, cf. [15]. In the following, for $g \in V_G$, H_g shows the copy of H in $G \circ H$ corresponding to g.

2. Main Results

The first result of this section is the proof of the conjecture mentioned in the previous section.

Theorem 1. *Let G and H be two graphs and $|V_G| = n$. Then,*

$$ns(H) \leq s(G \circ H) \leq ns(H) + \nu(G) + l, \quad (2)$$

where $\nu(G)$ is the size of a maximum matching M_G of the graph G and l is the number of M-unsaturated vertices of G.

Proof. First, we prove the upper bound. Let M_G be a maximum matching of G, and M_H be a maximal matching in H that $|M_H| = s(H)$. Also, suppose that vertices $g_1, \ldots, g_l$ are M-unsaturated vertices of G. There are two cases for H.

Case 1. H is a randomly matchable. Suppose that M_G' is a maximal matching in G that $|M_G| = s(G)$. Set

$$M = M_G' \cup \left(\cup_{i=1}^{n} M_{H_i} \right), \quad (3)$$

where M_{H_i} is the i^{th} copy of M_H, $i \in \{1, \ldots, n\}$, in H_i. Clearly, M is a maximal matching in $G \circ H$. Thus,

$$\begin{aligned} s(G \circ H) \leq |M| &= \left| M_G' \cup \left(\cup_{i=1}^{n} M_{H_i} \right) \right| \\ &= s(G) + ns(H) < ns(H) + \nu(G) + l. \end{aligned} \quad (4)$$

Case 2. H is not a randomly matchable. Thus, M_H is not a perfect matching. Suppose that h_j is a M_H-unsaturated of H. Set

$$M = M_G \cup \left(\cup_{i=1}^{n} M_{H_i} \right) \cup \left(\cup_{i=1}^{l} \left\{ h_j^i g_i \right\} \right), \quad (5)$$

where h_j^i is the copy of h_j in H_i corresponding to g_i. Easily one can check that M is a maximal matching in $G \circ H$. Therefore,

$$\begin{aligned} s(G \circ H) \leq |M| &= \left| M_G \cup \left(\cup_{i=1}^{n} M_{H_i} \right) \cup \left(\cup_{i=1}^{l} \left\{ h_i g_j^i \right\} \right) \right| \\ &= ns(H) + \nu(G) + l. \end{aligned} \quad (6)$$

Now, we prove the lower bound. Let M be a maximal matching of $G \circ H$. We consider two below cases for M.

Case 1. M does not have any edges e so that e has one end in G and one end in a copy of H. Hence, $|M \cap E_{H_i}| \geq s(H)$, and consequently, $ns(H) \leq s(G \circ H)$.

Case 2. Suppose $\left\{ g_1 h_{j_1}^1, g_2 h_{j_2}^2, \ldots, g_k h_{j_k}^k \right\}$ be all edges of M such that $g_i \in V_G$ and $h_{j_t}^i \in V_{H_i}$. Then, for each $i \notin \{1, \ldots, k\}$, we have $|M \cap E_{H_i}| = s(H)$. Also, for each $i \in \{1, \ldots, k\}$, we have $s(H) - 1 \leq |M \cap E_{H_i}| \leq s(H)$. On the other hand,

$$\{ g_1 h_1, g_2 h_2, \ldots, g_k h_k \} \cup \left(\cup_{i=1}^{n} \left(M \cap E_{H_i} \right) \right) \subseteq M. \quad (7)$$

Therefore, $ns(H) \leq s(G \circ H)$.

The next theorem gives the exact value of $s(G \circ H)$ for some family of graphs. □

Theorem 2. *Let G be a graph of order n. If H is a randomly matchable graph, then*

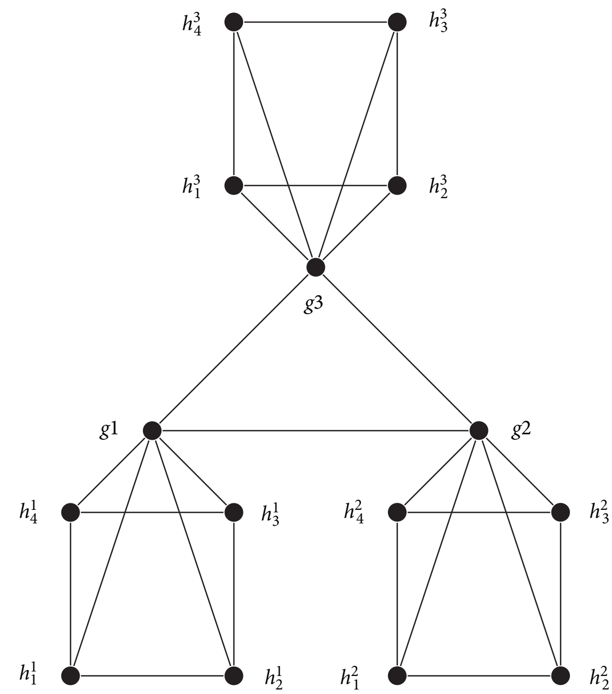

Figure 1: $C_3 \circ C_4$.

$$s(G \circ H) = ns(H). \quad (8)$$

Proof. By Theorem 1, we have $s(G \circ H) \geq ns(H)$. Then, it is sufficient to prove that $s(G \circ H) \leq ns(H)$. Suppose h_j^i is the copy of h_j in H_i corresponding to g_i. Let M_H is a maximal matching of H, and M_{H_i} is the i^{th} copy of M_H, $i \in \{1, \ldots, n\}$, in H_i. Assume that $V_G = \{g_1, \ldots, g_n\}$ and $h_l h_t \in M_H$. Set

$$M = \left\{ g_1 h_t^1, g_2 h_t^2, \ldots, g_n h_t^n \right\} \cup \left(\cup_{i=1}^{n} \left(M_{H_i} - \left\{ h_l^i h_t^i \right\} \right) \right). \quad (9)$$

(For more illustration, see Figure 1 which is $C_3 \circ C_4$. Suppose $C_4 := h_1, h_2, h_3, h_4, h_1$. Consider the maximal matching $M_{C_4} = \{h_1 h_2, h_3 h_4\}$. Since C_4 is a randomly matchable graph, then $M = \{g_1 h_1^1, g_2 h_1^2, g_3 h_1^3\} \cup \{h_3^1 h_4^1, h_3^2 h_4^2, h_3^3 h_4^3, h_3^4 h_4^4\}$). According to this fact that H is a randomly matchable graph, then M is a maximal matching in $G \circ H$. Thus,

$$\begin{aligned} s(G \circ H) &\leq |M| \\ &= \left| \left\{ g_1 h_t^1, g_2 h_t^2, \ldots, g_n h_t^n \right\} \cup \left(\cup_{i=1}^{n} \left(M_{H_i} - \left\{ h_l^i h_t^i \right\} \right) \right) \right|. \end{aligned} \quad (10)$$

On the other hand, $|\{\{g_1 h_t^1, g_2 h_t^2, \ldots, g_n h_t^n\} \cup (\cup_{i=1}^{n} (M_{H_i} - \{h_l^i h_t^i\}))| = n(s(H) - 1) + n = ns(H)$. Therefore, $s(G \circ H) \leq ns(H)$. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] C. Berge, *Graphs and Hypergraphs*, Open Access and Academic Publisher, Amsterdam, Netherlands, 1973.

[2] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, CRC Press, Boca Raton, FL, USA, 2010.

[3] I. Gutman and S. J. Cyvin, *Introduction to the Theory of Benzenoid Hydrocarbons*, Springer, Berlin, Heidelberg, 1989.

[4] M. Yannakakis and F. Gavril, "Edge dominating sets in graphs," *SIAM Journal on Applied Mathematics*, vol. 38, no. 3, pp. 364–372, 1980.

[5] M. Demange and T. Ekim, "Minimum maximal matching is NP-hard in regular bipartite graphs," in *Theory and Applications of Models of Computations*, pp. 364–374, Springer, Berlin, Germany, 2008.

[6] J. D. Horton and K. Kilakos, "Minimum edge dominating sets," *SIAM Journal on Discrete Mathematics*, vol. 6, no. 3, pp. 375–387, 1993.

[7] T. Biedl, E. D. Demaine, C. A. Duncan, R. Fleischer, and S. G. Kobourov, "Tight bounds on maximal and maximum matchings," *Discrete Mathematics*, vol. 285, no. 1-3, pp. 7–15, 2004.

[8] L. Lovász and M. D. Plummer, *Matching Theory*, american mathematical society, Providence, RI, USA, 1986.

[9] N. Tratnik and P. Žigert Pleteršek, "Saturation number of nanotubes," *Ars Mathematica Contemporanea*, vol. 12, no. 2, pp. 337–350, 2017.

[10] M. Zito, "Small maximal matchings in random graphs," *Theoretical Computer Science*, vol. 297, no. 1-3, pp. 487–507, 2003.

[11] V. Andova, F. Kardoš, and R. Škrekovski, "Sandwiching saturation number of fullerene graphs, MATCH Commun," *Journal of Computational Chemistry*, vol. 73, pp. 501–518, 2015.

[12] T. Došlić, "Saturation number of fullerene graphs," *Journal of Mathematical Chemistry*, vol. 43, no. 2, pp. 647–657, 2008.

[13] T. Došlić and I. Zubac, "Saturation number of benzenoid graphs, MATCH commun," *Journal of Computational Chemistry*, vol. 73, pp. 491–500, 2015.

[14] S. Alikhani and N. Soltani, "On the saturation number of graphs," *Iranian Journal of Mathematical Chemistry*, vol. 9, pp. 289–299, 2018.

[15] S. Klavžar and M. Tavakoli, "Dominated and dominator colorings over (edge) corona and hierarchical products," *Applied Mathematics and Computation*, vol. 390, Article ID 125647, 2021.