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Research Article

Generalized Derivations and Generalized Jordan Derivations on C^*-Algebras through Zero Products

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Let A be a unital C^*-algebra and X be a unitary Banach A-bimodule. In this paper, we characterize continuous generalized derivations and generalized Jordan derivations as form $D: A \longrightarrow X$ through the action on zero product. In other words, we show that under some conditions on elements of A, a linear map on A can be a generalized Jordan derivation.

1. Introduction and Preliminaries

Let A be a unital Banach algebra with the unit e_A and X be a unitary (Banach) A-bimodule. A linear map $D: A \longrightarrow X$ is said to be a derivation (resp., generalized derivation) if for each $a, b \in A$,

$$D(ab) = D(a) \cdot b + a \cdot D(b), \quad [\text{resp.}, D(ab) = D(a) \cdot b + a \cdot D(b) - a \cdot D(e_A) \cdot b], \tag{1}$$

where "$\circ$" denotes the Jordan product $a \circ b = ab + ba$ on A, and "$\cdot$" denotes the Jordan product on X defined through

$$a \cdot x = x \cdot a = a \cdot x + x \cdot a, \quad a \in A, \ x \in X. \tag{2}$$

The mentioned map is called a Jordan derivation (resp., generalized Jordan derivation) if

$$D(a \circ b) = D(a) \cdot b + a \cdot D(b), \quad [\text{resp.}, D(a \circ b) = D(a) \cdot b + a \cdot D(b) - a \cdot D(e_A) \cdot b - b \cdot D(e_A) \cdot a], \tag{3}$$

for all $a, b \in A$. By the usual polarization, the Jordan derivation (resp., generalized Jordan derivation) identity is equivalent to assuming that

$$D(a^2) = D(a) \cdot a, \quad [\text{resp.}, D(a^2) = D(a) \cdot a - a \, D(e_A)a], \ a \in A. \tag{4}$$

Clearly, each (generalized) derivation is a (generalized) Jordan derivation, but the converse is not true in general. There are plenty of known examples of Jordan derivations that are not a derivation and can be found in works of literature. For the correctness of the converse, Johnson in [1] (Theorem 6.3) proved that every continuous Jordan derivation from C^*-algebra A in any A-bimodule X is a derivation.

Recall from [2] that a C^*-algebra A is called a W^*-algebra if it is a dual space as a Banach space. Note that every W^*-algebra is unital.

The set of idempotents of given algebra A is denoted by $\mathcal{I}(A)$. Let $\mathrm{alg}\mathcal{I}(A)$ be the subalgebra of A generated by idempotents. We say that a Banach algebra A is generated by idempotents, if $A = \overline{\mathrm{alg}\mathcal{I}(A)}$, where $\overline{\mathrm{alg}\mathcal{I}(A)}$ denotes the closure of $\mathrm{alg}\mathcal{I}(A)$. Examples of Banach algebras with the last property include all W^*-algebras, the group algebra $L^1(G)$ for a compact group G, and also topologically simple Banach algebras containing a nontrivial idempotent [3]. In other words, such Banach algebras are generated by idempotents. Another classes of Banach algebras with the property that $A = \overline{\mathrm{alg}\mathcal{I}(A)}$ are available in [3].

Let A be a Banach algebra and X be an arbitrary Banach space. We say a continuous bilinear mapping $\phi\colon A \times A \longrightarrow X$ preserves zero products if

$$ab = 0 \Longrightarrow \phi(a,b) = 0, \quad a,b \in A. \tag{5}$$

The study of zero products preserving bilinear maps has been initiated by Alaminos et al. in [4] for a very special setting, and then, it was studied in [3] for the general case. Motivated by (5), the following concept was presented in [3].

Definition 1. A Banach algebra A has the property $(\mathbb{B})$ if for every continuous bilinear mapping $\phi\colon A \times A \longrightarrow X$, where X is an arbitrary Banach space, and condition (5) implies that $\phi(ab,c) = \phi(a,bc)$, for all $a,b,c \in A$.

It follows from [3] (Theorem 2.11) that C^*-algebras, group algebras, and Banach algebras that are generated by idempotents have the property $(\mathbb{B})$, see also [5] (Lemma 2.1) for group algebras in a different view.

It should be pointed out that the property $(\mathbb{B})$ is a powerful tool for characterizing homomorphisms, derivations, and Jordan derivations on that class of algebras through the action on zero products. We refer the reader to [3, 5–12] for a full account of the topic and references therein.

The next result which is closely related to the property $(\mathbb{B})$ plays a key role in the sequel.

Theorem 1 ([6], Theorem 2.2). *Let A be a C^*-algebra, X be a Banach space, and $\phi\colon A \times A \longrightarrow X$ be a continuous bilinear mapping such that*

$$ab = ba = 0 \Rightarrow \phi(a,b) = 0, \quad a,b \in A. \tag{6}$$

Then,

$$\phi(xa, yb) + \phi(bx, ay) = \phi(x, ayb) + \phi(bxa, y), \tag{7}$$

for all $a,b,x,y \in A$.

In this paper, we consider the subsequent conditions on a linear map D from a Banach algebra A into a Banach A-bimodule X for each $a,b,c \in A$.

(1) $[(\mathbb{G}1)]$ $ab = 0 \Rightarrow D(a)\cdot b + a\cdot D(b) = 0$
(2) $[(\mathbb{G}2)]$ $ab = bc = 0 \Rightarrow a\cdot D(b)\cdot c = 0$
(3) $[(\mathbb{J}1)]$ $ab = ba = 0 \Rightarrow D(a)\cdot b + a\cdot D(b) = 0$
(4) $[(\mathbb{J}2)]$ $ab = ba = 0 \Rightarrow D(a)\cdot b + a\cdot D(b) = 0$
(5) $[(\mathbb{J}3)]$ $ab = ba = 0 \Rightarrow a\cdot D(b) + b\cdot D(a) = 0.$

Our purpose is to investigate whether the conditions above can characterize generalized derivations and generalized Jordan derivations. Indeed, which of the above are sufficient conditions to be the generalized derivation of a continuous linear map.

2. Generalized Derivations on C^*-Algebras

In this section, we characterize generalized derivations from unital C^*-algebra A into unitary Banach A-bimodule X that satisfy condition $(\mathbb{G}1)$ or $(\mathbb{G}2)$, through their action on zero products.

Theorem 2. *Let A be a unital C^*-algebra. If $D\colon A \longrightarrow X$ is a continuous linear map satisfying $(\mathbb{G}1)$, then D is a generalized derivation.*

Proof. Define a continuous bilinear mapping $\phi\colon A \times A \longrightarrow X$ by $\phi(a,b) = D(a)\cdot b + a\cdot D(b)$. Then, $\phi(a,b) = 0$ whenever $ab = 0$, and so the property $(\mathbb{B})$ gives

$$\begin{aligned} D(a)\cdot bc + a\cdot D(bc) &= \phi(a,bc) = \phi(ab,c) \\ &= D(ab)\cdot c + ab\cdot D(c), \quad a,b,c \in A. \end{aligned} \tag{8}$$

Taking $c = e_A$ in (8), we get

$$D(a)\cdot b + a\cdot D(b) = D(ab) + ab\cdot D(e_A), \tag{9}$$

for all $a,b \in A$. Interchanging a into e_A in (9), we get $D(e_A)\cdot b = b\cdot D(e_A)$ for all $b \in A$. Thus, it follows from (9) that

$$D(ab) = D(a)\cdot b + a\cdot D(b) - a\cdot D(e_A)\cdot b, \tag{10}$$

for all $a,b \in A$. Therefore, D is a generalized derivation.

Note that if D is a generalized derivation, then the linear map $\delta\colon A \longrightarrow X$ defined by $\delta(a) = D(a) - D(e_A)\cdot a$ is a derivation, and thus, we get the following result. □

Corollary 1 ([4], Corollary 4.2). *Let A be a unital C^*-algebra. If $D\colon A \longrightarrow X$ is a continuous linear map satisfying $(\mathbb{G}1)$, then $a\cdot D(e_A) = D(e_A)\cdot a$ for all $a \in A$, and there is a derivation $\delta\colon A \longrightarrow X$ such that $D(a) = D(e_A)\cdot a + \delta(a)$.*

Assume that D is a continuous linear map from unital C^*-algebra A into unitary A-bimodule X such that $D(b)a + b\,D(a) = 0$ for all $a,b \in A$ with $ab = 0$. Then, D is a generalized Jordan derivation. Indeed, by a similar proof of the preceding theorem, we can obtain

$$D(ab) = D(b)\cdot a + b\cdot D(a) - D(e_A)\cdot ab, \quad a,b\in A. \tag{11}$$

Taking $a = b$, we find $D(a^2) = D(a)\cdot a - a\ D(e_A)a$ for all $a\in A$. Moreover, if $D(e_A) = 0$, then D is a Jordan derivation, and hence, it is a derivation by [1] (Theorem 6.3).

The following fundamental example has been demonstrated by Johnson in [1].

Example 1. Let

$$A = \left\{\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22}\end{bmatrix}: a_{11}, a_{12}, a_{22}\in\mathbb{C}\right\}. \tag{12}$$

We make $X = \mathbb{C}$ an A-bimodule by defining

$$\begin{aligned} a\lambda &= a_{22}\lambda, \\ \lambda a &= \lambda a_{11}, \\ \lambda &\in \mathbb{C},\ a\in A. \end{aligned} \tag{13}$$

Consider the linear map $D: A\longrightarrow X$ defined via $D(a) = a_{12}$. Then, $D(ab) = D(b)\cdot a + b\cdot D(a)$ for all $a,b\in A$, and $D(e_A) = 0$. In particular, D is a (generalized) Jordan derivation, but it is not a (generalized) derivation. Therefore, Johnson's result is not valid for unital Banach algebras instead of C^*-algebras in general.

In the next result, we characterize generalized derivations by using condition $(\mathbb{G}2)$.

Theorem 3. *Suppose that A is a unital C^*-algebra and $D: A\longrightarrow X$ is a continuous linear map such that condition $(\mathbb{G}2)$ holds. Then, D is a generalized derivation.*

Proof. Pick $a_0, b_0\in A$ such that $a_0b_0 = 0$ and define a continuous bilinear mapping $\phi: A\times A\longrightarrow X$ by $\phi(a,b) = aD(ba_0)b_0$, for all $a,b\in A$. Then, $\phi(a,b) = 0$ whenever $ab = 0$. Using the property $(\mathbb{B})$ with $c = e_A$, we obtain

$$abD(a_0)b_0 = \phi(ab, e_A) = \phi(a,b) = a\cdot D(ba_0)\cdot b_0, \quad a,b\in A. \tag{14}$$

We have derived this identity under the assumption that $a_0, b_0\in A$ such that $a_0b_0 = 0$. Let now $a_1, b_1\in A$ be fixed. We may apply the property $(\mathbb{B})$ for $\psi: A\times A\longrightarrow X$ given through

$$\psi(a,b) = a_1b_1\cdot D(a)\cdot b - a_1\cdot D(b_1a)\cdot b, \tag{15}$$

for all $a,b\in A$, and hence, we conclude that

$$a_1b_1\cdot D(ab) - a_1\cdot D(b_1ab)\cdot c = a_1b_1\cdot D(a)bc - a_1D(b_1a)\cdot bc. \tag{16}$$

Taking $a_1 = c = a = e_A$ in the above equality, we reach

$$D(b_1b) = D(b_1)\cdot b + b_1\cdot D(b) - b_1\cdot D(e_A)\cdot b, \tag{17}$$

for all $b_1, b\in A$. This completes the proof. □

3. Generalized Jordan Derivations on C^*-Algebras

In this section, we prove that each linear mapping D from unital C^*-algebra A into unitary Banach A-bimodule X that satisfies one of conditions $(\mathbb{J}1)$, $(\mathbb{J}2)$, and $(\mathbb{J}3)$ is a generalized Jordan derivation.

Our first main theorem is indicated as follows.

Theorem 4. *Let A be a W^*-algebra and $D: A\longrightarrow X$ be a continuous linear map satisfying $(\mathbb{J}1)$. Then, D is a generalized Jordan derivation.*

Proof. Define a bilinear mapping $\phi: A\times A\longrightarrow X$ by

$$\phi(a,b) = D(a)\cdot b + a\cdot D(b), \quad a,b\in A. \tag{18}$$

Then, $ab = ba = 0$ implies that $\phi(a,b) = 0$. Applying Theorem 1 by putting $x = y = e_A$, we arrive

$$\phi(a,b) + \phi(b,a) = \phi(e_A, ab) + \phi(ba, e_A), \tag{19}$$

for all $a,b\in A$. This means that

$$a\cdot D(b) + D(a)\cdot b = D(a\circ b) + D(e_A)\cdot ab + ba\cdot D(e_A). \tag{20}$$

Replacing b by a in (20) gives

$$2D(a^2) = 2(D(a)\cdot a) - D(e_A)\cdot a^2 - a^2\cdot D(e_A), \tag{21}$$

for all $a\in A$. Here, we claim that $a\cdot D(e_A) = D(e_A)\cdot a$ for all $a\in A$. Let p be an idempotent in A. Substituting a by p in (21), we have

$$\begin{aligned} 2D(p) &= 2p\cdot D(p) + 2D(p)\cdot p \\ &\quad - D(e_A)\cdot p - p\cdot D(e_A). \end{aligned} \tag{22}$$

We multiply (22) by p on the left, and we obtain

$$\begin{aligned} 2p\cdot D(p) &= 2p\cdot D(p) + 2p\cdot D(p)\cdot p \\ &\quad - p\cdot D(e_A)\cdot p - p\cdot D(e_A), \end{aligned} \tag{23}$$

and so

$$2p\cdot D(p)\cdot p - p\cdot D(e_A)\cdot p - p\cdot D(e_A) = 0. \tag{24}$$

Similarly, by multiplying both sides of (22) by p, we arrive at

$$2p\cdot D(p)\cdot p - D(e_A)\cdot p - p\cdot D(e_A)\cdot p = 0. \tag{25}$$

From (24) and (25), it follows that $p\cdot D(e_A) = D(e_A)\cdot p$ for all idempotent $p\in A$. By Lemma 1.7.5 and Proposition 1.3.1 of [2], every self-adjoint element $x\in A$ is the limit of a sequence of linear combinations of projections in A. Therefore, $x\cdot D(e_A) = D(e_A)\cdot x$ for each self-adjoint element x in A. Now, each arbitrary element $a\in A$ can be written as $a = x + iy$ where x, y are self-adjoint elements of A. Hence,

$$a \cdot D(e_A) = (x + iy) \cdot D(e_A) = D(e_A) \cdot (x + iy) \\ = D(e_A) \cdot a, \tag{26}$$

for all $a \in A$. Thus, equality (21) implies that

$$D(a^2) = D(a) \cdot a - a \cdot D(e_A) \cdot a, \tag{27}$$

for all $a \in A$. Consequently, D is a generalized Jordan derivation.

It is well-known that there are two products on A^{**}, the second dual space of a Banach algebra A, called the first and second Arens products which make A^{**} into a Banach algebra ([13], Definition 2.6.16). If these products coincide on A^{**}, then A is said to be Arens regular. It is shown in Chapter 2 of [13] that every C^*-algebra A is Arens regular. Moreover, the second dual of each C^*-algebra is a W^*-algebra.

According to [14], for each Banach A-bimodule X, X^{**} turns into a Banach A^{**}-bimodule where A^{**} equipped with the first Arens product. The module actions are defined by

$$\Phi \cdot u = w^* - \lim_i \lim_j a_i \cdot x_j, \\ u \cdot \Phi = w^* - \lim_j \lim_i x_j \cdot a_i, \\ \Phi \in A^{**}, u \in X^{**}, \tag{28}$$

where $\{a_i\}_{i \in I}$ and $\{x_i\}_{j \in I}$ are nested in A and X that converge, in w^*-topologies, to Φ and u, respectively. One may refer to the monograph of Dales [13] for a full account of Arens product and w^*-continuity of the above structures.

There exists another related concept of generalized derivation, which appeared in [15] for the first time. Let A be a Banach algebra and X be an A-bimodule. A linear operator D: $A \longrightarrow X$ is said to be a generalized derivation if there exists $\xi \in X^{**}$ such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) - a \cdot \xi \cdot b, \quad a, b \in A. \tag{29}$$

It should be noted that if A is unital and D is a generalized derivation, then by [3] (Proposition 4.2) equality (29) converts to

$$D(ab) = D(a) \cdot b + a \cdot D(b) - a \cdot D(e_A) \cdot b, \tag{30}$$

for all $a, b \in A$, and hence, D is a generalized derivation in the usual sense. Motivated by (29), we introduce the concept of generalized Jordan derivation as follows. A linear operator D: $A \longrightarrow X$ is said to be a generalized Jordan derivation if there exists $\xi \in X^{**}$ such that

$$D(a \circ b) = D(a) \cdot b + a \cdot D(b) - a \cdot \xi \cdot b - b \cdot \xi \cdot a, \tag{31}$$

for all $a, b \in A$. Similarly, if A is unital and D satisfies in (31) for some $\xi \in X^{**}$, then

$$D(ab) = D(a) \cdot b + a \cdot D(b) - a \cdot D(e_A) \\ \cdot b - b \cdot D(e_A) \cdot a, \quad a, b \in A. \tag{32}$$

In what follows, we prove Theorem 4 to the C^*-algebra case. First, note that the linear span of projections is dense in a unital C^*-algebra of real rank zero [16]; hence, the conclusion of Theorem 4 is also valid for such C^*-algebras.

Applying the techniques in the proof of Theorem 4.1 from [6], we have the upcoming result. □

Theorem 5. *Let A be a unital C^*-algebra and D: $A \longrightarrow X$ be a continuous linear map satisfying* ($\mathbb{J}1$). *Then, D is a generalized Jordan derivation.*

Proof. It follows from Theorem 1 that the bilinear map ϕ: $A \times A \longrightarrow X$ defined through

$$\phi(a, b) = D(a) \cdot b + a \cdot D(b) \tag{33}$$

fulfills

$$xa \cdot D(yb) + D(xa) \cdot yb + bx \cdot D(ay) \\ + D(bx) \cdot ay - x \cdot D(ayb) - D(x) \\ \cdot ayb - bax \cdot D(y) - D(bxa) \cdot y = 0, \tag{34}$$

for all $a, b, x, y \in A$. Furthermore, the Arens regularity of A, the w^*-w^*-continuity of D^{**}, and the separate weak continuity of the module operations on X^{**} necessitate that

$$xa \cdot D^{**}(yb) + D^{**}(xa) \cdot yb + bx \cdot D^{**}(ay) \\ + D^{**}(bx) \cdot ay - x \cdot D^{**}(ayb) - D^{**}(x) \\ \cdot ayb - bax \cdot D^{**}(y) - D^{**}(bxa) \cdot y = 0, \tag{35}$$

for all $a, b, x, y \in A^{**}$. Take $\xi = D^{**}(e_{A^{**}}) \in X^{**}$. Then, it follows from equality (35) by putting $x = y = e_{A^{**}}$ that

$$a \cdot D^{**}(b) + D^{**}(a) \cdot b = D^{**}(a \circ b) + \xi \cdot ab + ba \cdot \xi, \tag{36}$$

for all $a, b \in A^{**}$. In particular, we have

$$D(a \circ b) = a \cdot D(b) + D(a) \cdot b - \xi \cdot ab - ba \cdot \xi, \quad a, b \in A. \tag{37}$$

Similar to the proof of Theorem 4, we have $a \cdot \xi = \xi \cdot a$ for all $a \in A^{**}$. Hence, by (37),

$$D(a \circ b) = a \cdot D(b) + D(a) \cdot b - a \cdot \xi \cdot b - b \cdot \xi, \quad a, b \in A. \tag{38}$$

It remains to show that $\xi \cdot a \in X$ for all $a \in A$. In other words, it suffices to prove it for each positive element $a \in A$. Suppose that $a \in A$ be a positive element and $b \in A$ with $a = b^2$. According to (38),

$$\xi \cdot a = \xi \cdot b^2 = D(b) \cdot b - D(a) \in X. \tag{39}$$

Consequently, D is a generalized Jordan derivation. □

Corollary 2. *Let A be a unital C^*-algebra. If D: $A \longrightarrow X$ is a continuous linear map satisfying* ($\mathbb{J}1$)*, then D is a derivation if and only if $D(e_A) = 0$.*

Proof. It is obvious that if D is a derivation, then $D(e_A) = 0$. On the other hand, it follows from Theorem 5 that D is a Jordan derivation when $D(e_A) = 0$, and therefore, D is a derivation by Johnson's result.

Here, we mention that each condition ($\mathbb{G}1$) and the implication

$$a \circ b = 0 \Rightarrow D(a)b + aD(b) = 0, \quad a, b \in A \ (D1), \tag{40}$$

imply ($\mathbb{J}1$), and hence, Theorems 4, 5 and Corollary 2 still work with condition ($\mathbb{J}1$) replaced by ($\mathbb{G}1$) or ($D1$). ☐

Theorem 6. *Let A be a W^*-algebra and $D\colon A \longrightarrow X$ be a continuous linear map satisfying ($\mathbb{J}2$). Then, D is a generalized Jordan derivation.*

Proof. Since $p(e_A - p) = (e_A - p)p = 0$ for all idempotent $p \in A$, condition ($\mathbb{J}2$) implies that

$$2D(p) - 2p \cdot D(p) - 2D(p) \cdot p + p \cdot D(e_A) + D(e_A) \cdot p = 0. \tag{41}$$

Multiplying both sides of (41) by p on the left, we obtain

$$\begin{aligned} &2p \cdot D(p) - 2p \cdot D(p) - 2p \cdot D(p) \cdot p \\ &\quad + p \cdot D(e_A) \cdot p + p \cdot D(e_A) \cdot p = 0, \end{aligned} \tag{42}$$

and so

$$2p \cdot D(p) \cdot p - p \cdot D(e_A) \cdot p - p \cdot D(e_A) = 0. \tag{43}$$

Similarly, by multiplying (41) on the right by p, we arrive at

$$2p \cdot D(p) \cdot p - D(e_A) \cdot p - p \cdot D(e_A) \cdot p = 0. \tag{44}$$

From (43) and (44), we get $p \cdot D(e_A) = D(e_A) \cdot p$ for all idempotent $p \in A$. Now, similar to the proof of Theorem 4, we conclude that

$$a \cdot D(e_A) = D(e_A) \cdot a, \quad a \in A. \tag{45}$$

Applying Theorem 1 to the bilinear map $\phi\colon A \times A \longrightarrow X$ defined by

$$\phi(a, b) = D(a) \cdot b + a \cdot D(b), \quad a, b \in A, \tag{46}$$

we find

$$\begin{aligned} &xa \cdot D(yb) + D(xa) \cdot yb + bx \cdot D(ay) + D(bx) \cdot ay \\ &= x \cdot D(ayb) + D(x) \cdot ayb + bax \cdot D(y) + D(bxa) \cdot y, \end{aligned} \tag{47}$$

for all $a, b, x, y \in A$. Switching x, y by e_A and using (45), we reach

$$\begin{aligned} D(a) \cdot b + a \cdot D(b) = &D(ab) + ab \cdot D(e_A) \\ &+ D(ba) + ba \cdot D(e_A). \end{aligned} \tag{48}$$

Replacing b by a in (48), it concludes that

$$D(a^2) = D(a) \cdot a - a \cdot D(e_A) \cdot a, \tag{49}$$

for all $a \in A$. Thus, D is a generalized Jordan derivation.

Similar to the proof of Theorem 5, we can obtain the following corollary. ☐

Corollary 3. *Let A be a unital C^*-algebra and $D\colon A \longrightarrow X$ be a continuous linear map satisfying ($\mathbb{J}2$). Then, D is a generalized Jordan derivation.*

It should be pointed out that each of the conditions

$$\begin{aligned} ab = 0 &\Rightarrow D(a) \cdot b + a \cdot D(b) = 0, \quad a, b \in A \ (D2), \\ a \circ b = 0 &\Rightarrow D(a) \cdot b + a \cdot D(b) = 0, \quad a, b \in A \ (D3), \end{aligned} \tag{50}$$

implies ($\mathbb{J}2$), and therefore, Theorem 6 and Corollary 3 are true if the condition ($\mathbb{J}2$) replaced by either ($D2$) or ($D3$).

Theorem 7. *Let A be a W^*-algebra and $D\colon A \longrightarrow X$ be a continuous linear map such that the condition ($\mathbb{J}3$) holds. Then, $D(a) = a \cdot D(e_A)$ for all $a \in A$. In particular, D is a generalized derivation.*

Proof. It is obvious that for every idempotent $p \in A$, $p(e_A - p) = (e_A - p)p = 0$. Hence,

$$p \cdot D(e_A - p) + (e_A - p) \cdot D(p) = 0. \tag{51}$$

A simple calculation shows that $D(p) = p \cdot D(p)$. Let A_{sa} denote the set of self-adjoint elements of A and $x \in A_{\text{sa}}$. Then, $x = \sum_{k=1}^n \lambda_k p_k$, where $\{\lambda_k\}$ are real numbers and $\{p_k\}$ is an orthogonal family of projections in A. Since $p_i p_j = p_j p_i = 0$ for $i \neq j$, condition ($\mathbb{J}3$) implies that $p_i \cdot D(p_j) + p_j \cdot D(p_i) = 0$ for all i, j with $i \neq j$. Thus, for each $x \in A_{\text{sa}}$,

$$\begin{aligned} D(x^2) &= D\left(\sum_{k=1}^n \lambda_k^2 p_k^2\right) = \sum_{k=1}^n \lambda_k^2 D(p_k^2) \\ &= \left(\sum_{k=1}^n \lambda_k p_k\right) \cdot \left(\sum_{k=1}^n \lambda_k D(p_k)\right) = x \cdot D(x). \end{aligned} \tag{52}$$

It follows from the linearity of D that $D(xy + yx) = x \cdot D(y) + y \cdot D(x)$ for each $x, y \in A_{\text{sa}}$. Now, each arbitrary element $a \in A$ can be written as $a = x + iy$ for $x, y \in A_{\text{sa}}$. Therefore,

$$\begin{aligned} D(a^2) &= D(x^2 - y^2 + i(xy + yx)) \\ &= D(x^2) - D(y^2) + iD(xy + yx) \\ &= x \cdot D(x) - y \cdot D(y) + i(x \cdot D(y) + y \cdot D(x)) \\ &= a \cdot D(a). \end{aligned} \tag{53}$$

Thus, $D(a^2) = a \cdot D(a)$ for all $a \in A$. Replacing a by $a + e_A$, we obtain $D(a) = a\ D(e_A)$ for all $a \in A$.

The next result is a direct consequence of Theorem 7. We include it without proof. ☐

Corollary 4. *Let A be a unital C^*-algebra. If $D\colon A \longrightarrow X$ is a continuous linear map satisfying ($\mathbb{J}3$), then $D(a) = a \cdot D(e_A)$ for all $a \in A$. Moreover, if $D(e_A) = 0$, then $D \equiv 0$.*

One should remember that Theorem 1 has an important role in characterizing generalized Jordan derivations on C^*-algebras. The following result is another analogous criterion for Banach algebra with the property that $A = \overline{\text{alg}\mathcal{J}(A)}$.

Theorem 8 ([7], Corollary 3.6). *Let A be a unital Banach algebra, X be a Banach space, and ϕ: $A \times A \longrightarrow X$ be a continuous bilinear mapping such that*

$$ab = ba = 0 \Rightarrow \phi(a, b) = 0, \quad a, b \in A. \tag{54}$$

Then,

$$\phi(a, x) + \phi(x, a) = \phi(ax, e_A) + \phi(e_A, xa), \tag{55}$$

for all $a \in A$ and $x \in \mathcal{J}(A)$. In particular, if A is generated by idempotents, then

$$\phi(a, b) + \phi(b, a) = \phi(ab, e_A) + \phi(e_A, ba), \quad a, b \in A. \tag{56}$$

Since all Banach algebras which are generated by idempotents have the property ($\mathbb{B}$), Theorems 2 and 3 remain valid for such algebras. On the other hand, by using the preceding theorem, some results of the current paper can be proved for unital (Arens regular) Banach algebras with the property that $A = \overline{\text{alg}\mathcal{J}(A)}$.

Data Availability

Not applicable. In fact, all results are obtained without any software and found by manual computations. In other words, the manuscript is in the pure mathematics (mathematical analysis) category.

Conflicts of Interest

The authors do not have any conflicts of interest regarding this article.

Authors' Contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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