Research Article

Dynamic Analysis on a Diffusive Two-Enterprise Interaction Model with Two Delays

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The oscillation and instability of systems caused by time delays have been widely studied over the past several decades. In nature, the phenomenon of diffusion is universal. Therefore, it is necessary to investigate the dynamic behavior of reaction-diffusion systems with time delays. In this study, a two-enterprise interaction model with diffusion and delay effects is considered. By analyzing the distribution of the roots of the corresponding characteristic equation, some conditions for the stability of the unique positive equilibrium and the existence of Hopf bifurcation at the steady state are investigated. As the sum of the time delays changes, there are a series of periodic solutions at the trivial steady-state solution of the system. In addition, the direction of Hopf bifurcation and the stability of the periodic solutions are discussed by using the normal form theory and the center manifold reduction of partial functional differential equations. Finally, numerical simulation experiments are conducted to illustrate the validity of the theoretical conclusions.

1. Introduction

As we all know, it is of great significance for an enterprise to consider its developmental progress and growth trend, grasp its position, and comprehensively analyze its interactive relationships with other enterprises in the business ecosystem. In recent years, the dynamic relationships among enterprises described by corresponding mathematical models have been widely studied [1–6]. In [5], Xu proposed the following model for the dynamic competition and co-operation between two enterprises and applied coincidence degree theory to analyze the existence of periodic solutions:

$$\begin{cases} \dot{x}_1(t) = r_1 x_1(t) \left[1 - \frac{x_1(t)}{K_1} - \frac{\alpha(x_2(t) - c_2)^2}{K_2} \right], \\ \dot{x}_2(t) = r_2 x_2(t) \left[1 - \frac{x_2(t)}{K_2} + \frac{\beta(x_1(t) - c_1)^2}{K_1} \right], \\ x_1(0) \geq 0, x_2(0) \geq 0, \end{cases} \quad (1)$$

where $x_i(t)$ ($i = 1, 2$) denotes the output of two enterprises, respectively; r_i ($i = 1, 2$) represents the intrinsic growth rate of two enterprises, respectively; K_1 and K_2 are the carrying capacities of the natural market, i.e., load capacities of the two enterprises under unrestricted conditions; α is the consumption coefficient from the enterprise with output $x_2(t)$ to the one with output $x_1(t)$; β is the transformation coefficient from the enterprise with output $x_1(t)$ to the one with output $x_2(t)$; and c_i ($i = 1, 2$) represents the initial output of the two enterprises, respectively. It was found that the outputs of the two enterprises reach an equilibrium state and at least have a positive periodic solution when the intrinsic growth rates, natural market loads, competition coefficients, and initial productions of the two enterprises meet certain conditions. Considering that the real dynamic ecological system always depends on information from the historical system, Liao et al. [3, 4] proposed two different competition and cooperation models for two enterprises with a delay (or delays) in succession. They investigated the effect of the delay (or delays) on the stability of the positive equilibrium by choosing τ (or τ_1, τ_2) as the bifurcation

parameter (or parameters). On the basis of [3], Guerrini [1] further proposed that when the two delays are small, an approximate, simple method can be used to obtain the roots of the characteristic equation. An analysis of the dynamic behavior of a competition and cooperation model for two enterprises with two small delays was also provided. Considering the influence of periodic solutions on the development and operation of enterprises, Xu and Li [6] investigated almost periodic solutions for a competition and cooperation model of two enterprises with time-varying delays and feedback controls. The existence and global stability of the almost periodic solutions were obtained by establishing a suitable Lyapunov functional. In fact, the internal and external effects of time delays were found to be completely different for enterprises. Hence, Li et al. [2] proposed the following differential equations model with four different delays:

$$\begin{cases} \dot{x}_1(t) = r_1 x_1(t) \left[1 - \frac{x_1(t - \tau_1)}{K_1} - \frac{\alpha(x_2(t - \tau_2) - c_2)^2}{K_2} \right], \\ \dot{x}_2(t) = r_2 x_2(t) \left[1 - \frac{x_2(t - \tau_3)}{K_2} + \frac{\beta(x_1(t - \tau_4) - c_1)^2}{K_1} \right], \\ x_1(t) = \varphi(t), x_2(t) = \psi(t), t \in \left[-\max_{1 \leq i \leq 4} \{\tau_i\}, 0 \right], \end{cases} \quad (2)$$

where $\varphi(t), \psi(t) \in C([-\max_{1 \leq i \leq 4} \{\tau_i\}, 0], \mathbb{R})$, and $\tau_i \geq 0$ ($1 \leq i \leq 4$) represent the delay effects within and between enterprises, respectively. Specifically, τ_1 and τ_3 denote the gestation periods of the outputs for the two enterprises. τ_2 and τ_4 represent the block delay and promoting delay, respectively, imposed by one enterprise on another. One case where $\tau_i = \tau$ ($1 \leq i \leq 4$) in (2) was considered in [4], another case where $\tau_3 = \tau_1, \tau_4 = \tau_2$ was discussed in [3], and some corresponding results were obtained. In [2], Li et al. researched the stability of the positive equilibrium and the existence of Hopf bifurcation, deduced the formula determining the direction of Hopf bifurcation, and achieved the stability of bifurcating periodic solutions by setting $\tau_1 = \tau_3 = 0$ and choosing $\tau = \tau_2 + \tau_4 > 0$ as the parameter. It has shown

that time delay has an important impact on the steady-state equilibrium of the interactive relationship between enterprises. The latest research results of some delayed systems can be found in [7–13], which focus on multiple time delays and feedback control, fractional-order delay model, and discrete model with multiple delays.

However, most of these studies only considered the effect of time delays and lost sight of the effect of spatial diffusion. It is well known that diffusion factors are indispensable in the modeling of ecological and biological systems and often utilized to describe the spatial distributions of the densities of some certain substances, such as plants, animals, and other organisms. At present, there are a growing number of dynamic models described by reaction-diffusion equations with time delays in various application areas [14–27]. For example, some scholars [19, 23, 25–27] proposed several different diffusive predator-prey systems with delays to analyze the effect of diffusion on biological systems. Huang et al. [18] researched a plant-pollinator model with diffusion and time delay effects and discussed its bifurcation and temporal periodic patterns. Some scholars [16, 22] established delayed reaction-diffusion systems to investigate the spatiotemporal transmission of bacteria and viruses in epidemiology. As a result of the impact of the growth pole on the economic activities of industrial groups, Hu et al. [17] provided a Kaldor–Kalecki model of the business cycle with diffusion and time delay and described the influence of the diffusion coefficient on the spatial pattern of the system.

Generally, in the development of the regional economy, the enterprises located in the center of the city have the functions of production, information, transportation, trade, service, and decision-making, which act like a “magnetic pole” that attracts production from the surrounding areas and spreads its own production factors to the surrounding areas. Especially when the industrial economic development in the central area is at the mature stage, the diffusion effect continues to increase. Therefore, based on the above argument, we further consider spatial diffusion with zero-flux boundary conditions. A delayed reaction-diffusion system for a two-enterprise interaction should be described by the following model:

$$\begin{cases} \frac{\partial x_1}{\partial t} = D_1 \Delta x_1 + r_1 x_1(t, x) \left[1 - \frac{x_1(t, x)}{K_1} - \frac{\alpha(x_2(t - \tau_2, x) - c_2)^2}{K_2} \right], & x \in \Omega, t > 0, \\ \frac{\partial x_2}{\partial t} = D_2 \Delta x_2 + r_2 x_2(t, x) \left[1 - \frac{x_2(t, x)}{K_2} + \frac{\beta(x_1(t - \tau_1, x) - c_1)^2}{K_1} \right], & x \in \Omega, t > 0, \\ \frac{\partial x_1}{\partial \nu} = \frac{\partial x_2}{\partial \nu} = 0, & t \geq 0, x \in \partial\Omega, \\ x_1(t, x) = \varphi(t, x) \geq 0, x_2(t, x) = \psi(t, x) \geq 0, & (t, x) \in \left[-\max_{1 \leq i \leq 2} \{\tau_i\}, 0 \right] \times \overline{\Omega}, \end{cases} \quad (3)$$

where $x_i(t, x)$ ($i = 1, 2$) denotes the output of two enterprises at time t and spatial position x , respectively. $x_i(t - \tau_i, x)$ ($i = 1, 2$) represents the output of two enterprises in the gestation period at spatial state x , respectively. $(1 - x_i(t, x))/K_i$ ($i = 1, 2$) reflects the prevention effect on the output of enterprise i ($i = 1, 2$) caused by its own consumption of limited resources, respectively. As the two enterprises exist in the same environment, the impact of the delay, caused by enterprise 2 consuming the same limited resources in the process of development, on enterprise 1 is expressed by $-(\alpha(x_2(t - \tau_2, x) - c_2)^2)/K_2$, and the promoting delay by which enterprise 1 supports enterprise 2 is

denoted as $\beta(x_1(t - \tau_1, x) - c_1)^2/K_1$. In [3], it was suggested that the promoting delay and blocking delay in the gestation period of output for enterprises have important economic significance for their development through changing parameters α and β . Δ is the Laplacian operator, and $D_1, D_2 > 0$ denote the diffusion coefficients. Ω represents a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with a smooth boundary $\partial\Omega$, and ν is the outward unit normal vector on $\partial\Omega$. Similar to [2–4], by introducing the terms $\alpha_i = (r_i/K_i)$ ($i = 1, 2$), $b_1 = (\alpha r_1/K_2)$, $b_2 = (\beta r_2/K_1)$, $d_i = r_i - \alpha_i c_i$, $y_i(t, x) = x_i(t, x) - c_i$ ($i = 1, 2$), we obtain the following model:

$$\begin{cases} \frac{\partial y_1}{\partial t} = D_1 \Delta y_1 + (y_1(t, x) + c_1) [d_1 - a_1 y_1(t, x) - b_1 y_2^2(t - \tau_2, x)], & x \in \Omega, t > 0, \\ \frac{\partial y_2}{\partial t} = D_2 \Delta y_2 + (y_2(t, x) + c_2) [d_2 - a_2 y_2(t, x) + b_2 y_1^2(t - \tau_1, x)], & x \in \Omega, t > 0, \\ \frac{\partial y_1}{\partial \nu} = \frac{\partial y_2}{\partial \nu} = 0, & t \geq 0, x \in \partial\Omega, \\ y_1(t, x) = \varphi(t, x) \geq 0, y_2(t, x) = \psi(t, x) \geq 0, (t, x) \in \left[-\max_{1 \leq i \leq 2} \{\tau_i\}, 0 \right] \times \overline{\Omega}. \end{cases} \quad (4)$$

For simplicity, we reduce the spatial domain Ω to one dimension $(0, \pi)$. Although there is an abundant literature about the stability and Hopf bifurcation of delayed differential equation models with diffusion, stability analysis of the interactive behaviors among enterprises in the business ecosystem, especially reaction-diffusion financial systems with delays, is very rare. The main contributions of this study are as follows: first, the interaction between two enterprises is discussed from the perspective of ecology, and the factors such as multiple time delay and diffusion affecting the dynamic properties of the system are introduced into the model, which makes the given model more in line with the actual background in economics than the previous models. Second, the complex dynamic properties of the system are discussed, including the stability of the system and the qualitative analysis of the existence of Hopf bifurcation. The parameter calculation expressions of bifurcation direction, stability of bifurcation periodic solution, and period size are determined. Finally, we explain the results from the perspective of application and supplement the relevant conclusions of the existing literature.

The rest of this study is organized as follows. In Section 2, we consider the distribution of the roots of the characteristic equation associated linearized system and explore the stability of the positive steady state and the existence of Hopf

bifurcation of system (4) to satisfy some requirements. In Section 3, according to the normal form method and the center manifold reduction developed by Hassard et al. [28], we determine the direction of Hopf bifurcation, give an explicit rule, and analyze the stability of bifurcating periodic solutions. In addition, some numerical simulations are conducted in Section 4 to verify our theoretical predictions. The conclusions are presented in Section 5.

2. Stability and Hopf Bifurcation

In this section, taking τ as the varying parameter, we analyze the distribution of the roots of the corresponding characteristic equation of (4) at the positive constant steady state and the stability of the positive equilibrium point. Some conditions of Hopf bifurcation of (4) are provided.

According to [3, 4], there is a unique positive constant steady state $E(y_1^*, y_2^*)$ of system (4), which provides that the following condition holds:

$$(H1): a_2^2 d_1 > b_1 d_2^2. \quad (5)$$

Let $u_1(t, x) = y_1(t, x) - y_1^*$, $u_2(t, x) = y_2(t, x) - y_2^*$; then, (4) is equivalent to the following system:

$$\left\{ \begin{array}{l} \frac{\partial u_1(t, x)}{\partial t} = D_1 \Delta u_1(t, x) + (u_1(t, x) + y_1^* + c_1) [d_1 - a_1(u_1(t, x) + y_1^*) - b_1(u_2(t - \tau, x) + y_2^*)^2], \\ \frac{\partial u_2(t, x)}{\partial t} = D_2 \Delta u_2(t, x) + (u_2(t, x) + y_2^* + c_2) [d_2 - a_2(u_2(t, x) + y_2^*) + b_2(u_1(t, x) + y_1^*)^2], \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = 0, \quad t \geq 0, x \in \partial\Omega, \\ u_1(t, x) = \varphi(t, x) - y_1^*, u_2(t, x) = \psi(t, x) - y_2^*, (t, x) \in \left[-\max_{1 \leq i \leq 2} \{\tau_i\}, 0 \right] \times \overline{\Omega}, \end{array} \right. \quad (6)$$

where $\tau = \tau_1 + \tau_2$. (6) can be written as follows by setting $d_1 - a_1 y_1^* - b_1 (y_2^*)^2 = 0$ and $d_2 - a_2 y_2^* + b_2 (y_1^*)^2 = 0$.

$$\left\{ \begin{array}{l} \frac{\partial u_1(t, x)}{\partial t} = D_1 \Delta u_1(t, x) + (u_1(t, x) + y_1^* + c_1) [-a_1 u_1(t, x) - b_1 u_2^2(t - \tau, x) - 2b_1 y_2^* u_2(t - \tau, x)], \\ \frac{\partial u_2(t, x)}{\partial t} = D_2 \Delta u_2(t, x) + (u_2(t, x) + y_2^* + c_2) [-a_2 u_2(t, x) + b_2 u_1^2(t, x) + 2b_2 y_1^* u_1(t, x)]. \end{array} \right. \quad (7)$$

We assume that $\varphi, \psi \in \ell = C([- \tau, 0], X)$, and X is defined by

$$X = \left\{ (u_1, u_2): u_i \in W^{2,2}(0, \pi), \frac{\partial u_i(t, x)}{\partial \nu} = 0, i = 1, 2, x = 0, \pi \right\}, \quad (8)$$

with the inner product $\langle \cdot, \cdot \rangle$.

Denote $u_i(t) = u_i(t, \cdot)$ ($i = 1, 2$) and $U = (u_1, u_2)^T$. For $\phi = (\phi_1, \phi_2)^T \in \ell$, system (7) can be rewritten as the following abstract differential equation in the phase space $\ell = C([- \tau, 0], X)$:

$$\dot{U}(t) = D \Delta U(t) + L(U_t) + F(U_t), \quad (9)$$

where $D = \text{diag}(D_1, D_2)$ and $L: \ell \rightarrow X$, $F: \ell \rightarrow X$ are defined by the following:

$$\begin{aligned} L(\phi) &= \begin{pmatrix} A_1 & 0 \\ A_2 & B_2 \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + \begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-\tau) \\ \phi_2(-\tau) \end{pmatrix}, \\ F(\phi) &= \begin{pmatrix} F_1 + F_2 \\ F_3 + F_4 \end{pmatrix}, \end{aligned} \quad (10)$$

where

$$\begin{aligned} A_1 &= -a_1(y_1^* + c_1) < 0, \\ A_2 &= 2b_2 y_1^*(y_2^* + c_2) > 0, \\ B_1 &= -2b_1 y_2^*(y_1^* + c_1) < 0, \\ B_2 &= -a_2(y_2^* + c_2) < 0, \\ F_1 &= -a_1 \phi_1^2(0) - b_1(y_1^* + c_1) \phi_2^2(-\tau), \end{aligned}$$

$$\begin{aligned} F_2 &= -2b_1 y_2^* \phi_1(0) \phi_2(-\tau) - b_1 \phi_1(0) \phi_2^2(-\tau), F_3 \\ &= -a_2 \phi_2^2(0) + b_2(y_2^* + c_2) \phi_1^2(0), F_4 \\ &= 2b_2 y_1^* \phi_2(0) \phi_1(0) + b_2 \phi_2(0) \phi_1^2(0), \phi(\theta) \\ &= (\phi_1(\theta), \phi_2(\theta))^T \in \ell. \end{aligned} \quad (11)$$

It is easy to see that the linearization of (9) is given by

$$\dot{U}(t) = D \Delta U(t) + L(U_t), \quad (12)$$

and its characteristic equation is

$$\lambda y - D \Delta y - L(e^\lambda y) = 0, \quad (13)$$

where $y \in \text{dom}\{\Delta\} \setminus \{0\}$, $\text{dom}(\Delta) \subset X$.

According to the properties of the Laplacian operator on the bounded domain [14, 18], the operator Δ on X has the eigenvalues $-k^2$ and the relative eigenfunctions $\beta_k^1 = \begin{pmatrix} \cos kx \\ 0 \end{pmatrix}$, $\beta_k^2 = \begin{pmatrix} 0 \\ \cos kx \end{pmatrix}$, for $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. (β_k^1, β_k^2) is a basis of the phase space X . So, any element $y \in X$ can be expanded to a Fourier series as follows:

$$y = \sum_{k=0}^{\infty} Y_k^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix}, \quad (14)$$

where $Y_k^T = (\langle y, \beta_k^1 \rangle, \langle y, \beta_k^2 \rangle)$.

By using some simple computations, we have

$$L \left(\phi^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix} \right) = L(\phi^T) \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix}, \quad k \in \mathbb{N}_0. \quad (15)$$

Hence, (13) is equivalent to

$$\sum_{k=0}^{\infty} Y_k^T \left[(\lambda I_2 + Dk^2) - \begin{pmatrix} A_1 & B_1 e^{-\lambda \tau} \\ A_2 & B_2 \end{pmatrix} \right] \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix} = 0. \quad (16)$$

Therefore, all the characteristic roots of (13) are equivalent to the roots of the following equation:

$$\begin{aligned} \lambda^2 + (-A_1 + D_1 k^2 - B_2 + D_2 k^2) \lambda + (-A_1 + D_1 k^2)(-B_2 + D_2 k^2) \\ - A_2 B_1 e^{-\lambda \tau} = 0. \end{aligned} \quad (17)$$

Let $p = -(A_1 + B_2) + D_1 k^2 + D_2 k^2 > 0$, $q = -A_2 B_1 > 0$, and $r = (-A_1 + D_1 k^2)(-B_2 + D_2 k^2) > 0$; then, (17) can be rewritten as

$$\lambda^2 + p\lambda + r + qe^{-\lambda \tau} = 0. \quad (18)$$

It is clear that $\lambda = 0$ is not a root of (18) for $\forall k \in \mathbb{N}_0$.

When $\tau = 0$, (18) can be converted into

$$\lambda^2 + p\lambda + r + q = 0. \quad (19)$$

Suppose λ_1 and λ_2 are two roots of (19). For $\forall k \in \mathbb{N}_0$, it is easy to see that

$$\begin{aligned} \lambda_1 + \lambda_2 &= -[-(A_1 + B_2) + D_1 k^2 + D_2 k^2] < 0, \\ \lambda_1 \lambda_2 &= (-A_1 + D_1 k^2)(-B_2 + D_2 k^2) - A_2 B_1 > 0. \end{aligned} \quad (20)$$

Therefore, if (H1) holds, the unique positive equilibrium point $E(y_1^*, y_2^*)$ of (4) is asymptotically stable for $\tau = 0$.

Furthermore, suppose that $\tau > 0$ and $\lambda = i\omega$ ($\omega > 0$) is a solution of (18). Then, ω satisfies the following equation for some $k \in \mathbb{N}_0$:

$$-\omega^2 + i\omega p + r + q(\cos \omega \tau - i \sin \omega \tau) = 0. \quad (21)$$

By separating the imaginary and real parts of (21), we have

$$\begin{cases} \omega^2 - r = q \cos \omega \tau, \\ p\omega = q \sin \omega \tau. \end{cases} \quad (22)$$

It follows from (22) that

$$\omega^4 + (p^2 - 2r)\omega^2 + r^2 - q^2 = 0. \quad (23)$$

Let $z = \omega^2$; then, (23) becomes

$$z^2 + (p^2 - 2r)z + r^2 - q^2 = 0. \quad (24)$$

By performing some calculations, one can show that

$$\begin{aligned} p^2 - 2r &= (-A_1 + D_1 k^2)^2 + (-B_2 + D_2 k^2)^2, \\ r^2 - q^2 &= (-A_1 + D_1 k^2)^2 (-B_2 + D_2 k^2)^2 - A_2^2 B_1^2. \end{aligned} \quad (25)$$

Note that

$$r + q = (-A_1 + D_1 k^2)(-B_2 + D_2 k^2) - A_2 B_1 > 0. \quad (26)$$

Hence, if the following condition holds,

$$(H2): r - q = (-A_1 + D_1 k^2)(-B_2 + D_2 k^2) + A_2 B_1 > 0, \quad (27)$$

then (24) has no positive real roots, and the characteristic equation has no purely imaginary roots for all $\tau > 0$. Therefore, we have the following conclusion if (H1) and (H2) hold.

Theorem 1. Assume that conditions (H1) and (H2) hold; then, the positive equilibrium point $E(y_1^*, y_2^*)$ of (7) is locally asymptotically stable for all $\tau \geq 0$.

Now, we consider the existence of Hopf bifurcation at the positive equilibrium of (7).

When $k = 0$, we have

$$r^2 - q^2 = (A_1 B_2 + A_2 B_1)(A_1 B_2 - A_2 B_1), \quad (28)$$

where $A_1 B_2 - A_2 B_1 > 0$.

Hence, if $A_1 B_2 + A_2 B_1 < 0$, then $r^2 - q^2 < 0$, and (23) has a unique positive root ω_+^0 , where

$$\omega_+^0 = \left\{ \frac{1}{2} \left[2r - p^2 + \sqrt{(p^2 - 2r)^2 - 4(r^2 - p^2)} \right] \right\}^{1/2}, \quad (29)$$

and $z_0 = (\omega_+^0)^2$. Equation (18) with $k = 0$ has a pair of purely imaginary roots $\pm i\omega_+^0$. By (22), we can get the values for τ under this scenario:

$$\tau_j^0 = \frac{1}{\omega_+^0} \arctan \left\{ \frac{p\omega_+^0}{(\omega_+^0)^2 - r} \right\} + \frac{j\pi}{\omega_+^0}, \quad j \in \mathbb{N}_0. \quad (30)$$

Hence, all roots of (18), except $\pm i\omega_+^0$, have no zero real parts. According to Theorem 2.7 by Li et al. [2], we know that if (H1) and $A_1 B_2 + A_2 B_1 < 0$ hold, then the equilibrium $E(y_1^*, y_2^*)$ remains stable for $\tau < \tau_0^0 = \min\{\tau_j^0\}$, $j \in \mathbb{N}_0$, and we verify that

$$\left. \frac{d(\operatorname{Re} \lambda)}{d\tau} \right|_{\tau=\tau_0^0} > 0, \quad (31)$$

where $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ is a root of (18), $\alpha(\tau_j^0) = 0$, and $\omega(\tau_j^0) = \omega_0$.

Apparently, the transversality condition holds in this case, and there is a sign of Hopf bifurcation at $\omega = \omega_0$ and $\tau = \tau_0^0$.

Furthermore, we discuss the case when $k \geq 1$. Let $\lambda = i\omega_1$ ($\omega_1 > 0$) be a root of (18) with $k \geq 1$. Similarly, we have

$$\omega_1^4 + (p^2 - 2r)\omega_1^2 + r^2 - q^2 = 0. \quad (32)$$

By performing some calculations, we can get

$$r^2 - q^2 \geq (-A_1 + D_1)^2 (-B_2 + D_2)^2 - A_2^2 B_1^2. \quad (33)$$

It is easy to see that if the following condition holds,

$$(H3): (-A_1 + D_1)(-B_2 + D_2) > A_2 B_1, \quad (34)$$

then (18) has no purely imaginary roots for $k \geq 1$. Thus, the following conclusions are true.

Theorem 2. Assume that condition (H1) holds. ω_+^0 and τ_j^0 ($j \in \mathbb{N}_0$) be given by (29) and (30), respectively. Then, for system (7), the following conclusions are valid:

- (i) When (H2) holds, the positive equilibrium E is locally asymptotically stable for all $\tau \geq 0$
- (ii) When $A_1B_2 + A_2B_1 < 0$ and (H3) holds, the positive equilibrium E is locally asymptotically stable if $\tau \in [0, \tau_0^0)$ and unstable if $\tau > \tau_0^0$
- (iii) When $\tau = \tau_j^0$ ($j \in \mathbb{N}_0$), (7) occurs as Hopf bifurcation at E

By the same measure, we now make the following assumptions for a certain $k_0 \in \mathbb{N}_0 \setminus \{0\}$:

$$\begin{aligned} (H4) & (-A_1 + D_1k_0^2)(-B_2 + D_2k_0^2) + A_2B_1 < 0, \\ (H5) & (-A_1 + D_1(k_0 - 1)^2)(-B_2 + D_2(k_0 - 1)^2) > |A_2B_1|, \\ (H6) & (-A_1 + D_1(k_0 + 1)^2)(-B_2 + D_2(k_0 + 1)^2) > |A_2B_1|. \end{aligned} \quad (35)$$

Under condition (H4), equation (23) with $k = k_0$ has only a positive solution:

$$\omega_+^{k_0} = \frac{\sqrt{2}}{2} \left[2r - p^2 + \sqrt{(2r - p^2)^2 - 4(r^2 - q^2)} \right]_{k=k_0}^{1/2}. \quad (36)$$

Hence, when (H5) and (H6) hold, for $k \in \{1, 2, \dots, k_0 - 1\}$, we can deduce that

$$(-A_1 + D_1k^2)(-B_2 + D_2k^2) + A_2B_1 \geq (-A_1 + D_1(k_0 - 1)^2)(-B_2 + D_2(k_0 - 1)^2) - |A_2B_1| > 0, \quad (37)$$

and for $k \in \{k_0 + 1, k_0 + 2, \dots\}$,

$$(-A_1 + D_1k^2)(-B_2 + D_2k^2) + A_2B_1 \geq (-A_1 + D_1(k_0 + 1)^2)(-B_2 + D_2(k_0 + 1)^2) - |A_2B_1| > 0. \quad (38)$$

Therefore, it is easy to show that (18) with $k \in \mathbb{N}_0 \setminus \{k_0\}$ has no purely imaginary roots, and it follows from (22) that

$$\tau_j^{k_0} = \frac{1}{\omega_+^{k_0}} \arctan \left\{ \frac{p\omega_+^{k_0}}{(\omega_+^{k_0})^2 - r} \right\} + \frac{j\pi}{\omega_+^{k_0}}, \quad j \in \mathbb{N}_0. \quad (39)$$

Lemma 1. Assume that conditions (H1) and (H4)–(H6) hold; then, (18) with $k = k_0$ has a pair of purely imaginary roots $\pm i\omega_+^{k_0}$ with $\tau = \tau_j^{k_0}$ ($j \in \mathbb{N}_0$), and all roots of (18), except $\pm i\omega_+^{k_0}$, have no zero real parts.

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be a root of (18) near $\tau = \tau_j^{k_0}$ satisfying $\alpha(\tau_j^{k_0}) = 0$ and $\omega(\tau_j^{k_0}) = \omega_+^{k_0}$ for $j \in \mathbb{N}_0$; then, the following transversality condition is obtained.

Lemma 2. Assume that conditions (H1) and (H4)–(H6) hold. Then,

$$\text{sign} \left\{ \text{Re} \left(\frac{d\lambda}{d\tau} \right) \right\}_{\tau=\tau_j^{k_0}} > 0. \quad (40)$$

Proof. Differentiating both sides of (18) with respect to τ yields

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{\{2\lambda + [D_1k_0^2 + D_2k_0^2 - (A_1 + B_2)]\}e^{\lambda\tau}}{-\lambda A_2B_1} - \frac{\tau}{\lambda}. \quad (41)$$

It follows that

$$\begin{aligned} \text{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1}_{\tau=\tau_j^{k_0}} &= \text{Re} \left[\frac{\{2\lambda + [D_1k_0^2 + D_2k_0^2 - (A_1 + B_2)]\}e^{\lambda\tau}}{-\lambda A_2B_1} - \frac{\tau}{\lambda} \right]_{\tau=\tau_j^{k_0}} \\ &= \text{Re} \left\{ \frac{[2i\omega_+^{k_0} + (D_1k_0^2 + D_2k_0^2 - (A_1 + B_2))]e^{i\tau_j^{k_0}\omega_+^{k_0}}}{-i\omega_+^{k_0}A_2B_1} \right\} \\ &= \text{Re} \left\{ \frac{[-2\omega_+^{k_0} + i(D_1k_0^2 + D_2k_0^2 - (A_1 + B_2))]e^{i\tau_j^{k_0}\omega_+^{k_0}}}{\omega_+^{k_0}A_2B_1} \right\} \\ &= \frac{-2\omega_+^{k_0} \cos \tau_j^{k_0}\omega_+^{k_0} - (D_1k_0^2 + D_2k_0^2 - (A_1 + B_2)) \sin \tau_j^{k_0}\omega_+^{k_0}}{\omega_+^{k_0}A_2B_1}. \end{aligned} \quad (42)$$

According to the above analysis,

$$\text{sign} \left\{ \text{Re} \left(\frac{d\lambda}{d\tau} \right) \right\}_{\tau=\tau_j^{k_0}} = \text{sign} \left\{ \text{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right\}_{\tau=\tau_j^{k_0}} > 0. \quad (43)$$

From Lemmas 1 and 2, and the Hopf bifurcation theorem [28], we can have the following result. $\square$

Theorem 3. Assume that condition (H1) holds. $\omega_+^{k_0}$ and $\tau_j^{k_0}$ are given by (36) and (39), respectively. Then, for system (7), the following results are true:

- (i) When $\tau \in [0, \tau_0^{k_0})$, the positive equilibrium point E is asymptotically stable
- (ii) When $\tau > \tau_0^{k_0}$, the positive equilibrium point E is unstable

(iii) When $\tau = \tau_j^{k_0}$ ($j \in \mathbb{N}_0$), (7) occurs as Hopf bifurcation at E

3. Direction of Hopf Bifurcation and Stability of Bifurcating Periodic Solution

In the preceding section, we obtained a series of periodic solutions bifurcating from the positive constant steady state E of (7) when τ increases past the critical values τ_j^k ($k = 0, k_0; j \in \mathbb{N}_0$). In the present section, by using the normal form method and the center manifold theorem presented in [28–30], we compute the directions, stability, and periods of the bifurcating periodic solution. For brevity, we denote τ_j^k by $\hat{\tau}$ for some fixed $j \in \mathbb{N}_0$.

Let $u_i(t, x) = u_i(\tau t, x)$ ($i = 1, 2$) and $\tau = \hat{\tau} + \mu$. Then, system (7) can be transformed into

$$\begin{cases} \frac{\partial u_1(t, x)}{\partial t} = (\hat{\tau} + \mu) D_1 \Delta u_1(t, x) + (\hat{\tau} + \mu) (u_1(t, x) + y_1^* + c_1) [-a_1 u_1(t, x) - b_1 u_2^2(t-1, x) - 2b_1 y_2^* u_2(t-1, x)], \\ \frac{\partial u_2(t, x)}{\partial t} = (\hat{\tau} + \mu) D_2 \Delta u_2(t, x) + (\hat{\tau} + \mu) (u_2(t, x) + y_2^* + c_2) [-a_2 u_2(t, x) + b_2 u_1^2(t, x) + 2b_2 y_1^* u_1(t, x)]. \end{cases} \quad (44)$$

Let $u_i(t) = u_i(t, \cdot)$ ($i = 1, 2$) and $U = (u_1, u_2)^T$. Then, in the phase space $\ell = C([- \tau, 0], X)$, system (44) can be rewritten as

$$\frac{dU(t)}{dt} = \hat{\tau} D \Delta U(t) + L(\hat{\tau})(U_t) + F(U_t, \mu), \quad (45)$$

where $D = \text{diag}(D_1, D_2)$, $L: \ell \rightarrow X$, and $F: \ell \times \mathbb{R} \rightarrow X$ are given as

$$\begin{aligned} L(\mu)(\phi) &= \mu \begin{pmatrix} A_1 \phi_1(0) + B_1 \phi_2(-1) \\ A_2 \phi_1(0) + B_2 \phi_2(0) \end{pmatrix}, \\ F(\phi, \mu) &= \mu D \Delta \phi(0) + L(\mu)(\phi) + f(\phi, \mu), \end{aligned} \quad (46)$$

where

$$\begin{aligned} f(\phi, \mu) &= (\hat{\tau} + \mu) \begin{pmatrix} f_1 + f_2 \\ f_3 + f_4 \end{pmatrix}, \\ f_1 &= -a_1 \phi_1^2(0) - b_1 (y_1^* + c_1) \phi_2^2(-1), \\ f_2 &= -2b_1 y_2^* \phi_1(0) \phi_2(-1) - b_1 \phi_1(0) \phi_2^2(-1), \\ f_3 &= F_3, f_4 = F_4, \end{aligned} \quad (47)$$

for $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta))^T \in \ell$. From the discussion in the last section, we know that (44) has an equilibrium $(0, 0)$ and $\Lambda_0 = \{-i\omega_+^k \hat{\tau}, i\omega_+^k \hat{\tau}\}$ ($k = 0, k_0$) are a pair purely imaginary eigenvalues of characteristic equation of the linear equation

$$\dot{U}(t) = \hat{\tau} D \Delta U(t) + L(\hat{\tau})(U_t), \quad (48)$$

and the functional differential equation

$$\dot{z}(t) = L_k(\hat{\tau})(z_t), \quad (49)$$

where $L_k(\hat{\tau})$ is a continuous linear operator mapping $C([-1, 0], \mathbb{R}^2)$ into $\mathbb{R}^2$.

According to the Riesz representation theorem, for the ordinary functional differential equation,

$$\dot{X}(t) = -\hat{\tau} D k^2 X(t) + L(\hat{\tau})(X_t), \quad (50)$$

there exists a 2×2 matrix function of the bounded variation $\eta^{(k)}(\theta, \hat{\tau})$, $\theta \in [-1, 0]$, such that

$$-\hat{\tau} D k^2 \phi(0) + L(\hat{\tau})(\phi) = \int_{-1}^0 d[\eta^{(k)}(\theta, \hat{\tau})] \phi(\theta), \quad (51)$$

where $\phi(\theta) \in C([-1, 0], \mathbb{R}^2)$. In fact, we can take

$$\eta^{(k)}(\theta, \hat{\tau}) = \begin{cases} \hat{\tau} \begin{pmatrix} -D_1 k^2 + A_1 & 0 \\ A_2 & -D_2 k^2 + B_2 \end{pmatrix}, & \theta = 0, \\ 0, & \theta \in (-1, 0), \\ -\hat{\tau} \begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix}, & \theta = -1. \end{cases} \quad (52)$$

Let $A(\hat{\tau})$ be the infinitesimal generator corresponding to (50) and A^* be the formal adjoint of $A(\hat{\tau})$ under the bilinear inner product

$$\begin{aligned}
\langle \psi(s), \phi(\theta) \rangle &= \psi(0)\phi(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \psi(\xi - \theta) d[\eta^{(k)}(\theta, \widehat{\tau})] \phi(\xi) d\xi \\
&= \psi(0)\phi(0) + \widehat{\tau} \int_{-1}^0 \psi(\xi + 1) \begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix} \phi(\xi) d\xi,
\end{aligned} \tag{53}$$

for $\phi \in C([-1, 0], \mathbb{R}^2)$, and $\psi \in C([0, 1], \mathbb{R}^2)$. Then, $A(\widehat{\tau})$ and A^* are a pair of adjoint operators [31]. Since $\pm i\omega_+^k \widehat{\tau}$ are a pair of purely imaginary eigenvalues of $A(\widehat{\tau})$, they are also a pair of eigenvalues of A^* . Let P and P^* be the center spaces of $A(\widehat{\tau})$ and A^* , respectively, associated with Λ_0 . Thus, P^* is the adjoint space of P and $\dim P = \dim P^* = 2$ [30].

Lemma 3. *Let*

$$\begin{aligned}
\xi &= \frac{i\omega_+^k + D_1 k^2 - A_1}{B_1 e^{-i\omega_+^k \widehat{\tau}}}, \\
\eta &= \frac{-i\omega_+^k + D_1 k^2 + A_1}{A_2}.
\end{aligned} \tag{54}$$

Then, $p_1(\theta) = e^{i\omega_+^k \widehat{\tau} \theta} (1, \xi)^T$, $p_2(\theta) = \overline{p_1(\theta)}$, $\theta \in [-1, 0]$ is a basis of P associated with Λ_0 , and $q_1(s) = (1, \eta)e^{-i\omega_+^k \widehat{\tau} s}$, $q_2(s) = \overline{q_1(s)}$, $s \in [0, 1]$ is a basis of Q associated with Λ_0 .

Let $\Phi = (\Phi_1, \Phi_2)$ and $\Psi^* = (\Psi_1^*, \Psi_2^*)^T$ with $\Phi_1(\theta) = (p_1(\theta) + p_2(\theta))/2$, $\Phi_2(\theta) = (p_1(\theta) - p_2(\theta))/2i$, and $\Psi_1^*(s) = (q_1(s) + q_2(s))/2$, $\Psi_2^*(s) = (q_1(s) - q_2(s))/2i$.

Through calculation, we can get the following concrete expressions:

$$\begin{aligned}
\Phi_1(\theta) &= \begin{pmatrix} \operatorname{Re}\{e^{i\omega_+^k \widehat{\tau} \theta}\} \\ \operatorname{Re}\{\xi e^{i\omega_+^k \widehat{\tau} \theta}\} \end{pmatrix} = \begin{pmatrix} \cos \omega_+^k \widehat{\tau} \theta \\ \frac{M_1 \cos \omega_+^k \widehat{\tau} \theta - M_2 \sin \omega_+^k \widehat{\tau} \theta}{B_1} \end{pmatrix}, \\
\Phi_2(\theta) &= \begin{pmatrix} \operatorname{Im}\{e^{i\omega_+^k \widehat{\tau} \theta}\} \\ \operatorname{Im}\{\xi e^{i\omega_+^k \widehat{\tau} \theta}\} \end{pmatrix} = \begin{pmatrix} \sin \omega_+^k \widehat{\tau} \theta \\ \frac{M_1 \sin \omega_+^k \widehat{\tau} \theta + M_2 \cos \omega_+^k \widehat{\tau} \theta}{B_1} \end{pmatrix}, \\
\Psi_1^*(s) &= \begin{pmatrix} \operatorname{Re}\{e^{-i\omega_+^k \widehat{\tau} s}\} \\ \operatorname{Re}\{\eta e^{-i\omega_+^k \widehat{\tau} s}\} \end{pmatrix}^T = \begin{pmatrix} \cos \omega_+^k \widehat{\tau} s \\ -\frac{M_3}{A_2} \end{pmatrix}^T, \\
\Psi_2^*(s) &= \begin{pmatrix} \operatorname{Im}\{e^{-i\omega_+^k \widehat{\tau} s}\} \\ \operatorname{Im}\{\eta e^{-i\omega_+^k \widehat{\tau} s}\} \end{pmatrix}^T = \begin{pmatrix} -\sin \omega_+^k \widehat{\tau} s \\ -\frac{M_4}{A_2} \end{pmatrix}^T,
\end{aligned} \tag{55}$$

for $\theta \in [-1, 0]$ and $s \in [0, 1]$. Also,

$$\begin{aligned}
M_1 &= (D_1 k^2 - A_1) \cos \omega_+^k \widehat{\tau} - \omega_+^k \sin \omega_+^k \widehat{\tau}, \\
M_2 &= \omega_+^k \cos \omega_+^k \widehat{\tau} + (D_1 k^2 - A_1) \sin \omega_+^k \widehat{\tau}, \\
M_3 &= (D_1 k^2 + A_1) \cos \omega_+^k \widehat{\tau} s + \omega_+^k \sin \omega_+^k \widehat{\tau} s, \\
M_4 &= -(D_1 k^2 + A_1) \sin \omega_+^k \widehat{\tau} s + \omega_+^k \cos \omega_+^k \widehat{\tau} s.
\end{aligned} \tag{56}$$

Then, by (53) and some calculations, we can further obtain

$$\begin{aligned}
\langle \Psi_1^*, \Phi_1 \rangle &= 1 - \frac{(D_1 k^2 + A_1) M_1}{A_2 B_1} + \widehat{\tau} \left[\left(\frac{\sin \omega_+^k \widehat{\tau}}{2 \omega_+^k \widehat{\tau}} + \frac{\cos \omega_+^k \widehat{\tau}}{2} \right) M_1 + \frac{\sin \omega_+^k \widehat{\tau}}{2} M_2 \right], \\
\langle \Psi_1^*, \Phi_2 \rangle &= -\frac{(D_1 k^2 + A_1) M_2}{A_2} + \widehat{\tau} \left[\left(\frac{\sin \omega_+^k \widehat{\tau}}{2 \omega_+^k \widehat{\tau}} + \frac{\cos \omega_+^k \widehat{\tau}}{2} \right) M_2 - \frac{\sin \omega_+^k \widehat{\tau}}{2} M_1 \right], \\
\langle \Psi_2^*, \Phi_1 \rangle &= -\frac{\omega_+^k M_1}{A_2 B_1} - \widehat{\tau} \left[-\left(\frac{\cos \omega_+^k \widehat{\tau}}{2} - \frac{\sin \omega_+^k \widehat{\tau}}{2 \omega_+^k \widehat{\tau}} \right) M_2 + \frac{\sin \omega_+^k \widehat{\tau}}{2} M_1 \right], \\
\langle \Psi_2^*, \Phi_2 \rangle &= -\frac{\omega_+^k M_2}{A_2 B_1} - \widehat{\tau} \left[\left(\frac{\cos \omega_+^k \widehat{\tau}}{2} - \frac{\sin \omega_+^k \widehat{\tau}}{2 \omega_+^k \widehat{\tau}} \right) M_1 + \frac{\sin \omega_+^k \widehat{\tau}}{2} M_2 \right].
\end{aligned} \tag{57}$$

Then, we define

$$(\Psi^*, \Phi) = (\Psi_j^*, \Phi_k) = \begin{pmatrix} \langle \Psi_1^*, \Phi_1 \rangle & \langle \Psi_1^*, \Phi_2 \rangle \\ \langle \Psi_2^*, \Phi_1 \rangle & \langle \Psi_2^*, \Phi_2 \rangle \end{pmatrix}, \tag{58}$$

and structure a new basis Ψ for Q as

$$\Psi = (\Psi_1, \Psi_2)^T = (\Psi^*, \Phi)^{-1} \Psi^*. \tag{59}$$

Obviously, $(\Psi, \Phi) = I_2$ is a second-order unit matrix. Furthermore, we define $f_k = (\beta_k^1, \beta_k^2)$ and $c \cdot f_k = c_1 \beta_k^1 + c_2 \beta_k^2$, where $\beta_k^1 = \begin{pmatrix} \cos kx \\ 0 \end{pmatrix}$, $\beta_k^2 = \begin{pmatrix} 0 \\ \cos kx \end{pmatrix}$, and $c = (c_1, c_2)^T \in C([-1, 0], X)$. Thus, the center subspace of linear equation (48) is given as

$$P_{CN} \ell(\phi) = \Phi(\Psi, \langle \phi, f_k \rangle) \cdot f_k, \phi \in \ell, \tag{60}$$

where $\ell = P_{CN} \ell \oplus P_s \ell$ and $P_s \ell$ is the complementary subspace of $P_{CN} \ell$ in ℓ .

Let $A_{\widehat{\tau}}$ be the infinitesimal generator corresponding to (48); then, (44) can be rewritten in the following form:

$$\dot{U}_t = A_{\widehat{\tau}} U_t + R(\mu, U_t), \tag{61}$$

where

$$R(\mu, U_t) = \begin{cases} 0, & \theta \in [-1, 0], \\ F(U_t, \mu), & \theta = 0. \end{cases} \tag{62}$$

Using the decomposition $\ell = P_{CN} \ell \oplus P_s \ell$ and (60), the solution of (45) on the center manifold can be rewritten as

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_k + h(x_1, x_2, \mu), \tag{63}$$

where $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = (\Psi, \langle U_t, f_k \rangle)$, $h(x_1, x_2, \mu) \in P_s \ell$, $h(0, 0, 0) = 0$, and $Dh(0, 0, 0) = 0$. In fact, on the center manifold, the solution of (45) is given by

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_k + h(x_1, x_2, 0). \tag{64}$$

Let $z = x_1 - ix_2$, $\Psi(0) = (\Psi_1(0), \Psi_2(0))^T$. By the definitions of Φ_1 and Φ_2 , it is easy to obtain $p_1 = \Phi_1 + i\Phi_2$, and we further obtain

$$\Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_k = (\Phi_1, \Phi_2) \begin{pmatrix} \frac{1}{2}(z + \bar{z}) \\ \frac{i}{2}(z - \bar{z}) \end{pmatrix} \cdot f_k = \frac{1}{2}(p_1 z + \bar{p}_1 \bar{z}) \cdot f_k. \tag{65}$$

Therefore, (64) can be expressed as

$$U_t = \frac{1}{2} (p_1 z + \overline{p_1} \overline{z}) \cdot f_k + W(z, \overline{z}), \quad (66)$$

where $W(z, \overline{z}) = h((z + \overline{z})/2, i(z - \overline{z})/2, 0)$. By [30], we can deduce that z satisfies

$$\dot{z} = i\omega_+^k \widehat{\tau} z + g(z, \overline{z}), \quad (67)$$

where

$$g(z, \overline{z}) = (\Psi_1(0) - i\Psi_2(0)) \langle f(U_t, 0), f_k \rangle. \quad (68)$$

Let

$$W(z, \overline{z}) = W_{20} \frac{z^2}{2} + W_{11} z \overline{z} + W_{02} \frac{\overline{z}^2}{2} + \dots, \quad (69)$$

$$g(z, \overline{z}) = g_{20} \frac{z^2}{2} + g_{11} z \overline{z} + g_{02} \frac{\overline{z}^2}{2} + g_{21} \frac{z^2 \overline{z}}{2} + \dots. \quad (70)$$

From (47) and (64), (66) can be rewritten as

$$\begin{aligned} U_t &= \frac{1}{2} \left[z e^{i\omega_+^k \widehat{\tau} \theta} \begin{pmatrix} 1 \\ \xi \end{pmatrix} + \overline{z} e^{-i\omega_+^k \widehat{\tau} \theta} \begin{pmatrix} 1 \\ \xi \end{pmatrix} \right] \cdot f_k + W(z, \overline{z}) \\ &= \frac{1}{2} \begin{pmatrix} (z e^{i\omega_+^k \widehat{\tau} \theta} + \overline{z} e^{-i\omega_+^k \widehat{\tau} \theta}) \cos kx \\ (z \xi e^{i\omega_+^k \widehat{\tau} \theta} + \overline{\xi} \overline{z} e^{-i\omega_+^k \widehat{\tau} \theta}) \cos kx \end{pmatrix} + \begin{pmatrix} W^{(1)}(z, \overline{z}) \\ W^{(2)}(z, \overline{z}) \end{pmatrix}. \end{aligned} \quad (71)$$

Then,

$$\begin{aligned} u_{1t}(0) &= \frac{1}{2} (z + \overline{z}) \cos kx + W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0) z \overline{z} + W_{02}^{(1)}(0) \frac{\overline{z}^2}{2} + \dots; \\ u_{1t}(-1) &= \frac{1}{2} \left(z e^{-i\omega_+^k \widehat{\tau}} + \overline{z} e^{i\omega_+^k \widehat{\tau}} \right) \cos kx + W_{20}^{(1)}(-1) \frac{z^2}{2} + W_{11}^{(1)}(-1) z \overline{z} + W_{02}^{(1)}(-1) \frac{\overline{z}^2}{2} + \dots; \\ u_{2t}(0) &= \frac{1}{2} (z \xi + \overline{z} \overline{\xi}) \cos kx + W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z \overline{z} + W_{02}^{(2)}(0) \frac{\overline{z}^2}{2} + \dots; \\ u_{2t}(-1) &= \frac{1}{2} \left(z \xi e^{-i\omega_+^k \widehat{\tau}} + \overline{z} \overline{\xi} e^{i\omega_+^k \widehat{\tau}} \right) \cos kx + W_{20}^{(2)}(-1) \frac{z^2}{2} + W_{11}^{(2)}(-1) z \overline{z} + W_{02}^{(2)}(-1) \frac{\overline{z}^2}{2} + \dots. \end{aligned} \quad (72)$$

Therefore, we can have $f(U_t, 0) = \widehat{\tau} \begin{pmatrix} f_{11} \\ f_{22} \end{pmatrix}$, where

$$\begin{aligned} f_{11} &= -a_1 u_{1t}^2(0) - b_1 (y_1^* + c_1) u_{2t}^2(-1) - 2b_1 y_2^* u_{1t}(0) u_{2t}(-1) - b_1 u_{1t}(0) u_{2t}^2(-1), \\ f_{22} &= -a_2 u_{2t}^2(0) + b_2 (y_2^* + c_2) u_{1t}^2(0) + 2b_2 y_1^* u_{1t}(0) u_{2t}(0) + b_2 u_{1t}^2(0) u_{2t}(0), \\ \langle f(U_t, 0), f_k \rangle &= \frac{\widehat{\tau}}{2} \begin{pmatrix} M_{11} \\ M_{12} \end{pmatrix} \frac{1}{\pi} \int_0^\pi \cos^3 kx dx \frac{z^2}{2} + \frac{\widehat{\tau}}{4} \begin{pmatrix} M_{21} \\ M_{22} \end{pmatrix} \frac{1}{\pi} \int_0^\pi \cos^3 kx dx z \overline{z} \\ &\quad + \frac{\widehat{\tau}}{2} \begin{pmatrix} M_{31} \\ M_{32} \end{pmatrix} \frac{1}{\pi} \int_0^\pi \cos^3 kx dx \frac{\overline{z}^2}{2} + \widehat{\tau} \begin{pmatrix} M_{41} \\ M_{42} \end{pmatrix} \frac{z^2 \overline{z}}{2} + \dots, \end{aligned} \quad (73)$$

where

$$\begin{aligned}
M_{11} &= -a_1 - b_1(y_1^* + c_1)\xi^2 e^{-2i\omega_+^k \hat{\tau}} - 2b_1 y_2^* \xi e^{-i\omega_+^k \hat{\tau}}, \\
M_{12} &= -a_2 \xi^2 + b_2(y_2^* + c_2) + 2b_2 y_1^* \xi, \\
M_{21} &= -2a_1 - 2b_1(y_1^* + c_1)\xi \bar{\xi} - 2b_1 y_2^* \left(\xi e^{-i\omega_+^k \hat{\tau}} + \bar{\xi} e^{i\omega_+^k \hat{\tau}} \right), \\
M_{22} &= -2a_2 \xi \bar{\xi} + 2b_2(y_2^* + c_2) + 2b_2 y_1^* (\xi + \bar{\xi}), \\
M_{31} &= \overline{M_{11}}, M_{32} = \overline{M_{12}}, \\
M_{41} &= -a_1 \left\langle \frac{1}{2} W_{20}^{(1)}(0) + W_{11}^{(1)}(0) \right\rangle \cos kx, \cos kx \rangle
\end{aligned} \tag{74}$$

$$\begin{aligned}
& - b_1(y_1^* + c_1) \left\langle \frac{\bar{\xi}}{2} W_{20}^{(2)}(-1) e^{i\omega_+^k \hat{\tau}} + \xi W_{11}^{(2)}(-1) e^{-i\omega_+^k \hat{\tau}} \right\rangle \cos kx, \cos kx \rangle \\
& - 2b_1 y_2^* \left\langle \frac{1}{2} W_{11}^{(2)}(-1) + \frac{1}{4} W_{20}^{(2)}(-1) + \frac{\bar{\xi}}{4} W_{20}^{(1)}(0) e^{i\omega_+^k \hat{\tau}} + \frac{\xi}{4} W_{11}^{(1)}(0) e^{-i\omega_+^k \hat{\tau}} \right\rangle \cos kx, \cos kx \rangle \\
& - b_1 \left\langle \xi \bar{\xi} + \frac{\xi^2}{2} e^{-2i\omega_+^k \hat{\tau}} \right\rangle \cos kx, \cos kx \rangle, \\
M_{42} &= -a_2 \left\langle \xi W_{11}^{(2)}(0) + \frac{\bar{\xi}}{2} W_{20}^{(2)}(0) \right\rangle \cos kx, \cos kx \rangle + b_2(y_2^* + c_2) \left\langle W_{11}^{(2)}(0) + \frac{1}{2} W_{20}^{(1)}(0) \right\rangle \cos kx, \cos kx \rangle \\
& + 2b_2 y_1^* \left\langle \frac{1}{2} W_{11}^{(2)}(0) + \frac{1}{4} W_{20}^{(2)}(0) + \frac{\bar{\xi}}{4} W_{20}^{(1)}(0) + \frac{\xi}{2} W_{11}^{(1)}(0) \right\rangle \cos kx, \cos kx \rangle \\
& + b_2 \langle (\bar{\xi} + 2\xi) \cos kx, \cos kx \rangle,
\end{aligned} \tag{75}$$

$$\langle W_{ij}^{(n)}(\theta), \cos kx \rangle = \frac{1}{\pi} \int_0^\pi W_{ij}^{(n)}(\theta)(x) \cos kx dx, \quad i + j = 2, n = 1, 2. \tag{76}$$

Note that $\int_0^\pi \cos^3 kx dx = 0$ for $\forall k \in \mathbb{N}$.

Let $(\psi_1, \psi_2) = \Psi_1(0) - i\Psi_2(0)$. Comparing the coefficient with that of (70), we can obtain

$$\begin{aligned}
g_{20} &= \begin{cases} 0, & k \in \mathbb{N}, \\ \frac{\hat{\tau}}{2} (M_{11}\psi_1 + M_{12}\psi_2), & k = 0, \end{cases} \\
g_{11} &= \begin{cases} 0, & k \in \mathbb{N}, \\ \frac{\hat{\tau}}{4} (M_{21}\psi_1 + M_{22}\psi_2), & k = 0, \end{cases} \tag{77}
\end{aligned}$$

$$g_{02} = \overline{g_{20}},$$

$$g_{21} = \hat{\tau} M_{41} \psi_1 + \hat{\tau} M_{42} \psi_2.$$

Next, we compute the terms $W_{20}(\theta)$ and $W_{11}(\theta)$ that appear in the equation for g_{21} . Using a method to the one used in [32, 33] and [27], it follows from (69) that

$$\dot{W}(z, \bar{z}) = W_{20} z \dot{z} + W_{11} z \dot{\bar{z}} + W_{11} z \ddot{\bar{z}} + W_{02} \bar{z} \dot{z} + \dots, \tag{78}$$

$$A_{\hat{\tau}} W = A_{\hat{\tau}} W_{20} \frac{z^2}{2} + A_{\hat{\tau}} W_{11} z \bar{z} + A_{\hat{\tau}} W_{02} \frac{\bar{z}^2}{2} + \dots. \tag{79}$$

In addition, we can get from [30] that

$$\dot{W} = A_{\hat{\tau}} W + H(z, \bar{z}), \tag{80}$$

where

$$\begin{aligned}
H(z, \bar{z}) &= H_{20} \frac{z^2}{2} + H_{11} z \bar{z} + H_{02} \frac{\bar{z}^2}{2} + \dots = X_0 f(U_t, 0) \\
& - \Phi(\Psi, \langle X_0 f(U_t, 0), f_k \rangle) \cdot f_k.
\end{aligned} \tag{81}$$

Then, from (67) and (70)–(81), we have

$$\begin{cases} (2i\omega_+^k \hat{\tau} - A_{\hat{\tau}}) W_{20} = H_{20}, \\ -A_{\hat{\tau}} W_{11} = H_{11}, \\ (-2i\omega_+^k \hat{\tau} - A_{\hat{\tau}}) W_{02} = H_{02}. \end{cases} \tag{82}$$

Since $-A_{\hat{\tau}}$ has only two eigenvalues $\pm i\omega_+^k \hat{\tau}$, (82) has a unique solution W_{ij} in Q given by

$$\begin{cases} W_{20} = (2i\omega_+^k \hat{\tau} - A_{\hat{\tau}})^{-1} H_{20}, \\ W_{11} = -A_{\hat{\tau}}^{-1} H_{11}, \\ W_{02} = (-2i\omega_+^k \hat{\tau} - A_{\hat{\tau}})^{-1} H_{02}. \end{cases} \quad (83)$$

From (81), we can obtain, for $\theta \in [-1, 0)$, that

$$\begin{aligned} H(z, \bar{z}) &= -\Phi(\theta) \Psi(0) \langle f(U_t, 0), f_k \rangle \cdot f_k \\ &= -\left(\frac{p_1(\theta) + p_2(\theta)}{2}, \frac{p_1(\theta) - p_2(\theta)}{2i} \right) (\Psi_1(0), \Psi_2(0)) \langle f(U_t, 0), f_k \rangle \cdot f_k \\ &= -\frac{1}{2} [p_1(\theta) (\Psi_1(0) - i\Psi_2(0)) + p_2(\theta) (\Psi_1(0) + i\Psi_2(0))] \langle f(U_t, 0), f_k \rangle \cdot f_k \\ &= -\frac{1}{2} (p_1(\theta) g_{20} + p_2(\theta) \overline{g_{02}}) \cdot f_k \frac{z^2}{2} - \frac{1}{2} (p_1(\theta) g_{11} + p_2(\theta) \overline{g_{11}}) \cdot f_k z \bar{z} + \dots \end{aligned} \quad (84)$$

Hence, for $\theta \in [-1, 0)$, we have

$$H_{20}(\theta) = \begin{cases} 0, & k \in \mathbb{N}, \\ -\frac{1}{2} [p_1(\theta) g_{20} + p_2(\theta) \overline{g_{02}}] \cdot f_0, & k = 0, \end{cases} \quad (85)$$

$$H_{11}(\theta) = \begin{cases} 0, & k \in \mathbb{N}, \\ -\frac{1}{2} [p_1(\theta) g_{11} + p_2(\theta) \overline{g_{11}}] \cdot f_0, & k = 0, \end{cases} \quad (86)$$

$$H(z, \bar{z})(0) = f(U_t, 0) - \Phi(\Psi, \langle X_0 f(U_t, 0), f_k \rangle) \cdot f_k,$$

$$H_{20}(0) = \begin{cases} \frac{\hat{\tau}}{2} \begin{pmatrix} M_{11} \\ M_{12} \end{pmatrix} \cos^2 kx, & k \in \mathbb{N}, \\ \frac{\hat{\tau}}{2} \begin{pmatrix} M_{11} \\ M_{12} \end{pmatrix} - \frac{p_1(0) g_{20} + p_2(0) \overline{g_{02}}}{2} \cdot f_0, & k = 0, \end{cases} \quad (87)$$

$$H_{11}(0) = \begin{cases} \frac{\hat{\tau}}{4} \begin{pmatrix} M_{21} \\ M_{22} \end{pmatrix} \cos^2 kx, & k \in \mathbb{N}, \\ \frac{\hat{\tau}}{4} \begin{pmatrix} M_{21} \\ M_{22} \end{pmatrix} - \frac{p_1(0) g_{11} + p_2(0) \overline{g_{11}}}{2} \cdot f_0, & k = 0. \end{cases} \quad (88)$$

From the definition of $A_{\hat{\tau}}$ and (82), we can get

$$\dot{W}_{20}(\theta) = 2i\omega_+^k \widehat{\tau} W_{20}(\theta) + \frac{1}{2} [p_1(\theta)g_{20} + p_2(\theta)\overline{g_{02}}] \cdot f_k. \quad (89)$$

Note that $p_1(\theta) = p_1(0)e^{i\omega_+^k \widehat{\tau}\theta}$, $\theta \in [-1, 0]$. Therefore,

$$W_{20}(\theta) = \frac{1}{2} \left[\frac{ig_{20}}{\omega_+^k \widehat{\tau}} p_1(\theta) + \frac{i\overline{g_{02}}}{3\omega_+^k \widehat{\tau}} p_2(\theta) \right] \cdot f_k + E_1 e^{2i\omega_+^k \widehat{\tau}\theta} \quad (90)$$

where

$$\begin{aligned} & 2i\omega_+^k \widehat{\tau} \left[\frac{1}{2} \left(\frac{ig_{20}}{\omega_+^0 \widehat{\tau}} p_1(0) \cdot f_0 + \frac{i\overline{g_{02}}}{3\omega_+^0 \widehat{\tau}} p_2(0) \cdot f_0 \right) + E_1 \right] - \widehat{\tau} D\Delta \left[\frac{1}{2} \left(\frac{ig_{20}}{\omega_+^0 \widehat{\tau}} p_1(0) \cdot f_0 + \frac{i\overline{g_{02}}}{3\omega_+^0 \widehat{\tau}} p_2(0) \cdot f_0 \right) + E_1 \right] \\ & - L(\widehat{\tau}) \left[\frac{1}{2} \left(\frac{ig_{20}}{\omega_+^0 \widehat{\tau}} p_1(\theta) \cdot f_0 + \frac{i\overline{g_{02}}}{3\omega_+^0 \widehat{\tau}} p_2(\theta) \cdot f_0 \right) + E_1 e^{2i\omega_+^0 \widehat{\tau}\theta} \right] \\ & = \frac{\widehat{\tau}}{2} \begin{pmatrix} M_{11} \\ M_{12} \end{pmatrix} - \frac{1}{2} (p_1(0)g_{20} + p_2(0)\overline{g_{02}}) \cdot f_0. \end{aligned} \quad (92)$$

Note that

$$\begin{aligned} \widehat{\tau} D\Delta [p_1(0) \cdot f_0] + L(\widehat{\tau}) [p_1(\theta) \cdot f_0] &= i\omega_+^0 \widehat{\tau} p_1(0) \cdot f_0, \\ \widehat{\tau} D\Delta [p_2(0) \cdot f_0] + L(\widehat{\tau}) [p_2(\theta) \cdot f_0] &= i\omega_+^0 \widehat{\tau} p_2(0) \cdot f_0. \end{aligned} \quad (93)$$

Then, for $k \in \mathbb{N}_0$, we get

$$2i\omega_+^k \widehat{\tau} E_1 - \widehat{\tau} D\Delta E_1 - L(\widehat{\tau}) (E_1 e^{2i\omega_+^0 \widehat{\tau}\theta}) = \frac{\widehat{\tau}}{2} \begin{pmatrix} M_{11} \\ M_{12} \end{pmatrix} \cos^2 kx. \quad (94)$$

By (94), we have

$$E_1 = \frac{1}{2} E \begin{pmatrix} M_{11} \\ M_{12} \end{pmatrix} \cos^2 kx, \quad (95)$$

where

$$E = \begin{pmatrix} 2i\omega_+^k + a_1(y_1^* + c_1) + D_1 k^2 & -2b_1 y_2^*(y_1^* + c_1) e^{-2i\omega_+^k \widehat{\tau}} \\ -2b_2 y_1^*(y_2^* + c_2) & 2i\omega_+^k + a_2(y_2^* + c_2) + D_2 k^2 \end{pmatrix}^{-1}. \quad (96)$$

Similarly, we can also have

$$-\dot{W}_{11}(\theta) = -\frac{1}{2} [p_1(\theta)g_{11} + p_2(\theta)\overline{g_{11}}] \cdot f_k, \quad \theta \in [-1, 0], \quad (97)$$

$$W_{11}(\theta) = \frac{1}{2} \left[\frac{-ig_{11}}{\omega_+^k \widehat{\tau}} p_1(\theta) + \frac{i\overline{g_{11}}}{\omega_+^k \widehat{\tau}} p_2(\theta) \right] + E_2, \quad (98)$$

$$E_1 = \begin{cases} W_{20}(0), & k \in \mathbb{N}, \\ W_{20}(0) - \frac{1}{2} \left[\frac{ig_{20}}{\omega_+^k \widehat{\tau}} p_1(0) + \frac{i\overline{g_{02}}}{3\omega_+^k \widehat{\tau}} p_2(0) \right] \cdot f_0, & k = 0. \end{cases} \quad (91)$$

By using the definition of $A_{\widehat{\tau}}$ and combining (82) and (90), we can obtain the following equation:

where

$$\begin{aligned} E_2 &= \frac{1}{4} E' \begin{pmatrix} M_{21} \\ M_{22} \end{pmatrix} \cos^2 kx, \\ E' &= \begin{pmatrix} a_1(y_1^* + c_1) + D_1 k^2 & -2b_1 y_2^*(y_1^* + c_1) \\ -2b_2 y_1^*(y_2^* + c_2) & a_2(y_2^* + c_2) + D_2 k^2 \end{pmatrix}^{-1}. \end{aligned} \quad (99)$$

Substituting (90) and (98) into the expression of g_{21} , we can compute the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_+^k \widehat{\tau}} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{1}{2}g_{21}, \\ \mu_2 &= -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_j^k)\}}, \end{aligned} \quad (100)$$

$$\beta_2 = 2\operatorname{Re}\{c_1(0)\},$$

$$T_2 = -\frac{1}{\omega_+^k \widehat{\tau}} (\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_j^k)\}).$$

Through the above analysis, we can draw the following conclusions.

Theorem 3. Suppose conditions (H1)–(H6) hold. Then, the parameters μ_2 , β_2 , and T_2 determine the properties of Hopf bifurcation, where μ_2 determines the direction of the Hopf bifurcation, β_2 determines the stability of the bifurcating periodic solutions on the center manifold, and T_2 determines the period of the bifurcating periodic solutions, i.e.,

FIGURE 1: The positive constant steady state of the system with Neumann boundary is locally asymptotically stable for $\tau_1 = 0.9$, $\tau_2 = 0.9$, and $\tau = \tau_1 + \tau_2 < \tau_0^0 = 2.3633$. (a) Spatial distribution of $y_1(t, x)$. (b) Spatial distribution of $y_2(t, x)$.

FIGURE 2: The spatial pattern of the system with Neumann boundary for $\tau_1 = 0.9$, $\tau_2 = 0.9$, and $\tau = \tau_1 + \tau_2 < \tau_0^0 = 2.3633$. (a) Spatial pattern of $y_1(t, x)$. (b) Spatial pattern of $y_2(t, x)$.

FIGURE 3: The positive constant steady state of the system with Neumann boundary loses its stability, and Hopf bifurcation occurs for $\tau_1 = 0.9$, $\tau_2 = 1.5$, and $\tau = \tau_1 + \tau_2 > \tau_0^0 = 2.3633$. (a) Spatial distribution of $y_1(t, x)$. (b) Spatial distribution of $y_2(t, x)$.

FIGURE 4: The spatial pattern of the system with Neumann boundary for $\tau_1 = 0.9$, $\tau_2 = 1.5$, and $\tau = \tau_1 + \tau_2 > \tau_0^0 = 2.3633$. (a) Spatial pattern of $y_1(t, x)$. (b) Spatial pattern of $y_2(t, x)$.

FIGURE 5: The positive constant steady state of the system with Dirichlet boundary is locally asymptotically stable for $\tau_1 = 0.9$, $\tau_2 = 0.9$, and $\tau = \tau_1 + \tau_2 < \tau_0^0 = 2.3633$. (a) Spatial distribution of $y_1(t, x)$. (b) Spatial distribution of $y_2(t, x)$.

FIGURE 6: The spatial pattern of the system with Dirichlet boundary for $\tau_1 = 0.9$, $\tau_2 = 0.9$, and $\tau = \tau_1 + \tau_2 < \tau_0^0 = 2.3633$. (a) Spatial pattern of $y_1(t, x)$. (b) Spatial pattern of $y_2(t, x)$.

FIGURE 7: The positive constant steady state of the system with Dirichlet boundary loses its stability, and Hopf bifurcation occurs for $\tau_1 = 0.9$, $\tau_2 = 1.5$, and $\tau = \tau_1 + \tau_2 > \tau_0^0 = 2.3633$. (a) Spatial distribution of $y_1(t, x)$. (b) Spatial distribution of $y_2(t, x)$.

FIGURE 8: The spatial pattern of the system with Dirichlet boundary for $\tau_1 = 0.9$, $\tau_2 = 1.5$, and $\tau = \tau_1 + \tau_2 > \tau_0^0 = 2.3633$. (a) Spatial pattern of $y_1(t, x)$. (b) Spatial pattern of $y_2(t, x)$.

FIGURE 9: The bifurcation diagram of the system for y_1 with respect to τ . (a) Without diffusion. (b) With diffusion.

FIGURE 10: The bifurcation diagram of the system for y_2 with respect to τ . (a) Without diffusion. (b) With diffusion.

FIGURE 11: The spatial pattern of the system with Neumann boundary for y_1 with different D_i at times $t = 350-380$ (from top to bottom, $D_i = 0.01, 0.001, 0.0005, 0.0001$).

FIGURE 12: The spatial pattern of the system with Dirichlet boundary for y_2 with different D_i at times $t = 350-380$ (from top to bottom, $D_i = 0.01, 0.001, 0.0005, 0.0001$).

- (i) If $\mu_2 > 0$ ($\mu_2 < 0$), the direction of the Hopf bifurcation is forward (backward)
- (ii) If $\beta_2 < 0$ ($\beta_2 > 0$), the bifurcating periodic solutions are stable (unstable)
- (iii) If $T_2 > 0$ ($T_2 < 0$), the period increases (decreases)

4. Numerical Simulations

Example 1. We choose $D_1 = D_2 = 1, c_1 = c_2 = 1.5, d_1 = 1, d_2 = 0.6, a_1 = 0.4, a_2 = 0.9, b_1 = 0.5$, and $b_2 = 0.2$, which satisfy the above conditions. We take the initial conditions as

$$\begin{aligned} y_1(t, x) &= 1.2354 \times (1 + 0.2 \sin(3.7x) + 0.3 \sin(1.4x - 0.6)), \\ y_2(t, x) &= 1.0058 \times (1 + 0.2 \sin(2.7x) + 0.3 \sin(0.74x + 0.5)). \end{aligned} \quad (101)$$

Then, system (4) with the Neumann boundary has a unique positive constant steady state $E = (1.2354, 1.0058)$. By performing calculations, we can obtain $\omega_+^0 = 0.8838$ and $\tau_0^0 = 2.3633$. According to Theorem 2, E is locally asymptotically stable if $\tau \in [0, \tau_0^0)$ and is unstable if $\tau > \tau_0^0$. When τ crosses through the critical point τ_0^0 , E loses its stability, and Hopf bifurcation occurs. The simulation results are shown in

Figures 1–4. Based on formulas in Section 3, we can perform computations and obtain that $\text{Re}(c_1(0)) = -2.7368$, $\mu_2 = 1.8499 > 0$, and $\beta_2 = -5.4735 < 0$. Hence, when τ crosses through τ_0^0 , the bifurcation takes place, and a homogeneous, asymptotically stable periodic solution appears near the trivial steady-state solution E , both of which are shown in Figures 3 and 4.

Example 2. In this case, we choose diffusion coefficients $D_1 = D_2 = 0.001$, and we keep other parameter values and initial conditions unchanged. We consider Hopf bifurcation of the system with the Dirichlet boundary condition:

$$\begin{aligned} y_1(t, 0) &= y_1(t, \pi) = 1.2, \\ y_2(t, 0) &= y_2(t, \pi) = \tanh\left(1 + \frac{t}{3\pi}\right). \end{aligned} \quad (102)$$

The results of the simulation are shown in Figures 5–8. Apparently, when τ passes through the critical value τ_0^0 , Hopf bifurcation occurs, and the spatially homogeneous bifurcating periodic solution appears, which suggests that there are periodic oscillations in the interaction between two enterprises.

Figures 9 and 10 show the bifurcation state of the system with and without diffusion for y_1 and y_2 under the Neumann boundary, respectively. It is found that the diffusion can affect the amplitude of oscillation of the periodic solutions to a certain extent, and the bifurcating periodic solutions are regular and orbitally asymptotically stable.

To illustrate the spatial effect of diffusion on the system, we adopt the method used by references [14, 21] and choose different diffusion coefficients for two different boundary systems under the situations of asymptotically stable periodic solutions. The corresponding spatially dynamic behavior diagrams are shown in Figures 11 and 12.

5. Conclusions

Since the diffusions of information, technology, and products exist objectively, we deem that the model with reaction-diffusion and delays is more suitable for the interaction between two enterprises, which has no relevant results for system (3) at present. The aim of this study is first to introduce multiple delays and diffusion factors into the traditional two-enterprise interaction model and then to explore the asymptotic stability of the equilibrium and the periodicity of the solution by taking the sum of two time delays as the bifurcation parameter. It is shown that when the time delay passes through a critical value, the system changes from a stable state to an unstable state. The numerical simulations demonstrate that the diffusion effect causes rich, spatially dynamic behaviors.

We acknowledge that there are some economic implications. In the enterprise cluster, two enterprises can achieve a stable state and symbiosis of development via dynamic interaction. Although there exists competition, cooperation is very effective, and the enterprises can achieve long-term codevelopment through appropriate positive interactions. It is noteworthy that in addition to the varying economic environments, competition powers, mark-saturation degrees, and delay effects, the diffusion speed of products or technologies is an important factor in the self-development of enterprises. When enterprises are in a state of cyclical fluctuations, appropriate diffusion effects may change the amplitude of these fluctuations, which is beneficial to the enterprises to a certain extent. Furthermore, it is easy to find that differences in the initial resources or positions of an enterprise have a certain impact on its future self-development. Therefore, it is obvious that with the development of the digital economy and the emergence of industrial clusters, the internal relationships between enterprises are increasingly in line with the ecological model. As a consequence, it is of great practical significance to study the dynamic properties of these systems for understanding the evolution of market structure and taking corresponding regulatory strategies. Benefiting from the latest research studies, we will further consider the dynamic behaviors of discrete-time and fractional-order models in the future.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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