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Research Article

Optimal Control Problem Governed by an Evolution Equation and Using Bilinear Regular Feedback

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We solve an optimal control problem governed by an evolution equation using bilinear regular feedback. Using optimization techniques, we show how to approximate the flow of a reaction-diffusion bilinear system by a desired target. For application, we consider the regional flow problem constrained by a bilinear distributed system. The paper ends by an example illustrating the numerical approach of the proposed method.

1. Introduction

Bilinear systems form an important class of dynamic systems for several reasons. Many industrial or natural processes have a bilinear structure. For example, we can cite the transfer of heat by conduction convection, the neutron displacement in a nuclear reactor, and the dynamics of sense organs [1]. Research has shown that bilinear systems are sufficient to approach any nonlinear input-output behavior (see [1, 2]). The control has a double action in the system that allows the adaptation of the model at different levels of input signals. An example is provided by the functioning of sense organ (see [1]).

Optimal control methods continue to provide solutions to many real problems. We cite solutions of smoking models by Mahdy et al. [3] and COVID-19 prediction by Ahmed et al. [4]. Optimal control problems constrained by a distributed bilinear system are initiated by Bradley et al. and Lenhart [5, 6]. In [7], Joshi studies the case of regular velocity terms. Sonawane et al. [8] consider the optimal control for a vibrating string with axial variable. Rao et al. studied plant disease in [9].

Mall et al. propose a uniform method for optimal control problems with control and state constraints (see [10]). Chertovskih et al. in [11] give an indirect method for regular state-constrained optimal control problems in flow fields. Turgut et al. in [12] study an island-based crow search algorithm for solving optimal control problems. Al-Hawasy et al. in [13] consider the optimal control problems for triple elliptic partial differential equations. Bonnet and Frankowska in [14] characterize the necessary optimality conditions for optimal control problems in Wasserstein spaces. Granada and Kovtunenko in [15] consider a shape derivative for optimal control of the nonlinear Brinkman–Forchheimer equation.

For fractional systems, Saidi [16] discusses some results associated to first-order set-valued evolution problems with subdifferentials. Jajarmi and Baleanu [17] consider the fractional optimal control problems with a general derivative operator. Huixian et al. [18] study an averaging result for a class of impulsive fractional neutral stochastic evolution equations. Jafariet al. [19] propose a numerical approach for solving fractional optimal control problems with Mittag–Leffler kernel. Mehandiratta et al. [20] study fractional optimal control problems on a star graph. Heydari et al. [21] propose a numerical solution for an optimal control problems generated by Atangana–Riemann–Liouville fractal-fractional derivative.

The flow problems are one of the most important questions in mathematics. They have applications in several fields such as physics, biology, and engineering. We cite here the problem of controlling the blood flow in a vessel, where we need to calculate the gradient (flow) of the velocity of blood as a rate of change of the blood flow (see [22]).

Recently, many researchers focused on the study of flow problems using optimal control theory. They consider the gradient state of a distributed system and ask if there is an optimal control to reach a desired profile (see [23]). For this approach, one of the most important ideas is called the partial analysis. It has an objective to reach a target on a specific subdomain of the system domain, $\omega \subset D$ (see [24, 25]). For partial work on bilinear distributed systems, Ouzehra et al. [26, 27] study the exact and approximate controllability of reaction-diffusion equation using bilinear control. Zerrik and Ould Sidi [28–31] use partial control problems to orient the dynamics of infinite dimensional systems towards the desired state in a specific area. Zine and Ould Sidi in [32–34] deal with partial control problems in the case of hyperbolic systems. Ould Sidi and Beinane [35, 36] treat the partial flow control problems.

The objective of this paper is to control the flow of equation (1) towards a desired target using the penalization problem 3, and with a more regular spatiotemporal control function. In Section 2, we show the existence of a solution to the studied problems. Next, we give the characterization of its solution considering different types of actions. Section 3 is devoted to the study of the partial flow control problems constrained by bilinear distributed systems with regular optimal control time function. The paper ends by an example illustrating the numerical approach of the proposed method.

2. Flow Problem with Regular Control

Let us consider the system described by

$$\begin{cases} q_m(x,m) = q_{xx}(x,m) - v(x,m)q_x(x,m), & \Gamma, \\ q(x,0) = q_0(x), & D, \\ q = q_x = 0, & \Pi, \end{cases} \tag{1}$$

with a domain $D \subset IR^n$ $(n = 1,2,3)$ is open bounded, and its regular boundary is ∂D. Let $M > 0$ and $\Gamma = D \times]0, M[$, $\Pi = \partial D \times]0, M[$, where the space of control is $v \in L^2(0, M, H_0^1(D))$.

Let $q_0(x) \in L^2(D)$ and

$$S = \left\{ \frac{q \in L^2\left(0, M; H_0^1(D)\right)}{q_m \in L^2\left(0, M; H^{-2}(D)\right)} \right\}, \tag{2}$$

represents the state space (see [5]). The system dynamic is $q_{xx} = \Delta q = \sum_{i=1}^n \partial^2 q/\partial x_i^2$, and system (1) has a unique solution q_v in $S \cap L^\infty(0, M; L^2(D))$ (see [37]).

We consider the operator ∇:

$$\begin{aligned} \nabla &: H^1(D) \longrightarrow \left(L^2(D)\right)^n, \\ q &\longrightarrow \nabla q = \left(\frac{\partial q}{\partial x_1}, \ldots, \frac{\partial q}{\partial x_n}\right). \end{aligned} \tag{3}$$

The flow regular optimal control problem of system (1) is

$$\min_{v \in L^2\left(0, M, H_0^1(D)\right)} \Phi_\varepsilon(v), \tag{4}$$

with $\varepsilon > 0$, and Φ_ε is the cost penalty defined by

$$\begin{aligned} \Phi_\varepsilon(v) &= \frac{1}{2}\left\|\nabla q - q^d\right\|^2_{\left(L^2\left(0,M,H_0^1(D)\right)\right)^n} + \frac{\varepsilon}{2}\int_\Gamma \left[v_m^2(x,m) + v_x^2(x,m)\right] \mathrm{d}x\mathrm{d}m \\ &= \frac{1}{2}\sum_{i=1}^n \left\|\frac{\partial q}{\partial x_i} - q_i^d\right\|^2_{L^2\left(0,M,H_0^1(D)\right)} + \frac{\varepsilon}{2}\int_\Gamma \left[v_m^2(x,m) + v_x^2(x,m)\right] \mathrm{d}x\mathrm{d}m, \end{aligned} \tag{5}$$

where the desired flow is $q^d = (q_1^d, \ldots, q_n^d)$.

The main objective is to propose a method to steer the flow of (1) to $q^d(x)$, using the functional (5) and considering a more regular control space $v \in L^2(0, M; H_0^1(D))$. We characterize the solution of (4) through an extension of the Lagrangian method.

2.1. Existence of Solution. In the next theorem, we study the existence of a solution to the flow problem (4).

Theorem 1. *Let us consider q be the solution of the system*

$$\begin{cases} q_m = q_{xx} - vq_x, & \Gamma, \\ q(x,0) = q_0(x), & D, \\ q = q_x = 0, & \Pi. \end{cases} \tag{6}$$

Then, there exists an optimal control v, which is the minimum of (4).

Proof. Let us consider the set $\{\Phi_\varepsilon(v) | v \in L^2(0, M, H_0^1(D))\} \subset IR$, which is a positive nonempty and admits lower bounded. Thus, by choosing a minimizing sequence $(v_n)_n$ which verifies

$$\Phi^* = \lim_{n\longrightarrow+\infty} \Phi(v_n) = \inf_{v\in L^2(0,M,H_0^1(D))} \Phi_\varepsilon(v). \quad (7)$$

Then, the cost $\Phi_\varepsilon(v_n)$ is bounded, and it follows that $\|v_n\|_{L^2(0,M,H_0^1(D))} \le B$, with B as a positive constant.

We have

$$\begin{aligned} v_n &\rightharpoonup v, \quad L^2(0,M,H_0^1(D)),\\ q^n &\rightharpoonup q, \quad S,\\ q^n_{xx} &\rightharpoonup \chi, \quad S,\\ q^n_x &\rightharpoonup \Lambda, \quad S,\\ q^n_m &\rightharpoonup \Psi, \quad S. \end{aligned} \quad (8)$$

By passing to the limit in the equation $q^n_m(x,m) = q^n_{xx} - v_n q^n_x$, we deduce that $q_m(x,m) = \Psi$, $q\mapsto q_{xx}$, $q_{xx} = \chi$ and $vq_x = \Lambda$. Hence, we obtain

$$q_m = q_{xx} - v(x,m)q_x. \quad (9)$$

From the lower semicontinuity of $\Phi_\varepsilon(v)$:

$$\begin{aligned} \Phi_\varepsilon(v) &= \inf_n \sum_{i=1}^n \frac{1}{2}\int_0^M \int_D \left(\frac{\partial q_n}{\partial x_i} - q_i^d\right)^2 \mathrm{d}x + \frac{\varepsilon}{2}\int_\Gamma \left[v_m^2 + v_x^2\right]_n \mathrm{d}x\mathrm{d}m\\ &\le \lim_{n\longrightarrow 0} \Phi_\varepsilon(v_n) = \inf_v \Phi_\varepsilon(v). \end{aligned} \quad (10)$$

Therefore, v is a solution of (4). □

2.2. Characterization of Solution. In this section, the aim is to propose a formulation of the solution of our flow problem. Therefore, we should introduce the so-called optimal equation to find the differential of the functional $\Phi_\varepsilon(v)$ in (5). The following lemma mentions the differential of $\Phi_\varepsilon(v)$ with respecting v.

Lemma 2. *A differential of the map*

$$v \in L^2(0,M,H_0^1(D)) \longrightarrow q(v) \in S, \quad (11)$$

is

$$\frac{q(v+\varepsilon l) - q(l)}{\varepsilon} \rightharpoonup \mu, \quad (12)$$

where $\mu = \mu(q,l)$ verifies

$$\begin{cases} \mu_m = \mu_{xx} - v\mu_x - lq_x, & \Gamma,\\ \mu(x,0) = 0, & D,\\ \mu = \mu_x = 0, & \Pi, \end{cases} \quad (13)$$

where $q = q(v)$, $v \in L^2(0,M;H_0^1(D))$, and $d(q(v))l$ is the derivative of $v \longrightarrow q(v)$ with respect v.

Proof. We consider the solution of (13), verifying

$$\|\mu\|_S \le k_1 \|q\|_{L^\infty(0,M;H_0^1(D))} \|l\|_{L^2(0,M,H_0^1(D))}. \quad (14)$$

Also,

$$\|\mu'\|_S \le k_2 \|q\|_{L^\infty(0,M;H_0^1(D))} \|l\|_{L^2(0,M,H_0^1(D))}. \quad (15)$$

Thus,

$$\|\mu\|_{\mathscr{C}([0,M];H_0^1(D))} \le k_3 \|l\|_{L^2(0,M,H_0^1(D))}. \quad (16)$$

Then, we obtain that $l \in L^2(0,M;L^2(D)) \longrightarrow \mu \in \mathscr{C}((0,M);H_0^1(D))$ is bounded (see [5]).

If we put $q_l = q(v+l)$ and $\xi = q_l - q$, then ξ is the state of

$$\begin{cases} \xi_m(x,m) = \xi_{xx} - v(x,m)\xi_x(x,m) - l(x,m)(q_l)_x, & \Gamma,\\ \xi(x,0) = 0, & D,\\ \xi = \xi_x = 0, & \Pi. \end{cases} \quad (17)$$

Thus,

$$\|\xi\|_{L^\infty([0,M];H_0^1(D))} \le k_4 \|\theta\|_{L^2(0,M,H_0^1(D))}. \quad (18)$$

Let $\gamma = \xi - \mu$ which verifies the system

$$\begin{cases} \gamma_m = \gamma_{xx} + v(x,m)\gamma_x(x,m) + l(x,m)\xi_x, & \Gamma,\\ \gamma(x,0) = 0, & D,\\ \gamma = \gamma_x = 0, & \Pi, \end{cases} \quad (19)$$

$\gamma \in \mathscr{C}(0,M;H_0^1(D))$; consequently,

$$\|\gamma\|_{C([0,M];H_0^1(D))} \le k\|l\|^2_{L^2(0,M,H_0^1(D))}, \quad (20)$$

and we have

$$\|q(v+l) - q(v) - d(q(v))l\|_{C(0,M;H_0^1(D))} = \|\gamma\|_{C([0,M];H_0^1(D))} \le k\|l\|^2_{L^2(0,M;H_0^1(D))}, \quad (21)$$

where k_1, k_2, k_3, k_4, and k are a constant positive.

In the following, we define a family of optimal equations.

$$\begin{cases} -\dfrac{\partial p_i}{\partial m} = \dfrac{\partial^2 p_i}{\partial x^2} + \dfrac{\partial (v p_i)}{\partial x} + \left(\dfrac{\partial q}{\partial x_i} - q_i^d\right), & \Gamma, \\ v_x(x,0) = v_x(x,M) = 0, & D, \\ p_i(x,M) = 0, & D, \\ p_i = \dfrac{\partial p_i}{\partial x} = 0, & \Pi. \end{cases} \tag{22}$$

The next lemma characterizes the differential of $\Phi_\varepsilon(v)$. □

Lemma 3. *Let $v \in L^2(0, M, H_0^1(D))$ be the solution of (4), and we obtain*

$$\lim_{\beta \longrightarrow 0} \frac{\Phi_\varepsilon(v+\beta l) - \Phi_\varepsilon(v)}{\beta} = \sum_{i=1}^n \int_D \int_0^M \frac{\partial \mu(x,m)}{\partial x_i}\left(\frac{\partial q}{\partial x_i} - q_i^d\right) \mathrm{d}m\mathrm{d}x + \varepsilon \int_D \int_0^M \left[(v_m l_m) + (v_x l_x)\right] \mathrm{d}m\mathrm{d}x. \tag{23}$$

Proof. The cost $\Phi_\varepsilon(v)$ (5) can be expressed by

$$\Phi_\varepsilon(v) = \frac{1}{2}\sum_{i=1}^n \int_D \int_0^M \left(\frac{\partial q}{\partial x_i} - q_i^d\right)^2 \mathrm{d}m\mathrm{d}x + \frac{\varepsilon}{2}\int_D \int_0^M \left[v_m^2 + v_x^2\right] \mathrm{d}m\mathrm{d}x. \tag{24}$$

If we put $q_\beta = q(v+\beta l)$ and $q = q(v)$, using (59), we have

$$\begin{aligned} \lim_{\beta \longrightarrow 0} \frac{\Phi_\varepsilon(v+\beta l) - \Phi_\varepsilon(v)}{\beta} &= \lim_{\beta \longrightarrow 0} \sum_{i=1}^n \frac{1}{2}\int_D \int_0^M \frac{\left(\left(\partial q_\beta/\partial x_i\right) - q_i^d\right)^2 - \left(\left(\partial q/\partial x_i\right) - q_i^d\right)^2}{\beta} \mathrm{d}m\mathrm{d}x \\ &\quad + \lim_{\beta \longrightarrow 0} \frac{\varepsilon}{2\beta}\int_D \int_0^M \left[(v_m+\beta l_m)^2 - v_m^2 + (v_x+\beta l_x)^2 - v_x^2\right] \mathrm{d}m\mathrm{d}x, \end{aligned} \tag{25}$$

then

$$\begin{aligned} &\lim_{\beta \longrightarrow 0} \frac{\Phi_\varepsilon(v+\beta l) - \Phi_\varepsilon(v)}{\beta} \\ &= \lim_{\beta \longrightarrow 0} \sum_{i=1}^n \frac{1}{2}\int_D \int_0^M \frac{\left(\left(\partial q_\beta/\partial x_i\right) - \left(\partial q/\partial x_i\right)\right)}{\beta}\left(\frac{\partial q_\beta}{\partial x_i} + \frac{\partial q}{\partial x_i} - 2q_i^d\right) \mathrm{d}m\mathrm{d}x + \lim_{\beta \longrightarrow 0} \varepsilon \int_D \int_0^M \left[(v_m l_m) + (v_x l_x)\right](x,m) \mathrm{d}m\mathrm{d}x \\ &= \sum_{i=1}^n \int_D \int_0^M \frac{\partial \mu(x,m)}{\partial x_i}\left(\frac{\partial q(x,m)}{\partial x_i} - q_i^d\right) \mathrm{d}m\mathrm{d}x + \int_D \int_0^M \varepsilon\left[(v_m l_m) + (v_x l_x)\right](x,m) \mathrm{d}m\mathrm{d}x. \end{aligned} \tag{26}$$

The following theorem proposes a solution of the problem (4). □

Theorem 4. *Let $v \in L^2(0, M; H_0^1(D))$ be a solution of (4); then,*

$$v_{mm} + v_{xx} + \frac{1}{\varepsilon}(Div(p_i))q_x = 0, \quad (27)$$

where $q = q(v)$ is the output of (1), where $p_i = (p_1, \ldots, p_n)$ and $p_i \in C([0, M]; H_0^1(D))$ is the solution of (22).

Proof. Let $l \in L^2(0, M; H_0^1(D))$ and $v + \beta l \in L^2(0, M; H_0^1(D))$ for $\beta > 0$. The extremal of Φ_ε is realized at v; then,

$$0 \le \lim_{\beta \longrightarrow 0} \frac{\Phi_\varepsilon(v + \beta l) - \Phi_\varepsilon(v)}{\beta}. \quad (28)$$

Lemma 9 gives

$$0 \le \sum_{i=1}^{n} \int_D \int_0^M \frac{\partial \mu(x,m)}{\partial x_i} \left(\frac{\partial q(x,m)}{\partial x_i} - q_i^d \right) \mathrm{d}m\mathrm{d}x + \int_D \int_0^M \varepsilon[(v_m l_m) + (v_x l_x)](x,m)\mathrm{d}m\mathrm{d}x. \quad (29)$$

Therefore, using (22), we obtain

$$\begin{aligned} 0 \le & \sum_{i=1}^{n} \int_D \int_0^M \frac{\partial \mu(x,m)}{\partial x_i} \left(-\frac{\partial p_i(x,m)}{\partial m} - \frac{\partial^2 p_i(x,m)}{\partial x^2} - \frac{\partial v p_i(x,m)}{\partial x} \right) \mathrm{d}m\mathrm{d}x \\ & + \int_D \int_0^M \varepsilon[(v_m l_m) + (v_x l_x)]\mathrm{d}m\mathrm{d}x. \end{aligned} \quad (30)$$

By a simple calculus, we have

$$\begin{aligned} 0 \le & \sum_{i=1}^{n} \int_D \int_0^M \frac{\partial \mu(x,m)}{\partial x_i} \left(-\frac{\partial p_i(x,m)}{\partial m} - \frac{\partial^2 p_i(x,m)}{\partial x^2} + \frac{\partial v p_i(x,m)}{\partial x} \right) p_i(x,m)\mathrm{d}m\mathrm{d}x \\ & + \int_D \int_0^M \varepsilon[(v_m l_m) + (v_x l_x)]\mathrm{d}m\mathrm{d}x. \end{aligned} \quad (31)$$

From System(13), we obtain

$$\begin{aligned} 0 \le & \sum_{i=1}^{n} \int_D \int_0^M \frac{\partial}{\partial x_i} -(l(x,m)q_x) p_i(x,m)\mathrm{d}m\mathrm{d}x \\ & + \int_D \int_0^M \varepsilon[(v_m l_m) + (v_x l_x)]\mathrm{d}m\mathrm{d}x \\ = & \int_D \int_0^M \left[-l(x,m)q_x \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} p_i(x,m) \right) + \varepsilon(v_m l_m) + \varepsilon(v_x l_x) \right] \mathrm{d}m\mathrm{d}x. \end{aligned} \quad (32)$$

Moreover, if $l = l(t) \in L^2(0, M; H_0^1 D)$, we deduce

$$-l q_x \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} p_i(x,m) \right) - \varepsilon v_{mm} l - \varepsilon v_{xx} l = 0, \quad (33)$$

which allow us to introduce

$$v_{mm} + v_{xx} + \frac{1}{\varepsilon}(Div(p_i))q_x = 0, \quad (34)$$

that the solution v of (4) must satisfy. □

Remark 5. According to equation (2),

(1) If we consider a spatial control function $v = v(x,)$ then the variational formula becomes

$$v_{xx} = -\frac{1}{\varepsilon}(Div(p_i))q_x, \quad (35)$$

(2) If we consider a temporal control function $v = v(m)$, then the variational formula becomes

$$v_{mm} = -\frac{1}{\varepsilon}\left(Div\left(p_i\right)\right)q_x. \tag{36}$$

3. Partial Flow Control Problem

3.1. Problem Statement. We consider the bilinear distributed system (1), with a given $q_0 \in H^1(D)$. System (1) can be rewritten as follows:

$$q(m) = S(m)q_0 + \int_0^m S(m-s)v(s)q(s)\mathrm{d}s, \tag{37}$$

and the solution of (37) are often called the mild solution of (1).

The existence of a unique solution $q_v(x,m)$ in $L^2(0,M;H_0^1(D))$ satisfying (37) can be deduced from [37].

We choose $\omega \in D$, and

$$\begin{aligned}\chi_\omega\colon \left(L^2(D)\right)^n &\longrightarrow \left(L^2(D)\right)^n\\ q &\longrightarrow \chi_\omega q = q|_\omega,\end{aligned} \tag{38}$$

and χ_ω^*; its adjoint is given by

$$\chi_\omega^* q = \begin{cases} q \,\text{in}\, D,\\ 0 \in D\setminus\omega,\end{cases} \tag{39}$$

$$\begin{aligned}\tilde{\chi}_\omega\colon \left(L^2(D)\right) &\longrightarrow \left(L^2(\omega)\right)\\ q &\longrightarrow \tilde{\chi}_\omega q = q|_\omega.\end{aligned} \tag{40}$$

Definition 6. Equation (1) is called partial flow controllable on $\omega \subset D$, to $g^d \in (L^2(\omega))^n$ if there exists a control $v \in L^2(0,M,H_0^1(D))$ and $\varepsilon > 0$ such that

$$\left\|\chi_\omega \nabla q_v(M) - g^d\right\|_{(L^2(\omega))^n} \leq \varepsilon, \tag{41}$$

where $g^d = (y_1^d, \ldots y_n^d)$ is the desired flow in $(L^2(\omega))^n$.

Ouzehra in [26], studies the exact and approximate controllability of distributed bilinear systems. The partial flow control problem of (1) is

$$\min_{v\in L^2(0,M,H_0^1(D))}\Phi_\varepsilon(v), \tag{42}$$

where Φ_ε is presented for $\varepsilon > 0$ by

$$\begin{aligned}\Phi_\varepsilon(v) &= \frac{1}{2}\left\|\chi_\omega \nabla q(M) - g^d\right\|^2_{(L^2(\omega))^n} + \frac{\varepsilon}{2}\int_\Gamma \left[v_m^2(m)\right]\mathrm{d}m\\ &= \frac{1}{2}\sum_{i=1}^n \left\|\tilde{\chi}_\omega \frac{\partial q(M)}{\partial x_i} - y_i^d\right\|^2_{L^2(\omega)} + \frac{\varepsilon}{2}\int_\Gamma \left[v_m^2(m)\right]\mathrm{d}m.\end{aligned} \tag{43}$$

The objective of the presented problem is to command the flow of (1) to a target state $g^d(x)$, realizing (43), and find $v^* \in L^2(0,M,H_0^1(D))$, verifying

$$\Phi_\varepsilon(v^*) = \min_{v\in L^2(0,M,H_0^1(D))} \Phi_\varepsilon(v). \tag{44}$$

Remark 7. The existence of solutions for the partial flow control problem can be proved in the same way as in the proof of the previous section.

3.2. Characterization of Solution. Now, we are able to formulate the solution of the flow problem (42).

Lemma 8. *A differential of the map*

$$v \in L^2\left(0,M,H_0^1(D)\right) \longrightarrow q(v) \in S, \tag{45}$$

is

$$\frac{q(v+\varepsilon l) - q(l)}{\varepsilon} \longrightarrow \mu, \tag{46}$$

where $\mu = \mu(q,l)$ verifies

$$\begin{cases}\mu_m = \mu_{xx} - v\mu_x - lq_x, & \Gamma,\\ \mu(x,0) = 0, & D,\\ \mu = \mu_x = 0, & \Pi,\end{cases} \tag{47}$$

where $q = q(v)$, $v \in L^2(0,M;H_0^1(D))$, and $d(q(v))l$ is the derivative of $v \longrightarrow q(v)$ with respect v.

Proof. The output of equation (13) satisfies

$$\|\mu\|_S \leq k_1\|q\|_{L^\infty(0,M;H_0^1(D))}\|l\|_{L^2(0,M,H_0^1(D))}. \tag{48}$$

Also,

$$\left\|\mu'\right\|_S \leq k_2\|q\|_{L^\infty(0,M;H_0^1(D))}\|l\|_{L^2(0,M,H_0^1(D))}. \tag{49}$$

Thus,

$$\|\mu\|_{\mathscr{C}([0,M];H_0^1(D))} \leq k_3\|l\|_{L^2(0,M,H_0^1(D))}. \tag{50}$$

Then, we obtain that $l \in L^2(0,M;L^2(D)) \longrightarrow \mu \in \mathscr{C}((0,M);H_0^1(D))$ is bounded (see [5]).

If we put $q_l = q(v+l)$ and $\xi = q_l - q$, then ξ is the state of

$$\begin{cases}\xi_m(x,m) = \xi_{xx} - v(x,m)\xi_x(x,m) - l(x,m)\left(q_l\right)_x, & \Gamma,\\ \xi(x,0) = 0, & D,\\ \xi = \xi_x = 0, & \Pi.\end{cases} \tag{51}$$

Thus,

$$\|\xi\|_{L^\infty([0,M]];H_0^1(D))} \leq k_4\|l\|_{L^2(0,M,H_0^1(D))}. \tag{52}$$

Let $\gamma = \xi - \mu$ which verifies the system

$$\begin{cases}\gamma_m = \gamma_{xx} + v(x,m)\gamma_x(x,m) + l(x,m)\xi_x, & \Gamma,\\ \gamma(x,0) = 0, & D,\\ \gamma = \gamma_x = 0, & \Pi,\end{cases} \tag{53}$$

$\gamma \in \mathscr{C}(0,M;H_0^1(D))$; consequently,

$$\|\gamma\|_{\mathscr{C}([0,M];H_0^1(D))} \leq k\|l\|_{L^2(0,M,H_0^1(D))}^2, \quad (54)$$

and we have

$$\|q(v+l) - q(v) - d(q(v))l\|_{\mathscr{C}(0,M;H_0^1(D))} = \|\gamma\|_{\mathscr{C}([0,M];H_0^1(D))} \leq k\|l\|_{L^2(0,M,H_0^1(D))}^2. \quad (55)$$

We introduce the family of optimal systems in the case of partial controllability

$$\begin{cases} -(p_i)_m(x,m) = (p_i)_{xx}(x,m) + (v(m)p_i)_x(x,m), & \Gamma, \\ p_i(x,M) = \left(\dfrac{\partial q(M)}{\partial x_i} - \widetilde{\chi}_\omega^* y_i^d\right), & D, \\ p_i(x,m) = (p_i)_x(x,m) = 0, & \Pi, \end{cases} \quad (56)$$

where $\widetilde{\chi}_\omega^*$ is the adjoint of $\widetilde{\chi}_\omega$ defined from $L^2(\omega) \longrightarrow L^2(D)$ by

$$\widetilde{\chi}_\omega^* q(x) = \begin{cases} q(x), & x \in \omega, \\ 0, & x \in D\backslash\omega. \end{cases} \quad (57)$$

The following lemma mentions the differential of $\Phi_\varepsilon(v)$ with respecting v. □

Lemma 9. *If $v \in L^2(0,M)$ is the control realizing (42), μ is the output of (47), and p_i is the solution of (56), we deduce*

$$\lim_{\beta\longrightarrow 0} \frac{\Phi_\varepsilon(v+\beta l) - \Phi_\varepsilon(v)}{\beta} = \sum_{i=1}^n \int_\omega \widetilde{\chi}_\omega^* \widetilde{\chi}_\omega \left[\int_0^M \frac{\partial p_i}{\partial m}\frac{\partial \mu(x,m)}{\partial x_i} dm + \int_0^M p_i \frac{\partial}{\partial x_i}\left(\frac{\partial \mu}{\partial m}\right) dm\right] dx + \int_0^M 2\varepsilon l_m v_m dm. \quad (58)$$

Proof. The functional $\Phi_\varepsilon(v)$ given by (43) can take the form

$$\Phi_\varepsilon(v) = \frac{1}{2}\sum_{i=1}^n \int_\omega \left(\widetilde{\chi}_\omega \frac{\partial q}{\partial x_i} - y_i^d\right)^2 dx + \varepsilon \int_0^M v_m^2(m) dm. \quad (59)$$

Let $q_\beta = q(v+\beta l)$ and $q = q(v)$, using $(59)_2$ we have

$$\begin{aligned} \lim_{\beta\longrightarrow 0} \frac{\Phi_\varepsilon(v+\beta l) - \Phi_\varepsilon(v)}{\beta} &= \lim_{\beta\longrightarrow 0} \sum_{i=1}^n \frac{1}{2}\int_\omega \frac{\left(\widetilde{\chi}_\omega(\partial q_\beta/\partial x_i) - y_i^d\right)^2 - \left(\widetilde{\chi}_\omega(\partial q/\partial x_i) - y_i^d\right)^2}{\beta} dx \\ &+ \lim_{\beta\longrightarrow 0} \frac{\varepsilon}{\beta}\int_0^M \left[(v_m + \beta l_m)^2 - v_m^2\right] dm. \end{aligned} \quad (60)$$

Consequently,

$$\begin{aligned} \lim_{\beta\longrightarrow 0} \frac{\Phi_\varepsilon(v+\beta l) - \Phi_\varepsilon(v)}{\beta} &= \lim_{\beta\longrightarrow 0} \sum_{i=1}^n \frac{1}{2}\int_\omega \widetilde{\chi}_\omega \frac{\left((\partial q_\beta/\partial x_i) - (\partial q/\partial x_i)\right)}{\beta}\left(\widetilde{\chi}_\omega(\partial q_\beta/\partial x_i) + \widetilde{\chi}_\omega(\partial q/\partial x_i) - 2y_i^d\right) dx \\ &+ \lim_{\beta\longrightarrow 0}\int_0^M \left(2\varepsilon l_m v_m + \varepsilon\beta l_m^2\right) dm = \sum_{i=1}^n \int_\omega \widetilde{\chi}_\omega \frac{\partial \mu(x,M)}{\partial x_i}\widetilde{\chi}_\omega\left(\frac{\partial q(x,M)}{\partial x_i} - \widetilde{\chi}_\omega^* y_i^d\right) dx + \int_0^M 2\varepsilon l_m v_m dm \\ &= \sum_{i=1}^n \int_\omega \widetilde{\chi}_\omega \frac{\partial \mu(x,M)}{\partial x_i}\widetilde{\chi}_\omega p_i(x,M) dx + 2\varepsilon\int_0^M l_m v_m dm. \end{aligned} \quad (61)$$

From (56) and (61), we deduce that

$$\lim_{\beta\longrightarrow 0}\frac{\Phi_\varepsilon(v_\varepsilon+\beta l)-\Phi_\varepsilon(v_\varepsilon)}{\beta}=\sum_{i=1}^{n}\int_\omega \tilde{\chi}_\omega^*\tilde{\chi}_\omega\left[\int_0^M \frac{\partial p_i}{\partial m}\frac{\partial\mu(x,m)}{\partial x_i}dm+\int_0^M p_i\frac{\partial}{\partial x_i}\left(\frac{\partial\mu}{\partial m}\right)dm\right]dx+\int_0^M 2\varepsilon l_m v_m dm. \tag{62}$$

Now, we will deduce the solution of (42), exploiting the family of optimal systems. □

Theorem 10. *Let $v\in L^2(0,M;H_0^1)$ be the solution of the partial flow problem, and $q=q(v)$ is its corresponding state of (1), we show that*

$$\sum_{i=1}^{n}\langle\tilde{\chi}_\omega\frac{\partial q_x}{\partial x_i};\tilde{\chi}_\omega p_i\rangle_{L^2(\omega)}-2\varepsilon v_{mm}=0, \tag{63}$$

is a solution of problem (42), where $p_i\in C([0,M];H_0^1(D))$ is the unique solution of the adjoint system (56).

Proof. Let $l\in L^2(0,M;H_0^1(D))$ with $v+\beta l\in L^2(0,M;H_0^1(D))$ for $\beta>0$. The functional Φ_ε get its minimum at v, and we deduce

$$0\leq\lim_{\beta\longrightarrow 0}\frac{\Phi_\varepsilon(v+\beta l)-\Phi_\varepsilon(v)}{\beta}. \tag{64}$$

Using Lemma 9, replacing $\partial\mu/\partial m$ in system (47), we have

$$\begin{aligned}
0&\leq\lim_{\beta\longrightarrow 0}\frac{v_\varepsilon(q_\varepsilon+\beta l)-v_\varepsilon(q_\varepsilon)}{\beta}\\
&=\sum_{i=1}^{n}\int_\omega\tilde{\chi}_\omega^*\tilde{\chi}_\omega\left[\int_0^M\frac{\partial\mu}{\partial x_i}\frac{\partial p_i}{\partial m}dm+\int_0^M\frac{\partial}{\partial x_i}(\mu_{xx}-v\mu_x-lq_x)p_i dm\right]dx+\int_0^M 2\varepsilon l_m v_m dm\\
0&\leq\sum_{i=1}^{n}\int_\omega\tilde{\chi}_\omega^*\tilde{\chi}_\omega\left[\int_0^M\frac{\partial\mu}{\partial x_i}\left(\frac{\partial p_i}{\partial m}+\frac{\partial^2 p_i}{\partial x^2}+v(m)\frac{\partial p_i}{\partial x}\right)+l(m)\frac{\partial q_x}{\partial x_i}p_i dm\right]dx+\int_0^M 2\varepsilon l_m v_m dm\\
&=\sum_{i=1}^{n}\int_0^M l(m)\langle\tilde{\chi}_\omega\frac{\partial q_x}{\partial x_i};\tilde{\chi}_\omega p_i\rangle_{L^2(\omega)}dm+\int_0^M 2\varepsilon l_m v_m dm\\
&=\int_0^M\left[l(m)\sum_{i=1}^{n}\langle\tilde{\chi}_\omega\frac{\partial q_x}{\partial x_i};\tilde{\chi}_\omega p_i\rangle_{L^2(\omega)}+2\varepsilon l_m v_m\right]dm.
\end{aligned} \tag{65}$$

Consequently, for an arbitrary control $l=l(m)$ we conclude

$$l(m)\sum_{i=1}^{n}\langle\tilde{\chi}_\omega\frac{\partial q_x}{\partial x_i};\tilde{\chi}_\omega p_i\rangle_{L^2(\omega)}-2\varepsilon v_{mm}l(m)=0. \tag{66}$$

Then,

$$\sum_{i=1}^{n}\langle\tilde{\chi}_\omega\frac{\partial q_x}{\partial x_i};\tilde{\chi}_\omega p_i\rangle_{L^2(\omega)}-2\varepsilon v_{mm}=0. \tag{67}$$

Consequently,

$$v_{mm}=\frac{-1}{2\varepsilon}\sum_{i=1}^{n}\langle\tilde{\chi}_\omega\frac{\partial q_x}{\partial x_i};\tilde{\chi}_\omega p_i\rangle_{L^2(\omega)}. \tag{68}$$

□

4. Example

In this section, we propose the numerical approach to computing the solution of our method (68). We consider the one dimensional bilinear equation

$$\begin{cases}\dfrac{\partial q}{\partial t}+\alpha\dfrac{\partial^2 q}{\partial x^2}=\beta v(x,m)\dfrac{\partial q}{\partial x}, & [0,1]\times[0,1],\\ q(x,0)=q_0(x)=2x, & [0,1],\\ q=0, & \text{at } x=0,1.\end{cases} \tag{69}$$

The operator $-\alpha(\partial^2/\partial x^2)$ admits a set of eigenfunctions $\phi_n(\cdot)$ associated to the eigenvalues λ_n given by

$$\begin{aligned}\phi_n(x)&=\sqrt{2}\sin(n\pi x);\\ \lambda_n&=\alpha n^2\pi^2,\\ n&\geq 1.\end{aligned} \tag{70}$$

While the operator $-\alpha(\partial^2/\partial x^2)$ of system (69) and the perturbation $\beta v(x,m)\partial q/\partial x$ commute, using Pazy [37], the solution of (69) can be written as

Step 1: Choose
The desired targ y^d et.
The convergence accuracy ζ.
The subregion ω and time M.
Step 2: Until $\|v^{n+1} - v^n\| \leq \zeta$ repeat
Using (71), compute q^n associated to v^n.
Using (72), compute p^n associated to v^n.
Using (74) and (75), compute v^{n+1}.
Step 3: v^n such that $\|v^{n+1} - v^n\| \leq \zeta$ is the minimum of (76).

Algorithm 1:Algorithm for calculating the solution of the problem (50).

$$q(x,m) = \sum_{n=1}^{n=N} e^{\alpha n^2 \pi^2 t} < e^{\beta \int_0^m (\partial v/\partial x)(x,m)\mathrm{d}m} q_0, \sqrt{2}\sin(n\pi x) > \sqrt{2}\sin(n\pi x). \tag{71}$$

The one dimensional adjoint system can be written as

$$\begin{cases} -p_m(x,m) = p_{xx}(x,m) + (v(m)p)_x(x,m), & [0,1]\times[0,1], \\ p(x,1) = \left(\dfrac{\partial q(M)}{\partial x} - \widetilde{\chi}_\omega^* y^d\right), & [0,1], \\ p(0,m) = (p)_x(0,m) = 0, & [0,1]. \end{cases} \tag{72}$$

We define the perturbation function

$$f(q,p) = \langle \widetilde{\chi}_\omega \frac{\partial^2 q(m)}{\partial x^2}; \widetilde{\chi}_\omega p(m)\rangle_{L^2(\omega)}. \tag{73}$$

Using (68) and a finite difference schema, the optimal control v can be found by solving

$$\begin{cases} v_{mm}(x,m) = \dfrac{-1}{2\varepsilon} f(q,p), & [0,1]\times[0,1], \\ v(0) = v_m(1) = 0, & [0,1]. \end{cases} \tag{74}$$

By choosing $\varepsilon = 1/n$, we define the following sequence of control $(v^n)_n$ solution of

$$\begin{cases} v_{mm}^{n+1}(x,m) = \dfrac{-n}{2} f(q^n,p^n), & [0,1]\times[0,1], \\ v(0) = v_m(1) = 0, & [0,1], \end{cases} \tag{75}$$

where q^n and p^n are, respectively, the solution of (71) and (72) perturbed by v^n with $v^0 = 0$.

The penalty cost (43) becomes

$$\Phi_n(v^n) = \frac{1}{2}\left\|\chi_\omega \nabla q^n - y^d\right\|^2_{L^2\left(0,1,H_0^1([0,1])\right)} + \frac{1}{2n}\int_0^1\int_0^1 \left[v_m^n(x,tm)\right]^2 \mathrm{d}x\mathrm{d}m. \tag{76}$$

The following convergent Algorithm 1 allows the implementation of our results.

Remark 11.

(1) The distributed bilinear systems (1), are considered with the feedback map $v(x,m)q_x(x,m)$ as multiplication of the control by the velocity of the state system. One can consider another different type of perturbation.

(2) In the case of partial controllability, we use in general temporal control feedback. This type of control is compatible with real applications.

(3) For the simulation point of view, the obtained control formula is easy to calculate numerically. This encourages us to establish numerical approaches and simulations of the proposed problems using Algorithm 1..

5. Conclusion

We consider the flow optimal control problem constrained by a bilinear distributed system. The chosen optimal controls are regular, and the existence of solutions is proved and characterized using optimization techniques. Our method is applied to the partial flow control problem allowing us to control a flow on a specific subdomain of the system domain.

Finally, as an example, we present the numerical approach, which makes it possible to concretize the obtained results.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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