Assouad and Lower Dimensions of Some Homogeneous Cantor Sets

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We compute the Assouad dimensions and the lower dimensions of a class of homogeneous Cantor sets without the condition that the smallest compression ratio $C_\ast > 0$ and find that the lower dimension of a homogeneous Cantor set $E$ may be any number in the interval $[0, \dim_H E]$ and the Assouad dimension of $E$ may be any number in the interval $[\dim_H E, 1]$.

1. Introduction

Let us begin with the definition of the Assouad dimension and the lower dimension. The Assouad dimension of a nonempty set $F \subseteq \mathbb{R}^d$ is defined by

$$\dim_A F = \inf \{ \alpha \geq 0 : \text{there exists a constant } c > 0 \text{ such that}$$

$$\text{for any } 0 < r < R, \text{ and } x \in F,$$

$$N_r (B(x, R) \cap F) \leq c \left( \frac{R}{r} \right)^{\alpha} \}.$$  \hfill (1)

The lower dimension of $F$ is defined by

$$\dim_L F = \sup \{ \alpha \geq 0 : \text{there exists a constant } c > 0 \text{ such that}$$

$$\text{for any } 0 < r < R \leq |F|, \text{ and } x \in F,$$

$$N_r (B(x, R) \cap F) \geq c \left( \frac{R}{r} \right)^{\alpha} \}.$$

$N_r (B(x, R) \cap E)$ is the smallest number of open sets with a diameter less than $r$ needed to cover $B(x, R) \cap E$.

The Assouad dimension which is introduced by Assouad [1] has recently received an enormous interest in the mathematical literature due to its connections with the doubling property. The dual notion of the Assouad dimension is the lower dimension which was introduced by Larman [2]. Just like the Assouad dimension, the lower dimension has also received an enormous interest in the mathematical literature due to its connections with the uniform property of metric spaces. As a result of this, a large number of papers have investigated the Assouad dimension and the lower dimension of different classes of fractal sets. Fraser [3] has a detailed discussion of the Assouad dimension, the lower dimension, and its use in fractal geometry. Olsen [4] computed the Assouad dimension of a graph-directed Moran fractal satisfying the open-set conditions which are Ahlfors regular. However, in general, it is difficult to obtain the Assouad dimensions of sets which are not Ahlfors regular. Mackay [5] calculated the Assouad dimension of the self-affine carpets of Bedford and McMullen, and his main result solved the problem posed by Olsen [4]. For the Moran sets introduced by Wen [6] which are not Ahlfors regular, Li et al. [7] obtained the Assouad dimensions of Moran sets under suitable condition. Li [8] also proved that the Assouad dimensions of some Moran sets coincide with their packing and upper box dimensions under the condition that the smallest compression ratio $C_\ast > 0$ and therefore gave a conjecture that the conclusion remains true if the condition $C_\ast > 0$ is removed. Xiao [9] proved that the lower dimensions of a class of Moran sets coincide with their Hausdorff dimensions under the condition that the compression sequence and the compression ratio sequence are smoothly changing and found some homogeneous Cantor sets whose Assouad dimension is not
equal to their upper box dimensions and packing dimensions and therefore gave a negative answer to the conjecture in the paper [8]. In this paper, we compute the Assouad dimensions and the lower dimensions of a class of homogeneous Cantor sets without the condition that the smallest compression ratio $C_\ast > 0$ and find that the lower dimension of a homogeneous Cantor set $E$ may be any a number in the interval $[0, \dim_E E]$ and the Assouad dimension of $E$ may be any a number in the interval $[\dim_H E, 1]$.

2. Homogeneous Cantor Set

Firstly, let us recall the definition of Moran sets introduced by Wen [5]. Let $\{n_k\}_{k \geq 1} \subset \mathbb{N}$ be a sequence of positive integer (we assume $n_k \geq 2$). Define $D_0 = \emptyset$, and for any $k \geq 1$, set $D_{m,k} = \{\sigma_m \sigma_{m+1} \cdots \sigma_k: 1 \leq \sigma_j \leq n_j, m \leq j \leq k\}$, $D_k = D_{1,k}$ and $D = \bigcup_{k \geq 0} D_k$. If $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k \in D_k$ and $\tau = \tau_1 \tau_2 \cdots \tau_m \in D_{k+1,m}$, we define $\sigma \ast \tau = \sigma_1 \cdots \sigma_1 \tau_1 \cdots \tau_m$.

Definition 1. Suppose that $J \subset \mathbb{R}^d$ is a compact set with $\overline{m}J = J$. Let $\{\Phi_k\}$ be a sequence of positive real vectors with $\{\Phi_k\} = (c_{k,1}, c_{k,2}, \ldots, c_{k,n_k}), \sum_{j=1}^{n_k} c_{k,j} \leq 1, k \in \mathbb{N}$. We say the collection $\mathcal{F} = \{J; \sigma \in \Phi\}$ of closed subsets of $J$ possesses the Moran structure if it satisfies the following Moran structure conditions (MSC):

1. For $\sigma \in D$, $J_\sigma$ is geometrically similar to $J$; that is, there exists a similarity $S_\sigma: \mathbb{R}^d \to \mathbb{R}^d$ such that $J_\sigma = S_\sigma(J)$. For convenience, we write $J_\emptyset = J$.
2. For all $k \geq 0$ and $\sigma, \tau \in \Phi_k$ if $J_{\sigma\tau}$ is a subset of $J_\tau$ and satisfy that $\text{int}[J_{\sigma\tau}] \cap \text{int}[J_{\sigma\tau}'] = \emptyset$ whenever $\sigma \neq \tau$.
3. For any $k \geq 1$ and $\sigma \in D_{k+1}, 1 \leq j \leq n_k$,

$$\frac{|J_{\sigma\tau}|}{|J_{\sigma}|} = c_{k,j},$$

where $|A|$ denotes the diameter of $A$.

Suppose that $\mathcal{F} = \{J; \sigma \in \Phi\}$ is a collection of closed subsets of $J$ possessing the Moran structure, set

$$E_k = \bigcup_{\sigma \in D_k} J_\sigma,$$
$$E = \bigcap_{k \geq 0} E_k.$$

It is ready to see that $E$ is a nonempty compact set. The set $E = E(\mathcal{F})$ is called the Moran set associated with the collection $\mathcal{F}$.

Let $\mathcal{F}_k = \{J; \sigma \in D_k\}$, and $\mathcal{F} = \bigcup_{k \geq 0} \mathcal{F}_k$. The elements of $\mathcal{F}_k$ are called kth-level basic sets of $E$, and the elements of $\mathcal{F}$ are called basic sets of $E$. Suppose the set $J$ and the sequences $\{n_k\}, \{\Phi_k\}$ are given. We denote by $\mathcal{M} = \mathcal{M}(\{n_k\}, \{\Phi_k\})$ the class of the Moran sets satisfying the MSC. We call $\mathcal{M}(\{n_k\}, \{\Phi_k\})$ the Moran class associated with the triplet $(J, \{n_k\}, \{\Phi_k\})$.

Definition 2. Suppose $J$ is the interval $[0, 1]$ and $c_{k,j} = c_k$ for any $k > 0, 1 \leq j \leq n_k$ in Definition 1. For all $k > 0, \sigma \in D_k$,

$$\text{dist}(J_{\sigma(1)}, J_{\sigma(j+1)}) = \text{dist}(J_{\sigma(j)}, J_{\sigma(j+1)}) \quad (1 \leq j \leq n_k - 1),$$

and the left endpoint of $J_{\sigma(1)}$ is the left endpoint of $J_\sigma$ and the right endpoint of $J_{\sigma(n_k)}$ is the right endpoint of $J_\sigma$. The set $E(\mathcal{M}(\{n_k\}, \{c_k\}))$ is called the homogeneous Cantor set.

Write $E = \mathcal{E}(J, \{n_k\}, \{c_k\})$.

Remark 1. Let $C_\ast = \text{inf} c_k$. In the present paper, the author obtains formulas for the Assouad dimension and the lower dimension of sets belonging to this class of Moran fractals without assuming the condition $C_\ast > 0$.

Theorem 1 (see [10]). Suppose the set $E(= \mathcal{E}(J, \{n_k\}, \{c_k\}))$ is a homogeneous Cantor set. Then,

$$\dim_H E = \liminf_{k \to \infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_k}$$
$$\dim_P E = \dim_H E = \limsup_{k \to \infty} \frac{\log n_1 n_2 \cdots n_{k+1}}{-\log c_1 c_2 \cdots c_k + \log n_{k+1}},$$

where $\dim_H E$, $\dim_P E$, and $\overline{\dim}_E E$ denote the Hausdorff, packing, and upper box dimensions of $E$, respectively.

3. Statement and Proof of Results

Now, consider four conditions that $\{n_k\}$ and $\{c_k\}$ satisfy.

Condition 1. Suppose $\inf_i n_i = n_0 > 0$ and there are two strictly increasing sequence $(l_i)$, $(k_i)$ of positive integers with $l_i \leq k_i$ such that

(a) $\lim c_{l_i+1} \cdots c_{k_i} = 0$

(b) The following limit

$$z = \lim_{i \to \infty} \frac{\log n_1 \cdots n_k}{-\log c_{l_i} \cdots c_k}$$

exists (note that $z$ is defined as the value of the limit in (6))

(c) For each $\varepsilon > 0$, there is an $r_0$ (only depending on $\varepsilon$) such that

$$\frac{\log n_1 \cdots n_k}{-\log c_{l_i} \cdots c_k} > z - \varepsilon,$$

for each pair of positive integers $l$ and $k$ with $l \leq k$ and $c_{l+1} \cdots c_k < r_0$.

Condition 2. Suppose $\lim c_{l_i+1} \cdots c_{k_i} = 0$.

Condition 3. Suppose $\inf_i n_i = n_0 < + \infty$ and there are two strictly increasing sequence $(l_i)$, $(k_i)$ of positive integers with $l_i \leq k_i$ such that

(a) $\lim c_{l_i+1} \cdots c_{k_i} = 0$

(b) The following limit

$$\overline{z} = \lim_{i \to \infty} \frac{\log n_1 \cdots n_k}{-\log c_{l_i+1} \cdots c_k}$$

exists (note that $\overline{z}$ is defined as the value of the limit in (8))

(c) For each $\varepsilon > 0$, there is an $r_0$ (only depending on $\varepsilon$) such that

$$\frac{\log n_1 \cdots n_k}{-\log c_{l_i+1} \cdots c_k} > \overline{z} - \varepsilon,$$
exists (note that \( 3 \) is defined as the value of the limit in (8))

(c) For each \( \varepsilon > 0 \), there is an \( r_0 \) (only depending on \( \varepsilon \)) such that

\[
\log \frac{n_{i+1} \cdots n_k}{-\log \epsilon} < 3 + \varepsilon,
\]

for each pair of positive integers \( l \) and \( k \) with \( l \leq k \) and \( c_{i+1} \cdots c_k < r_0 \).

**Condition 4.** \( \lim \sup \tau_n = \infty \)

**Theorem 2.** Suppose the set \( E = \mathcal{C}(J, [n_k], [c_k]) \) is a homogeneous Cantor set. Then,

\[
\dim_{f} E = \begin{cases} 
3, & \text{if Condition 1 is satisfied;} \\
0, & \text{if Condition 2 is satisfied.}
\end{cases}
\]

\[
\dim_{E} E = \begin{cases} 
3, & \text{if Condition 3 is satisfied;} \\
1, & \text{if Condition 4 is satisfied.}
\end{cases}
\]

**Proof** of Theorem 2. For two positive numbers \( r \) and \( R \) with \( r < R \leq 1 \), there exist \( l, k \in \mathbb{N} \) and \( l \leq k \) such that

\[
c_1 c_2 \cdots c_l \leq R < c_1 c_2 \cdots c_l,
\]

\[
c_1 c_2 \cdots c_k \leq r < c_1 c_2 \cdots c_k.
\]

For \( \sigma \in \mathcal{D}_k \), we consider the relationship between \( r \) and \( N_r (J \cap E) \). Notice that

\[
c_1 c_2 \cdots c_{k+1} + \frac{c_1 c_2 \cdots c_k (1 - n_k c_{k+1})}{n_{k+1} - 1} = \frac{c_1 c_2 \cdots c_k (1 - c_{k+1})}{n_{k+1} - 1},
\]

where \( c_1 c_2 \cdots c_{k+1} \) is the length of the basic elements of order \( k + 1 \) of the homogeneous Cantor set \( E \), and \( c_1 c_2 \cdots c_k (1 - n_k c_{k+1})/(n_{k+1} - 1) \) is the length of the interval among the basic elements of order \( k + 1 \) of the homogeneous Cantor set \( E \). By a simple calculation, we obtain the relationship between \( r \) and \( N_r (J \cap E) \). The results are shown in Table 1. where \( 1 \leq q \leq n_{k+1} - 1 \).

Secondly, let \( x \in E \), we consider the relationship between \( R \) and \( \#\{r \in D_{1,1} \mid B(x, R) \cap E \cap J \neq \emptyset \} \). Denote \( \#\{r \in D_{1,1} \mid B(x, R) \cap E \cap J \neq \emptyset \} \) by \( B_r \). By a simple calculation, we obtain the relationship between \( R \) and \( B_r \). The results are shown in Table 2. where \( 1 \leq p \leq n_{k+1} - 1 \).

**Table 1:** The calculation of \( N_r (J \cap E) \).

<table>
<thead>
<tr>
<th>( r \mid c_1 c_2 \cdots c_{k+1} c_1 c_2 \cdots c_k (1 - c_{k+1})/(n_{k+1} - 1) )</th>
<th>( N_r (J \cap E) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>c_1 c_2 \cdots c_{k+1} c_1 c_2 \cdots c_k (1 - c_{k+1})/(n_{k+1} - 1) + c_1 c_2 \cdots c_k (1 - c_{k+1})/(n_{k+1} - 1) + c_1 c_2 \cdots c_{k+1} )</td>
</tr>
<tr>
<td>(</td>
<td>c_1 c_2 \cdots c_{k+1} c_1 c_2 \cdots c_k (1 - c_{k+1})/(n_{k+1} - 1) + c_1 c_2 \cdots c_{k+1} + c_1 c_2 \cdots c_k (1 - c_{k+1})/(n_{k+1} - 1) )</td>
</tr>
<tr>
<td>(</td>
<td>(q - 1)c_1 c_2 \cdots c_{k+1} c_1 c_2 \cdots c_k (1 - c_{k+1})/(n_{k+1} - 1) + c_1 c_2 \cdots c_k (1 - c_{k+1})/(n_{k+1} - 1) + c_1 c_2 \cdots c_{k+1}</td>
</tr>
<tr>
<td>(</td>
<td>(n_{k+1} - 1)c_1 c_2 \cdots c_{k+1} c_1 c_2 \cdots c_k (1 - c_{k+1})/(n_{k+1} - 1) + c_1 c_2 \cdots c_{k+1} )</td>
</tr>
</tbody>
</table>

Using (13) and Tables 1–3, we obtain

\[
\frac{1}{4} B_r N_r (J \cap E) n_{l+1} \leq 2p \left( \frac{n_{k+1} + 1}{q} \right) n_{l+2} \cdots n_k \leq \frac{4 p (n_{k+1})}{q} n_{l+2} \cdots n_k,
\]

for entry 1 in Table 3, by a simple calculation, we obtain

\[
\frac{-2 \log 2 + p - \log q + \log n_{l+2} \cdots n_{k+1} \log((p - 1)(1 - c_{l+1})/(n_{l+1} - 1) + c_{l+1}) - \log((q - 1)(1 - c_{k+1})/(n_{k+1} - 1) + c_{k+1}) - \log c_{l+1} \cdots c_k}{\log R/r} \leq \frac{2 \log 2 + p - \log q + \log n_{l+2} \cdots n_{k+1} \log((p - 1)(1 - c_{l+1})/(n_{l+1} - 1) + c_{l+1}) - \log((q - 1)(1 - c_{k+1})/(n_{k+1} - 1) + c_{k+1}) - \log c_{l+1} \cdots c_k}{\log R/r}
\]
Table 2: The calculation of $B_r$.

<table>
<thead>
<tr>
<th>$R$</th>
<th>$B_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${c_1c_2\cdots c_{i+1}, c_1c_2\cdots c_n(1-c_{i+1})/(n_{i+1} - 1)}$</td>
<td>$1 \leq B_r \leq 2$</td>
</tr>
<tr>
<td>${c_1c_2\cdots c_i(1-c_{i+1})/(n_{i+1} - 1), c_1c_2\cdots c_i(1-c_{i+1})/(n_{i+1} - 1) + c_i c_1 \cdots c_{i+1}}$</td>
<td>$2 \leq B_r \leq 4$</td>
</tr>
<tr>
<td>${c_1c_2\cdots c_i(1-c_{i+1})/(n_{i+1} - 1) + c_i c_1 \cdots c_{i+1}, c_1c_2\cdots c_i(1-c_{i+1})/(n_{i+1} - 1)}$</td>
<td>$3 \leq B_r \leq 6$</td>
</tr>
<tr>
<td>${p-1} c_1c_2\cdots c_i(1-c_{i+1})/(n_{i+1} - 1) + c_i c_1 \cdots c_{i+1}, pc_1c_2\cdots c_i(1-c_{i+1})/(n_{i+1} - 1) + c_i c_1 \cdots c_{i+1}}$</td>
<td>$p \leq B_r \leq 2p$</td>
</tr>
<tr>
<td>${p-1} c_1c_2\cdots c_i(1-c_{i+1})/(n_{i+1} - 1), pc_1c_2\cdots c_i(1-c_{i+1})/(n_{i+1} - 1) + c_i c_1 \cdots c_{i+1}}$</td>
<td>$p + 1 \leq B_r \leq 2(p + 1)$</td>
</tr>
<tr>
<td>${p-1} c_1c_2\cdots c_i(1-c_{i+1})/(n_{i+1} - 1), c_1c_2\cdots c_i}$</td>
<td>$n_{i+1} \leq B_r \leq 2n_{i+1}$</td>
</tr>
</tbody>
</table>

Table 3: The calculation of $R/r$.

<table>
<thead>
<tr>
<th>$R$</th>
<th>$r$</th>
<th>$R/r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 {p-1} c_1 \cdots c_i(1-c_{i+1})/(n_{i+1} - 1) + c_i \cdots c_{i+1}$</td>
<td>$(q-1) c_1 \cdots c_k(1-c_{k+1})/(n_{k+1} - 1) + c_k \cdots c_{k+1}$</td>
<td>$(p-1) (1-c_{i+1})/(n_{i+1} - 1) + c_{i+1}/((q-1)(1-c_{k+1})/(n_{k+1} - 1) + c_{k+1})c_k \cdots c_k$</td>
</tr>
<tr>
<td>$2 (p-1) c_1 \cdots c_i(1-c_{i+1})/(n_{i+1} - 1) + c_i \cdots c_{i+1}$</td>
<td>$q_1 c_1(1-c_{k+1})/(n_{k+1} - 1)$</td>
<td>$(p-1) (1-c_{i+1})/(n_{i+1} - 1) + c_{i+1}/(q_1(1-c_{k+1})/(n_{k+1} - 1) + c_{k+1})c_k \cdots c_k$</td>
</tr>
<tr>
<td>$3 pc_1 \cdots c_i(1-c_{i+1})/(n_{i+1} - 1)$</td>
<td>$(q-1) c_1 \cdots c_k(1-c_{k+1})/(n_{k+1} - 1) + c_k \cdots c_{k+1}$</td>
<td>$p(1-c_{i+1})/(n_{i+1} - 1)/((q-1)(1-c_{k+1})/(n_{k+1} - 1) + c_{k+1})c_k \cdots c_k$</td>
</tr>
<tr>
<td>$4 pc_1 \cdots c_i(1-c_{i+1})/(n_{i+1} - 1)$</td>
<td>$qc_1 \cdots c_k(1-c_{k+1})/(n_{k+1} - 1)$</td>
<td>$p(1-c_{i+1})/(n_{i+1} - 1)/q(1-c_{k+1})/(n_{k+1} - 1) + c_{k+1})c_k \cdots c_k$</td>
</tr>
</tbody>
</table>

Table 4: The calculation of $\alpha$.

<table>
<thead>
<tr>
<th>Number</th>
<th>$N_r (B(x,R) \cap E) = \infty$</th>
<th>$\alpha(\infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$pm_{k+1}/qm_{i+1} \cdots n_k$</td>
<td>log $p - \log q + \log n_{i+1} \cdots n_k$ if $p \neq 1, q \neq 1$</td>
</tr>
<tr>
<td>2</td>
<td>$pm_{k+1}/qm_{i+1} \cdots n_k$</td>
<td>log $p - \log q + \log n_{i+1} \cdots n_k$ if $p = 1, q \neq 1$</td>
</tr>
<tr>
<td>3</td>
<td>$pm_{k+1}/qm_{i+1} \cdots n_k$</td>
<td>log $p - \log q + \log n_{i+1} \cdots n_k$ if $p \neq 1, q = 1$</td>
</tr>
<tr>
<td>4</td>
<td>$pm_{k+1}/qm_{i+1} \cdots n_k$</td>
<td>log $p - \log q + \log n_{i+1} \cdots n_k$ if $p = 1, q = 1$</td>
</tr>
</tbody>
</table>

for $q \neq 0$ and $p \neq 0$.

In order to simplify expression, the notation $A \geq B$ is introduced. If there exist two constant positive numbers $a_1, a_2$ such that $a_1 A \leq B \leq a_2 A$ we remark $A \geq B$. Denote log $N_r (B(x,R) \cap E)/\log R/r$ by $\alpha$.

Using (13) and Tables 1 and 2, one gets

$$1 \frac{pm_{i+1}}{q+1} n_{i+1} \cdots n_k \leq N_r (B(x,R) \cap E) \leq \frac{4(p+1) pm_{k+1}}{q} n_{i+1} \cdots n_k$$

(16)

for $1 \leq p \leq n_{i+1} - 1$ and $1 \leq q \leq n_{k+1} - 1$. It follows that $N_r (B(x,R) \cap E) = pm_{k+1}/qm_{i+1} \cdots n_k$ where the constants $a_1 = (1/8)$ and $a_2 = (8)$ be chosen uniformly in $r, R$ and $x$. By a simple calculation that is similar to entry 1 in Table 3, we obtain Table 4.

Notice that log $p/\log (p - 1 - c_{i+1})/(n_{i+1} - 1) + c_{i+1}$ is decreasing for $1 \leq p \leq c_{i+1} - 1$ and log $q/\log (1 - c_{k+1})/(n_{k+1} - 1)$ is decreasing for $1 \leq q \leq c_{k+1} - 1$, using (16) and Table 4, we obtain that

$$-3 \log 2 - \log (n_{k+1} - 1) + \log n_{i+1} \cdots n_k \leq \log (1 - c_{i+1}) - \log (n_{i+1} - 1) + \log c_{i+1} \cdots c_k$$

(17)

If Condition 1 is satisfied, we prove dim $E = \infty$. Using (c) in Condition 1 and (17) and notice that $\lim_{i \to \infty} n_{i+1} / n_{i+1} > 0$, it follows that for all small $\varepsilon > 0$, there exists $r_0 > 0$ such that
\[ N_r(B(x, R) \cap E) = \left( \frac{R}{r} \right)^{\frac{\alpha}{r}} \geq \left( \frac{R}{r} \right)^{-\log(\log(n_{k+1}-1)+\log(n_{k+2}-n_{k+1})/\log(1-c_{i+1})-\log(n_{i+1}-1)-\log(c_{i+1}-c_i))} > \left( \frac{R}{r} \right)^{-\varepsilon}, \tag{18} \]

for \( c_i \cdots c_{k+1} < r_0 \). If \( c_i \cdots c_{k+1} \geq r_0 \), notice that \( R/r \leq 1/c_i \cdots c_{k+1} \leq 1/r_0 \),

\[ N_r(B(x, R) \cap E) \geq \frac{1}{\log R_0} \left( \frac{R}{r} \right)^{\varepsilon} \geq \frac{1}{\log R_0} \left( \frac{R}{r} \right)^{-\varepsilon}. \tag{19} \]

By combining (18) and (19), we obtain that \( \dim E \geq \varepsilon \).

To prove that \( \dim E \leq \varepsilon \), it suffices to find a constant \( c > 0 \) and a sequence of points \( x_i \in E \) and scales \( 0 < r_i < R_i \) such that \( R_i/r_i \to \infty \) and for all \( i \)

\[ N_r(B(x_i, R_i) \cap E) \leq 2n_i \cdots n_{k+1} = 2 \left( \frac{R_i}{r_i} \right)^{\log n_i \cdots n_{k+1}/\log c_i \cdots c_{k+1}} \leq 2 \left( \frac{R_i}{r_i} \right)^{-\varepsilon}, \tag{21} \]

for all \( i > i_0 \). The proof of \( \dim E = \varepsilon \) is finished.

If Condition 2 is satisfied, there exists a subsequence \( \{k_i\} \)

such that \( \lim_{i \to \infty} n_{k_i}c_{k_i} = 0 \). Let \( x_i \) be the left endpoint of some basic elements of order \( k_i \), take \( R_i = c_1c_2 \cdots c_{k+1} + c_1c_2 \cdots c_k (1-n_{k+1}c_{k+1})/(n_{k+1} - 1) \) and \( r_i = c_1c_2 \cdots c_{k+1} \). It is obvious that

\[ N_r(B(x, R) \cap E) = \left( \frac{R}{r} \right)^{\frac{\alpha}{r}} \leq \left( \frac{R}{r} \right)^{(-\log(\log(n_{k+1}-1)+\log(n_{k+2}-n_{k+1})/\log(1-c_{i+1})-\log(n_{i+1}-1)-\log(c_{i+1}-c_i))} < \left( \frac{R}{r} \right)^{-\varepsilon}, \tag{22} \]

for all \( c_i \cdots c_{k+1} < r_0 \). If \( c_i \cdots c_{k+1} \geq r_0 \), notice that \( R/r \leq 1/c_i \cdots c_{k+1} \leq 1/r_0 \),

\[ N_r(B(x, R) \cap E) \leq \frac{1}{\log R_0} \left( \frac{R}{r} \right)^{\varepsilon} \leq \frac{1}{\log R_0} \left( \frac{R}{r} \right)^{-\varepsilon}. \tag{23} \]

By combining (23) and (24), we obtain that \( \dim E \geq \varepsilon \).

To prove that \( \dim E \leq \varepsilon \), it suffices to find a constant \( c > 0 \) and a sequence of points \( x_i \in E \) and scales \( 0 < r_i < R_i \) such that \( R_i/r_i \to \infty \) and for all \( i \)

\[ N_r(B(x_i, R_i) \cap E) \geq n_i \cdots n_{k+1} = \left( \frac{R_i}{r_i} \right)^{\log n_i \cdots n_{k+1}/\log c_i \cdots c_{k+1}} \geq \left( \frac{R_i}{r_i} \right)^{-\varepsilon}, \tag{24} \]

for all \( i > i_0 \). The proof of \( \dim E = \varepsilon \) is finished.

Finally, we prove \( \dim E = 1 \) if \( \limsup_{i \to \infty} n_{k_i} \) is \( +\infty \). There exist a subsequence \( \{k_i\} \) such that \( \lim_{i \to \infty} n_{k_i} = +\infty \). Let \( x_i \) be the left endpoint of some basic elements of order \( k_i \), take \( R_i = c_1c_2 \cdots c_k \) and \( r_i = c_1c_2 \cdots c_k (1-c_{k+1})/(n_{k+1} - 1) \). It is obvious that

\[ N_r(B(x_i, R_i) \cap E) \geq n_{k+1} \geq n_k \frac{1-c_{k+1}}{n_{k+1}-1} \frac{R_i}{r_i} \geq 2 \frac{R_i}{r_i}. \tag{25} \]

Notice that \( (R_i)/r_i = (n_{k+1} - 1)/(1-c_{k+1}) \to \infty \)

when \( i \to \infty \); therefore, \( \dim E = 1 \). \( \square \)
Corollary 1. Give a number $\mu$ and $0 < \mu < 1$, for any number $\lambda$ which satisfies that $0 \leq \lambda \leq \mu$ there exists a homogeneous Cantor set $E$ such that

\[
\dim H E = \mu, \quad \dim L E = \lambda.
\]

Proof of Corollary 1. \hfill \square

Case 1. $0 < \lambda \leq \mu$.

Let $\{T_k\}_{k \geq 1}$ be a sequence of integers such that

\[
T_1 = 1,
\]
\[
T_2 = 2,
\]
\[
T_{2k+1} = T_{2k} + k^2,
\]
\[
T_{2k+2} = T_{2k+1} + k.
\]

Define the family of parameters $n_i, c_i$ as follows:

\[
n_i = \begin{cases} 
3, & \text{if } T_{2k} < i \leq T_{2k+1}; \\
4, & \text{if } T_{2k+1} < i \leq T_{2k+2}.
\end{cases}
\]

\[
c_i = \begin{cases} 
\frac{1}{3}^{1/\mu}, & \text{if } T_{2k} < i \leq T_{2k+1}; \\
\frac{1}{3}^{1/\lambda}, & \text{if } T_{2k+1} < i \leq T_{2k+2}.
\end{cases}
\]

By Theorem 1 and Theorem 2, it is obvious that $\dim H E = \mu, \dim L E = \lambda$.

Case 2. $\lambda = 0$.

Let $\{T_k\}_{k \geq 1}$ be a sequence of integers such that

\[
T_1 = 1,
\]
\[
T_2 = 2,
\]
\[
T_{2k+1} = T_{2k} + k^2,
\]
\[
T_{2k+2} = T_{2k+1} + 1.
\]

Define the family of parameters $n_i, c_i$ as follows:

\[
n_i = \begin{cases} 
3, & \text{if } T_{2k} < i \leq T_{2k+1}; \\
4, & \text{if } T_{2k+1} < i \leq T_{2k+2}.
\end{cases}
\]

\[
c_i = \begin{cases} 
\frac{1}{3}^{1/\mu}, & \text{if } T_{2k} < i \leq T_{2k+1}; \\
\frac{1}{3}^{1/\lambda}, & \text{if } T_{2k+1} < i \leq T_{2k+2}.
\end{cases}
\]

By Theorem 1 and Theorem 2, it is obvious that $\dim H E = \mu, \dim L E = \lambda$.

Corollary 2. Give a number $\mu$ and $0 < \mu < 1$, for any number $\lambda$ which satisfies that $\mu \leq \lambda \leq 1$ there exists a homogeneous Cantor set $E$ such that

\[
\dim P E = \dim B E = \mu, \dim A E = \lambda.
\]

Proof of Corollary 2. \hfill \square

Case 3. $\mu \leq \lambda < 1$.

Let $\{T_k\}_{k \geq 1}$ be a sequence of integers such that

\[
T_1 = 1,
\]
\[
T_2 = 2,
\]
\[
T_{2k+1} = T_{2k} + k^2,
\]
\[
T_{2k+2} = T_{2k+1} + k.
\]

Define the family of parameters $n_i, c_i$ as follows:

\[
n_i = \begin{cases} 
3, & \text{if } T_{2k} < i \leq T_{2k+1}; \\
4, & \text{if } T_{2k+1} < i \leq T_{2k+2}.
\end{cases}
\]

\[
c_i = \begin{cases} 
\frac{1}{3}^{1/\mu}, & \text{if } T_{2k} < i \leq T_{2k+1}; \\
\frac{1}{3}^{1/\lambda}, & \text{if } T_{2k+1} < i \leq T_{2k+2}.
\end{cases}
\]

By Theorem 1 and Theorem 2, it is obvious that $\dim P E = \dim B E = \mu, \dim A E = \lambda$. 

Case 4. $\lambda = 1$.

For Case 2 in Corollary 1, notice that $\limsup_{n_k} n_k = +\infty$, it is obvious that $\dim_\mu E = \overline{\dim}_\mu E = \mu, \dim_A E = 0$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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