Hindawi
Journal of Mathematics
Volume 2022, Article ID 3559035, 14 pages
https://doi.org/10.1155/2022/3559035

Research Article

Fractional Brownian Motion for a System of Fuzzy Fractional Stochastic Differential Equation

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Received 26 July 2022; Revised 8 September 2022; Accepted 24 September 2022; Published 11 October 2022

Academic Editor: Heng Liu

We study fractional Brownian motion– (FBM–) driven fuzzy stochastic fractional evolution equations. These equations can be used to model fuzziness, long-range dependence, and unpredictability in hybrid real-world systems. Under various assumptions regarding the coefficients, we investigate the existence-uniqueness of the solution using an approximation method to the fractional stochastic integral. We can solve an equation with linear coefficients, for example, in financial models Application to a model of population dynamics is also illustrated. An example is propounded to show the applicability of our results.

1. Introduction

Fractional stochastic differential equations (FSDEs) play an important role in the modeling of numerous complicated processes in several sectors of science and engineering. FSDE theory and applications were examined. Furthermore, numerous academics have produced interest in systems with memory or aftereffects.

There appears to be some difficulty in modeling a variety of modern-world systems, such as trying to characterize the physical system and differing viewpoints on its properties. The fuzzy set theory will be utilized to resolve this issue [1]. It can handle linguistic claims like "big" and "less" mathematically using this approach. A fuzzy set provides the ability to examine fuzzy differential equations (FDEs) in representing a variety of phenomena, including imprecision. For example, fuzzy stochastic differential equations (FSDEs) could be used to explore a wide range of economic and technical problems involving two types of uncertainty: randomness and fuzziness.

Deterministic fuzzy differential equations were developed as a result of research into dynamic systems with inadequate or ambiguous parameter information. They are increasingly used in system models in biology, engineering, civil engineering, bioinformatics, and computational biology, quantum optics and gravity, hydraulic and mechanical system modeling. Many studies in this field have been conducted utilizing various ways of expressing differential problems in a fuzzy framework. The Hukuhara derivative of a set-valued function was used as the foundation for the first approach to deterministic FDEs.

There are several articles on FSDEs, each of which takes a different approach. The fuzzy stochastic ItÃ´ integral was defined by the author in [2]. The fuzzy It$\hat{o}$ stochastic integral was driven by fuzzy non-anticipating stochastic processes and the Wiener process. Malinowski [3–5] worked on the application, properties, Ito type strong solutions, and with delay to stochastic fuzzy differential equations. To construct a fuzzy random variable, the method involves embedding crisp It$\hat{o}$ stochastic integral into fuzzy space. Guo et al. [6], Li et al. [7], Deng et al. [8], and Ahmad [9] worked on the stability of fractional stochastic differential equation. Hussain et al. [10] studied stochastic modeling of COVID-19. Abbas et al. [11, 12] solve ordinary differential equation. In the study of Niazi et al. [13], Iqbal et al. [14], Shafqat et al. [15], Alnahdi [16], Khan [17], and Abuasbeh et al. [18–21], existence and uniqueness of the fuzzy fractional evolution equations were investigated.

FBM has been used to describe the behavior of asset prices and stock market volatility. This process is a good fit for describing these values because of its long-range dependence on self-similarity qualities. For a general discussion of the applications of FBM to model financial quantities, see Shiryaev [22]. Several writers have proposed a fractional Black and Scholes model to replace the traditional Black and Scholes model, which is memoryless and depends on the so-called fractional Black and Scholes model of geometric Brownian motion. The risky asset's market stock price is given by this model:

$$\mathbf{S}_\nu = \mathbf{S}_0 \exp\left(\mu\nu + \sigma\mathscr{B}_\nu^{\mathscr{H}} - \frac{\sigma^2}{2}\nu^{2\mathscr{H}}\right), \tag{1}$$

where $\mathscr{B}^{\mathscr{H}}$ is an FBM with the Hurst parameter $\mathscr{H}$, μ is the mean rate of return, and $\sigma > 0$ is the volatility, and at time ν, the price of non-risky assets is $e^{r\nu}$, where r is the interest rate.

In modeling many stochastic systems, on the other hand, the FBM, which exhibits long-range dependency, is proposed to replace Brownian motion as a driving mechanism. The FBM is a Gaussian process with favorable qualities such as long-range dependency, self-similarity, and increment stationarity when $H \in (1, 2)$ is used as the Hurst parameter. This method is well suited to the study of phenomena with long-range and scale-invariant correlations. When $H \neq 3/2$, however, FBM is not semimartingale.

Jafari et al. [23] worked on FSDEs driven by FBM. Inspired by the above paper, we introduce fuzzy fractional stochastic differential equation (FFSDEs) in relation to FBM in this study for order (1,2). These equations can be used to simulate unpredictability, fuzziness, and long-range dependence in hybrid dynamic systems. To determine the explicit answers, we use an approximation approach to fractional stochastic integrals. To investigate existence-uniqueness of strong solutions, we use Liouville form of FBM with parameter $H \in (3/2, 2)$. Furthermore, we discuss using the equations in financial models:

$$\begin{aligned} {}_c^0D_\nu^\alpha X(\nu) &= \mathfrak{f}\left(\nu, X(\nu), D^\beta X(\nu)\right)ds + g\left(\nu, X(\nu)D^\beta X(\nu)\right)d\mathscr{B}_H(\nu), \nu \in [0, \mathfrak{T}], \\ X(0) + m(X) &= X_0, \\ X'(0) &= X_1, \end{aligned} \tag{2}$$

where $\alpha \in (1, 2)$, $\beta \in (0, 1)$, and $\mathscr{B}_H$ are FBM and $f, g : [0, \mathfrak{T}] \longrightarrow \mathbf{R}^\mathbf{m}$ is the continuous function.

There has been a recent interest in input noises lacking independent increments and exhibiting long-range dependence and self-similarity qualities, which has been motivated by some applications in hydrology, telecommunications, queueing theory, and mathematical finance. When the covariances of a stationary time series converge to zero like a power function and diverge so slowly that their sums diverge, this is known as long-range dependence. The self-similarity property denotes distribution invariance when the scale is changed appropriately. FBM, a generalization of classical Brownian motion, is one of the simplest stochastic processes that are Gaussian, self-similar, and exhibit stationary increments. When the Hurst parameter is more than 1/2, the FBM exhibits long-range dependency, as we will see later. In this note, we look at some of the features of FBM and discuss various strategies for constructing a stochastic calculus for this process. We will also go through some turbulence and mathematical finance applications. The remaining of this paper is as follows. Section 2 discusses the definition of FBM and Liouville form of this process. Then, some introductory material on fuzzy stochastic processes and fuzzy stochastic integrals is reviewed. Section 3 introduces a class of FFSDEs driven by FBM. Furthermore, the existence-uniqueness of solutions is proven using an approximation approach. In Section 4, some findings are presented. In section 5, application to a model of population dynamics is also illustrated. Finally, in Section 6, a conclusion is given.

2. Preliminaries

A Gaussian process $\mathscr{B}^{\mathscr{H}} = \{\mathscr{B}^{\mathscr{H}}(\nu)$ is called FBM of Hurst parameter $\mathscr{H} \in (0, 1)$ if it has mean zero and the covariance function:

$$R_{\mathscr{B}^{\mathscr{H}}}(\nu, s) = E\left(\mathscr{B}^{\mathscr{H}}(\nu)\mathscr{B}^{\mathscr{H}}(s)\right) = \frac{1}{2}\left(s^{2\mathscr{H}} + \nu^{2\mathscr{H}} - |\nu - s|^{2\mathscr{H}}\right). \tag{3}$$

This phenomenon was first described in [24] and investigated in [25], where a Brownian motion-based stochastic integral representation was constructed. For $\mathscr{H} > 3/2$, this process' long-range dependence and self-similarity qualities give suitable driving noise in stochastic models like networks, finance, and physics. Because$\mathscr{B}^{\mathscr{H}}$is not semimartingale if$\mathscr{H} \neq 3/2$. In terms of FBM, classical Itô theory cannot be used to generate stochastic integral. Two approaches were used to define stochastic integrals about FBM. In the situation of $\mathscr{H} > 3/2$, Young's integral [26] can be used to define the Riemann-Stieltjes stochastic integral. The Malliavin calculus is used in a second way to define stochastic integral concerning FBM (see [27–30]). The following is an illustration of $\mathscr{B}_{\mathscr{H}}(\nu)$ given in [25]:

$$\mathscr{B}^{\mathscr{H}}(\nu) = \frac{1}{\Gamma(1+\alpha)}\left(\int_{-\infty}^0 [(\nu - s)^\alpha - (-s)^\alpha]dW(s) + \mathscr{B}_{\mathscr{H}}(\nu)\right), \tag{4}$$

where W is a Browian motion, $\alpha = \mathscr{H} - 3/2$ and $\mathscr{B}_{\mathscr{H}}(\nu) = \int_0^\nu (\nu - s)^\alpha dW(s)$. A FBM in Liouville form (LFBM) is the process $\mathscr{B}_{\mathscr{H}}(\nu)$ with $\mathscr{H} \in (1, 2)$, which has many of the same qualities as the FBM except for the non-stationary increments. In [27], a semimartingale process was utilized to approximate $\mathscr{B}_{\mathscr{H}}(\nu)$ using the Malliavin calculus technique:

$$\mathscr{B}_{\mathscr{H},\varepsilon}(\nu) = \int_0^\nu (\nu - s + \varepsilon)^\alpha dW(s), \varepsilon > 0red. \tag{5}$$

Furthermore,

$$\mathscr{B}_{\mathscr{H},\varepsilon}(\nu) = \alpha\int_0^\nu \psi^\varepsilon(s)ds + \varepsilon^\alpha W(\nu), \tag{6}$$

where

$$\psi^\varepsilon(\nu) = \int_0^\nu (\nu - s + \varepsilon)^{\alpha-1} dW(s). \tag{7}$$

The process $\mathscr{B}_{\mathscr{H},\varepsilon}(\nu)$ converges to $\mathscr{B}_{\mathscr{H}}(\nu)$ in $L^2(\Omega)$ when ε tends to zero [31].

Preliminaries on FRVs, fuzzy stochastic processes (FSP), and fuzzy stochastic integrals are provided in this section (see [4, 29, 32]). The family of all nonempty, compact, and convex subsets of $\mathbf{R}^\mathbf{m}$ is denoted by $\mathscr{G}(\mathbf{R}^\mathbf{m})$. The Hausdorff metric, abbreviated as $d_{\mathscr{H}}$, is defined as follows:

$$d_{\mathscr{H}}(\mathscr{N},\mathscr{B}) = \max\left\{\sup_{n\in\mathscr{N}}\inf_{b\in\mathscr{K}}\|n-b\|, \sup_{b\in\mathscr{B}}\inf_{n\in\mathscr{N}}\|n-b\|\right\}red. \tag{8}$$

With regard to $d_{\mathscr{H}}$, the space $\mathscr{G}(\mathbf{R}^\mathbf{m})$ is a full and separable metric space. If $\mathscr{N},\mathscr{B}$, and $\mathscr{Q}$ are equal to $\mathscr{G}(\mathbf{R}^\mathbf{m})$, then

$$d_{\mathscr{H}}(\mathscr{N} + \mathscr{Q}, \mathscr{B} + \wp) = d_{\mathscr{H}}(\mathscr{N},\mathscr{B}). \tag{9}$$

A probability space is defined $(\Omega,\mathscr{A},\wp)$. If mapping $\mathfrak{F} : \Omega \longrightarrow \mathscr{G}(\mathbf{R}^\mathbf{m})$ satisfies the following conditions, it is called $\mathscr{N}$-measurable.

$$\{\omega\in\Omega : \mathfrak{F}(\omega)\cap Q \neq \varphi\} \in \mathscr{N}. \tag{10}$$

$\mathscr{Q}\subset\mathbf{R}^\mathbf{m}$ for every closed set, let $M(\Omega,\mathscr{N};\mathscr{G}(\mathbf{R}^\mathbf{m}))$ denote a set of $\mathscr{A}$-measurable multifunctions with $\mathscr{G}(\mathbf{R}^\mathbf{m})$ values. For $\wp\geq 1$, a multifunction $\mathfrak{F}\in M$ is said to be $\mathscr{L}^\wp$-integrably bounded if $h\in\mathscr{L}^\wp(\Omega,\mathscr{N},\wp;\mathbf{R}^\mathbf{m}{}_+)$ exists that is $|||\mathfrak{F}|||\leq h\wp - a.e, \mathbf{R}^\mathbf{m}{}_+ = [0,\infty)$, and

$$|||\mathfrak{F}||| = d_{\mathscr{H}}(\mathfrak{F},\{0\}) = \sup_{\mathfrak{f}\in\mathfrak{F}}|\mathfrak{f}|. \tag{11}$$

$\mathfrak{F}\in M$ is known to be $\mathscr{L}^\wp$-integrably bounded if and only if $|||\mathfrak{F}|||\in\mathscr{L}^\wp(\Omega,\mathscr{N},\wp;\mathbf{R}^\mathbf{m}{}_+)$ (see [31]). Let us put it this way:

$$\mathscr{L}^\wp(\Omega,\mathscr{N},\wp;\mathscr{G}(R))) = \{\mathfrak{F}\in M(\Omega,\mathscr{N};\mathscr{G}(R)) : |||\mathfrak{F}|||\in\mathscr{L}^\wp(\Omega,\mathscr{N},\wp;\mathbf{R}^\mathbf{m}{}_+)\}. \tag{12}$$

For fuzzy set $u\in\mathbf{R}^\mathbf{m}$, membership function $u : \mathbf{R}^\mathbf{m}\longrightarrow[1,2]$ is defined, where $u(\mathrm{x})$ denotes degree of membership of x in fuzzy set u. Assume fuzzy sets $u : \mathbf{R}^\mathbf{m}\longrightarrow[1,2]$ denoted by $\mathfrak{F}(\mathbf{R}^\mathbf{m})$ that is $[u]^\alpha\in\mathscr{G}(\mathbf{R}^\mathbf{m})$ for every $\alpha\in[1,2]$, where $[u]^\alpha = \{\mathrm{x}\in\mathbf{R}^\mathbf{m} : u(\mathrm{x})\geq\alpha\}$. Define $d_\alpha : \mathfrak{F}(\mathbf{R}^\mathbf{m})\times\mathfrak{F}(\mathbf{R}^\mathbf{m})\longrightarrow[0,\infty)$ by

$$d_\infty(u,\nu) = \sup_{1\leq\alpha\leq 2} d_{\mathscr{H}}\{[\nu]^\alpha,[\nu]^\alpha\}red. \tag{13}$$

Therefore, in $\mathfrak{F}(\mathbf{R}^\mathbf{m})$, d_∞ is metric, and $(\mathfrak{F}(\mathbf{R}^\mathbf{m}), d_\infty)$ is complete metric space. We have below properties for any $u, \nu, w, z\in\mathfrak{F}(\mathbf{R}^\mathbf{m}), \lambda\in\mathbf{R}^\mathbf{m}$:

(i) $d_\infty(u+w,\nu+w) = d_\infty(u+\nu)$

(ii) $d_\infty(u+w,\nu+z) = d_\infty(u+w) + d_\infty(w,\nu)$

(iii) $d_\infty \leq d_\infty(u,w) + d_\infty(w,\nu)$

(iv) $d_\infty(\lambda u,\lambda\nu = |\lambda| d_\infty(u,\nu)$

We use $\langle 1\rangle\in\mathfrak{F}(\mathbf{R}^\mathbf{m})$ as $\langle 1\rangle = 2_{\{1\}}$, where for $y\in\mathbf{R}^\mathbf{m}$, $2_{\{y\}}(\mathrm{x}) = 2$ if $\mathrm{x} = y$ and $2_y(\mathrm{x}) = 1$ if $\mathrm{x}\neq y$.

Definition 1 See ([33]). A probability space is defined as $(\Omega,\mathscr{N},\wp)$. If mapping $[X]^\alpha : \Omega\longrightarrow\mathscr{G}(\boldsymbol{R}^{\boldsymbol{m}})$ is an $\mathscr{N}$ -measurable multifunction for all $\alpha\in[1,2]$, FRV is function $X : \Omega\longrightarrow\mathscr{G}(\boldsymbol{R}^{\boldsymbol{m}})$.

Assume metric ρ in set $\mathfrak{F}(\mathbf{R}^\mathbf{m})$ and η-algebra $\mathscr{B}_\rho$, which is created by topology induced by $\mathscr{N}$ fuzzy random variable (FRV) can be thought of as measurable mapping between two measurable spaces $(\Omega,\mathscr{N})$ and $(\mathfrak{F}(\mathbf{R}^\mathbf{m}),\mathscr{B}_\rho)$, which we refer to as X is $\mathscr{N}|\mathscr{B}_\rho$-measurable. Take a look at the below metric:

$$d_s(u,\nu) = \max\left\{\inf_{\lambda\in\wedge}\sup_{\nu\in[1,2]}\|\lambda(\nu),\nu\|, \sup_{\nu\in[1,2]} d_{\mathscr{H}}(X_u(\nu), X_u(\lambda(\nu)))\right\}, \tag{14}$$

where $\wedge$ represents set of strictly increasing continuous functions: $\lambda : [1,2]\longrightarrow[1,2]$, where $\lambda(1) = 1, \lambda(2) = 2$, and X_u, $X_\nu : [1,2]\longrightarrow\mathfrak{F}(\mathbf{R}^\mathbf{m})$ are cÃ dlÃ g representations for fuzzy sets $u,\nu\in\mathfrak{F}(\mathbf{R}^\mathbf{m})$ (see [34]). The space $(\mathfrak{F}(\mathbf{R}^\mathbf{m}), d_s)$ is a Polish metric space, and space $(\mathfrak{F}(\mathbf{R}^\mathbf{m}), d_\infty)$ is complete and non-separable.

We have

â€"X is FRV if and only if X is $\mathscr{N}|\mathscr{B}_{d_s}$-measurable for mapping $X : \longrightarrow\mathfrak{F}(\mathbf{R}^\mathbf{m})$ on probability space $(\Omega,\mathscr{N},\wp)$.

If X is $\mathscr{N}|\mathscr{B}_{d_s}$-measurable, it is FRV, but not the other way.

Definition 2. If $[X]^\alpha\in\mathscr{L}^\wp(\Omega,\mathscr{N},\wp;\mathscr{G}(\boldsymbol{R}^{\boldsymbol{m}}))$, for any $\alpha\in[1,2]$, FRV $X : \Omega\longrightarrow\mathfrak{F}(\boldsymbol{R}^{\boldsymbol{m}})$ is $\mathscr{L}^\wp$-integrably bounded for $\wp\geq 2$.

Let us denote by $\mathscr{L}^\wp(\Omega,\mathscr{N},\wp;\mathfrak{F}(\mathbf{R}^\mathbf{m}))$ set of all $\mathscr{L}^\wp$ -integrably bounded FRVs. The random variables $X, Y\in\mathscr{L}^\wp(\Omega,\mathscr{N},\wp;\mathfrak{F}(\mathbf{R}^\mathbf{m}))$ are identical if $\wp(d_\infty(X,Y) = 1) = 2$. For FRV $X : \Omega\longrightarrow\mathfrak{F}(\mathbf{R}^\mathbf{m})$, and $\wp\geq 1$, the below conditions are equivalent:

$$\begin{aligned} &X \in \mathscr{L}^{\wp}(\Omega, \mathscr{N}, \wp; \mathfrak{F}(\mathbf{R^m})), \\ &[X]^0 \in \mathscr{L}^{\wp}(\Omega, \mathscr{N}, \wp; \mathscr{G}(\mathbf{R^m})), \\ &\left|\left|\left|[X]^0\right|\right|\right| \in \mathscr{L}^{\wp}(\Omega, \mathscr{N}, \wp; \mathbf{R^m}_+) red. \end{aligned} \tag{15}$$

Assume $j := [0, \mathfrak{T}]$, and $(\Omega, \mathscr{N}, \wp)$ be complete probability space with filtration $\{\mathscr{N}_\nu\}_{\nu \in j}$ satisfying hypotheses, an increasing and right continuous family of sub ρ-algebras of $\mathscr{N}$, and containing all $\wp$-null sets.

Definition 3. If mapping $X(\nu)$: $\Omega \longrightarrow \mathfrak{F}(\boldsymbol{R^m})$, for every $\nu \in j$, is FRV, then $X : j \times \Omega \longrightarrow \mathfrak{F}(\boldsymbol{R^m})$ is FSP.

Definition 4. A FSP X is d_∞ -continuous, if almost all its trajectories, that is mappings $X(.,\omega)$: $j \times \Omega \longrightarrow \mathfrak{F}(\boldsymbol{R^m})$ are d_∞ -continuous functions.

A FSP X is measurable, if $[X]^\alpha : j\Omega \longrightarrow \mathscr{G}(\mathbf{R^m})$ is $\mathscr{B}(j) \otimes \mathscr{N}$-measurable multifunction for all $\alpha \in [1, 2]$, where $\mathscr{B}(j)$ denotes Borel ρ-algebra of subsets of j.

A process X is nonanticipating if and only if for every $\alpha \in [1, 2]$, multifunction $[X]^\alpha$ is measurable with respect to ρ-algebra $\mathscr{A}$, where it is defined as follows:

$$\mathscr{A} := \{\mathscr{N} \in \mathscr{K} \otimes \mathscr{N} : \mathscr{N}^\nu \in \mathscr{N}_\nu foreveryv \in j\}, \tag{16}$$

where $\mathscr{N}^\nu = \{\omega : (\nu, \omega) \in \mathscr{N}\}$.

Definition 5. A FSP X is called $\mathscr{L}^{\wp}$ -integrably bounded ($\wp \geq 2$), if there exists real-valued stochastic process $h \in \mathscr{L}^{\wp}(j\Omega, \mathscr{A}; \mathfrak{F}(\boldsymbol{R^m}))$, the Fubini theorem, fuzzy integral is defined by

$$\int_0^{\mathfrak{T}} X(s, \omega) ds, \tag{17}$$

for $\omega \in \Omega \setminus \mathscr{A}_x$, where $\mathscr{A}_x \in \mathscr{N}$ and $P(\mathscr{A}_x) = 1$. The fuzzy integral $\int_0^{\mathfrak{T}} X(s, w) ds$ can be defined level-wise. For every $\alpha \in [1, 2]$, and $\omega \in \Omega \setminus \mathscr{A}_x$, Aumann integral $\int_0^{\mathfrak{T}} [X(s, \omega)]^\alpha ds$ belongs to $\mathscr{G}(\mathbf{R^m})$, so FRV $\int_0^{\mathfrak{T}} X(s, \omega)$ belongs to $\mathfrak{F}(\mathbf{R^m})$, so FRV $\int_0^{\mathfrak{T}} X(s, \omega) ds$ belongs to $\mathfrak{F}(\mathbf{R^m}) \forall \omega \in \Omega \setminus \mathscr{A}_x$.

Definition 6. The fuzzy stochastic Lebesgue-Aumann integral of $X \in \mathscr{L}^1(j \times \Omega, \mathscr{A}; \mathfrak{F}(\boldsymbol{R^m}))$ is defined as

$$\mathscr{L}_x(\nu, \omega) = \begin{cases} \displaystyle\int_0^{\mathfrak{T}} 2_{[0,\nu]}(s) X(s, \omega) ds f oreveryw \in \Omega \setminus \mathscr{A}_x, \\ 0 f oreveryw \in \mathscr{A}_x. \end{cases} \tag{18}$$

Proposition 7 See ([5]). *The properties of the integral $\mathscr{L}_x$ can be demonstrated as follows:*

Suppose $\wp \geq 2$. If $X \in \mathscr{L}^{\wp}(j \times \Omega, \mathscr{A}; \mathfrak{F}(\mathbf{R^m}))$, then $\mathscr{L}_x(.,.) \in \mathscr{L}^{\wp}(j \times \Omega, \mathscr{A}; \mathfrak{F}(\mathbf{R^m}))$.

(i) *Suppose $X \in \mathscr{L}^1(j \times \Omega, \mathscr{A}; \mathfrak{F}(\mathbf{R^m}))$, then $\{\mathscr{L}_x(\nu)\}_{\nu \in j}$ is d_∞-continuous*

(ii) *Suppose $X, Y \in \mathscr{L}^{\wp}(j \times \Omega, \mathscr{A}; \mathfrak{F}(\mathbf{R^m}))$, for $\wp \geq 1$, then*

(iii) $\sup_{u \in [0,\nu]} d_\infty^{\wp}(\mathscr{L}_{\nu,x}(u), \mathscr{L}_{\nu,y}(u)) \leq \nu^{p-1} \int_0^\nu d_\infty^p(X(s), Y(s)) ds$.

Let us define an embedding of $\mathbf{R^m}$ into $\mathfrak{F}(\mathbf{R^m})$ by (.): $\mathbf{R^m} \longrightarrow \mathfrak{F}(\mathbf{R^m})$, which is for $r' \in \mathbf{R^m}$,

$$k'(a) = \begin{cases} 1 f orn = k', \\ 0 f orn \in \mathbf{R^m} \setminus \left\{k'\right\}. \end{cases} \tag{19}$$

If $X : \Omega \longrightarrow \mathbf{R^m}$ is random variable on probability space $(\Omega, \mathscr{N}, \wp)$, now $X : \Omega \longrightarrow \mathfrak{F}(\mathbf{R^m})$ is FRV. The same property exists for stochastic processes.

We define fuzzy stochastic Itô integral by using FRV as $\int_0^{\mathfrak{T}} X(s) dW(s)$, where W is Wiener process. [5] will be beneficial for the following properties.

Proposition 8. *Assume $X \in \mathscr{L}^2(j \times \Omega, \mathscr{A}; \boldsymbol{R^m})$, then $\{\langle \int_0^\nu X(s) dW(s) \rangle\}_{\nu \in j}$ is FSP, and we have $\langle \int_0^\nu X(s) dW(s) \rangle \in \mathscr{L}^2(j \times \Omega, \mathscr{A}; \mathfrak{F}(\boldsymbol{R^m}))$.*

Proposition 9. *Suppose $X \in \mathscr{L}^2(j \times \Omega, \mathscr{A}; \mathbf{R^m})$, then $\{\langle \int_0^\nu X(s) dW(s) \rangle\}_{\nu \in j}$ is d_∞ -continuous.*

3. Application to Fuzzy Stochastic Differential Equation

The following is a list of FFSDEs driven by FBM that we will investigate in this section:

$$\begin{aligned} X(\nu) = {} & \mathscr{Q}_q(\nu)(X_0 - m(X)) + \mathscr{G}_q(\nu) X_1 \\ & + \frac{1}{\sqrt{\gamma}} \left[\int_0^\nu (\nu - s)^{q-1} P_q(\nu - s) \mathfrak{f}\left(s, X(s), D^\beta X(s)\right) ds \right. \\ & \left. + \left\langle \int_0^\nu (\nu - s)^{q-1} P_q(\nu - s) g\left(s, X(s), D^\beta X(s)\right) d\mathscr{B}_H(s) \right\rangle \right], \end{aligned} \tag{20}$$

such that

$$\begin{aligned} \mathscr{Q}_q(\nu) = M_q(\theta) C(t^q \theta) d\theta, \mathscr{G}_q(\nu) = \int_0^\nu C_q s ds, P_q(\nu) \\ = \int_0^\infty q\theta M_q(\theta) C(t^q \theta) d\theta, \end{aligned} \tag{21}$$

where $\mathscr{Q}_q(\nu)$ and $\mathscr{G}_q(\nu)$ are continuous with $C(0) = I$ and $K(0) = I$, $|\mathscr{Q}_q(\nu)| \leq c, c > 1$ and $|\mathscr{G}_q(\nu)| \leq c, c > 1, \forall \nu \in [0, T]$. Equation (20) shows that the Liouville form of FBM with $H \in (1, 2), X_0 : \Omega \longrightarrow \mathfrak{F}(\mathbf{R^m})$ is an FRV, $g : j \times \Omega \times \mathfrak{F}(\mathbf{R^m})$

$\longrightarrow \mathbf{R^m}$. The approximate corresponding equation (20) is

$$X^{\varepsilon}(\nu)=\mathcal{Q}_q(\nu)(X_0-m(X))+\mathcal{G}_q(\nu)X_1 + \frac{1}{\sqrt{\gamma}}\left[\left[\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f}\left(s,X^{\varepsilon}\left(s,D^{\beta}X^{\varepsilon}(s)\right)\right)ds + \left\langle\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X^{\varepsilon}(s),D^{\beta}X^{\varepsilon}(s)\right)d\mathscr{B}_H^{\varepsilon}(s)\right\rangle\right].\tag{22}$$

Assumption 10. Consider the below assumptions about coefficients of equation:

$A1)$ The mappings $\mathfrak{f}: j\times\Omega\times\mathfrak{F}(\mathbf{R^m})\longrightarrow\mathfrak{F}(\mathbf{R^m})$ and $g: j\times\Omega\times\mathfrak{F}(\mathbf{R^m})\longrightarrow\mathbf{R^m}$ are $\mathcal{N}\otimes\mathcal{A}_{d_s}|\mathscr{B}_{d_s}|$-measurable and $\mathcal{A}\otimes\mathscr{B}_{d_s}|\mathscr{B}(\mathbf{R^m})|$-measurable, respectively

$A2)$ There exists constant $\mathscr{L}>0\forall u,\nu\in\mathfrak{F}(\mathbf{R^m})$ and every $\nu\in j$ such that

$$\max\{d_{\infty}(f(\nu,\omega,u),f(\nu,\omega,\nu)),|g(\nu,\omega,u)-g(\nu,\omega,\nu)|\} \leq \mathscr{L}d_{\infty}(u,\nu))\tag{23}$$

$A3)$ There exists constant $\mathcal{Q}>0\forall u,\nu\in\mathfrak{F}(\mathbf{R^m})$ and every $\nu\in j$,

$$\max\{d_{\infty}(f(\nu,\omega,u),0),|g(\nu,\omega,u)|\}\leq\mathcal{Q}(1+d_{\infty}(u,0))\tag{24}$$

Proposition 11. See [5]. *Suppose* $X,Y\in\mathscr{L}^2(j\times\Omega,\mathcal{A};\boldsymbol{R^m})$, *then*

$$E\sup_{u\in[0,\nu]}d_{\infty}^2\left(\left\langle\int_0^u X(s)dW(s)\right\rangle,\left\langle\int_0^u Y(s)dW(s)\right\rangle\right) \leq 4E\int_0^{\nu}d_{\infty}^2(\langle X(s)\rangle,\langle Y(s)\rangle)ds,\tag{25}$$

for every $\nu\in j$.

Theorem 12. *Assume* $\mathfrak{f}: j\times\Omega\times\mathfrak{F}(\boldsymbol{R^m})\longrightarrow\mathfrak{F}(\boldsymbol{R^m})$ *and* $g: j\times\Omega\times\mathfrak{F}(\boldsymbol{R^m})\longrightarrow\boldsymbol{R^m}$ *as mappings satisfy assumptions* $(A1)-(A3)$ *and* $X_0\in\mathscr{L}^2(\Omega,\mathcal{N}_0,\wp;\mathfrak{F}(\boldsymbol{R^m}))$. *Then, Equation (22) has a strong unique solution.*

Proof. Assume SDE (22),

$$X^{\varepsilon}(\nu)=\mathcal{Q}_q(\nu)(X_0-m(X))+\mathcal{G}_q(\nu)X_1 + \int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f}\left(s,X^{\varepsilon}(s),D^{\beta}X^{\varepsilon}(s)\right)ds + \left\langle\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X^{\varepsilon}(s),D^{\beta}X^{\varepsilon}(s)\right)d\mathscr{B}_H^{\varepsilon}(s)\right\rangle.\tag{26}$$

By Equation (6), we can write

$$X^{\varepsilon}(\nu)=\mathcal{Q}_q(\nu)(X_0-m(X))+\mathcal{G}_q(\nu)X_1 + \int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f}\left(s,X^{\varepsilon}(s),D^{\beta}X^{\varepsilon}(s)\right)ds + \left\langle\int_0^{\nu}\alpha\varphi^{\varepsilon}(s)(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X^{\varepsilon}(s),D^{\beta}X^{\varepsilon}(s)\right)ds + \int_0^{\nu}\varepsilon^{\alpha}(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X^{\varepsilon}(s),D^{\beta}X^{\varepsilon}(s)\right)dW(s)\right\rangle.\tag{27}$$

□

Let us consider the Picard iterations

$$X_n^{\varepsilon}(\nu)=\mathcal{Q}_q(\nu)(X_0-m(X))+\mathcal{G}_q(\nu)X_1 + \int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f}\left(s',X_{n-1}^{\varepsilon}(s),D^{\beta}X_{n-1}^{\varepsilon}(s)\right)ds + \left\langle\int_0^{\nu}\alpha\varphi^{\varepsilon}(s)(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X_{n-1}^{\varepsilon}(s)D^{\beta}X_{n-1}^{\varepsilon}(s)\right)ds + \int_0^{\nu}\varepsilon^{\alpha}(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X_{n-1}^{\varepsilon}(s)D^{\beta}X_{n-1}^{\varepsilon}(s)\right)dW(s)\right\rangle,\tag{28}$$

for n =1,2,..., and for every $\nu\in j$, and $X_0(\nu)=X_0$. For $\nu\in j$ and $n\in\mathcal{A}$ we denote

$$\begin{aligned}
\ell_1(\nu) &= E\sup_{u\in[0,\nu]}d_{\infty}^2\left(\int_0^u(\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f}\left(s,X_0^{\varepsilon}(s),D^{\beta}X_0^{\varepsilon}(s)\right)ds + \left\langle\int_0^u\alpha\varphi^{\varepsilon}(s)(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X_0^{\varepsilon}(s),D^{\beta}X_0^{\varepsilon}(s)\right)ds + \int_0^u\varepsilon^{\alpha}(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X_0^{\varepsilon}(s),D^{\beta}X_0^{\varepsilon}(s)\right)dW(s)\right\rangle,0\right)\\
&\leq 3E\sup_{u\in[0,\nu]}d_{\infty}^2\left(\int_0^u(\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f}\left(s,X_0^{\varepsilon}(s),D^{\beta}X_0^{\varepsilon}(s)\right)ds,\langle 0\rangle\right)\\
&\quad+3E\sup_{u\in[0,\nu]}d_{\infty}^2\left(\left\langle\int_0^u\alpha\varphi^{\varepsilon}(s)(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X_0^{\varepsilon}(s),D^{\beta}X_0^{\varepsilon}(s)\right)ds\right\rangle,\langle 0\rangle\right)\\
&\quad+3E\sup_{u\in[0,\nu]}d_{\infty}^2\left(\left\langle\int_0^u\varepsilon^{\alpha}(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X_0^{\varepsilon}(s),D^{\beta}X_0^{\varepsilon}(s)\right)dW(s)\right\rangle,\langle 0\rangle\right)\\
&\leq 3\nu E\int_0^{\nu}d_{\infty}^2\left((\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f}\left(s,X_0^{\varepsilon},D^{\beta}X_0^{\varepsilon}(s)\right),\langle 0\rangle\right)ds\\
&\quad+3\alpha^2E\sup_{u\in[0,\nu]}d_{\infty}^2\left(\left\langle\int_0^u\varphi^{\varepsilon}(s)(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X_0^{\varepsilon}(s),D^{\beta}X_0^{\varepsilon}(s)\right)ds\right\rangle,\langle 0\rangle\right)\\
&\quad+12\varepsilon^{2\alpha}E\int_0^{\nu}d_{\infty}^2\left(\left\langle(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X_0^{\varepsilon}(s),D^{\beta}X_0^{\varepsilon}(s)\right)\right\rangle,\langle 0\rangle\right)ds\\
&\leq 6\mathcal{Q}^2\left(T+4\varepsilon^{2\alpha}\right)\left(1+E\left|\left|\left|[X^{\varepsilon}]^0\right|\right|\right|^2\right)\nu+3\alpha^2E\sup_{u\in[0,\nu]}d_{\infty}^2\\
&\quad\cdot\left(\left\langle\int_0^u\varphi^{\varepsilon}(s)(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X_0^{\varepsilon}(s),D^{\beta}X_0^{\varepsilon}(s)\right)ds\right\rangle,\langle 0\rangle\right),
\end{aligned}\tag{29}$$

for $\alpha=H-3/2>1$. Hence

$$\ell_1(\nu)\leq 6\mathcal{Q}^2\left(T+4\varepsilon^{2\alpha}\right)\left(1+E|||[X^{\varepsilon}]|||^2\right)\nu+3\alpha^2E\sup_{u\in[0,\nu]}d_{\infty}^2 \cdot\left(\left\langle\int_0^u\varphi^{\varepsilon}(s)(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X_0^{\varepsilon}(s),D^{\beta}X_0^{\varepsilon}(s)\right)ds\right\rangle,\langle 0\rangle\right).\tag{30}$$

We have

$$E\sup_{u\in[0,\nu]} d_\infty^2\left(\left\langle\int_0^u \varphi^\varepsilon(s)(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X_0^\varepsilon(s),D^\beta X_0^\varepsilon(s)\right)ds\right\rangle,\langle 0\rangle\right)$$
$$\le E\sup_{u\in[0,\nu]} d_H^2\left(\left\{\int_0^u \varphi^\varepsilon(s)(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X_0^\varepsilon(s),D^\beta X_0^\varepsilon(s)\right)ds\right\},0\right)$$
$$\le E\sup_{u\in[0,\nu]}\left(\int_0^u \varphi^\varepsilon(s)(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X_0^\varepsilon(s),D^\beta X_0^\varepsilon(s)\right)ds\right)^2. \tag{31}$$

By applying (7)–(31) and Holder inequality, we have

$$E\sup_{u\in[0,\nu]}\left(\int_0^u \varphi^\varepsilon(s)(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X_0^\varepsilon(s),D^\beta X_0^\varepsilon(s)\right)ds\right)^2$$
$$=E\sup_{u\in[0,\nu]}\left(\int_0^u\left(\int_0^s\left(s-k'+\varepsilon\right)^{\alpha-1}dW\left(k'\right)\right)(\nu-s)^{q-1}P_q(\nu-s)g\right.$$
$$\left.\times\left(s,X_0^\varepsilon(s),D^\beta X_0^\varepsilon(s)\right)ds\right)^2$$
$$=E\sup_{u\in[0,\nu]}\left(\int_0^u\int_0^u(\nu-s)^{q-1}P_q(\nu-s)g\left(k',X_0^\varepsilon\left(k'\right),D^\beta X_0^\varepsilon\right.\right.$$
$$\left.\left.\times\left(\left(k'\right)\right)\right)\left(k'-s+\varepsilon\right)^{\alpha-1}dk'dW(s)\right)^2$$
$$\le 4E\int_0^\nu\left(\int_s^\nu(\nu-s)^{q-1}P_q(\nu-s)g\left(k',X_0^\varepsilon\left(k'\right),D^\beta X_0^\varepsilon\right.\right.$$
$$\left.\left.\times\left(\left(k'\right)\right)\left(k'-s+\varepsilon\right)^{\alpha-1}dk'\right)^2ds$$
$$\le 4E\int_0^\nu\left(\int_s^\nu(\nu-s)^{q-1}P_q(\nu-s)g^2\left(k',X_0^\varepsilon\left(k'\right)\right)\right.$$
$$\left.\times\left(k'-s+\varepsilon\right)^{\alpha-1}dk'\right)\left(\int_s^\nu\left(r'-s+\varepsilon\right)^{\alpha-1}dr'\right)ds$$
$$\le\frac{4}{\alpha^2}(\nu+\varepsilon)^\alpha E\int_0^\nu(\nu-s)^{q-1}P_q(\nu-s)g^2\left(k',X_0^\varepsilon\left(k'\right),D^\beta X_0^\varepsilon\right.$$
$$\times\left(\left(k'\right)\right)\left(k'+\varepsilon\right)^\alpha dk'\le\frac{4}{\alpha^2}(\nu+\varepsilon)^{2\alpha}E\int_0^\nu(\nu-s)^{q-1}P_q(\nu-s)g^2$$
$$\times\left(k',X_0^\varepsilon\left(k'\right),D^\beta X_0^\varepsilon\left(\left(k'\right)\right)dk'\le\frac{8\mathcal{Q}^2}{\alpha^2}(T+\varepsilon)^{2\alpha}\left(1+E\|\|[X_0^\varepsilon]\|\|^2\right)\nu. \tag{32}$$

Hence, from (30) and (32), we obtain

$$j_1(\nu)\le 6\mathcal{Q}^2\left(T+4\varepsilon^{2\alpha}+4(T+\varepsilon)^{2\alpha}\right)\left(1+E\left\|\left|[X_0^\varepsilon]^0\right|\right\|^2\right)\nu\nu, \tag{33}$$

for every $\nu\in I$. Then, similarly,

$$\ell_{n+1}(\nu)\le 3\left(\nu+4\varepsilon^{2\alpha}+4(\nu+\varepsilon)^{2\alpha}\right)L^2E\int_0^\nu d_\infty^2(X_n^\varepsilon(u),X_{n-1}^\varepsilon(u))ds$$
$$\le 3\left(\nu+4\varepsilon^{2\alpha}+4(\nu+\varepsilon)^{2\alpha}\right)L^2\int_0^\nu E\sup_{u\in[0,s]}d_\infty^2(X_n^\varepsilon(u),X_{n-1}^\varepsilon(u))ds$$
$$\le 3\left(\nu+4\varepsilon^{2\alpha}+4(\nu+\varepsilon)^{2\alpha}\right)L^2\int_0^\nu\ell_n(s)ds. \tag{34}$$

Therefore,

$$\ell_n(\nu)\le 2\mathcal{Q}^2 3^n\left(\nu+4\varepsilon^{2\alpha}+4(\nu+\varepsilon)^{2\alpha}\right)^n$$
$$\cdot\left(1+E\left\|\left|[X_0^\varepsilon]^0\right|\right\|^2\right)L^{2(n-1)}\frac{t^n}{n!},\nu\in j,n\in\mathcal{N}. \tag{35}$$

Using Chebyshev inequality, we can determine

$$P\left(\sup_{u\in I}d_\infty^2(X_n^\varepsilon(u),X_{n-1}^\varepsilon(u))>\frac{1}{2^n}\right)\le 2^n\ell_n(\mathfrak{T}). \tag{36}$$

The series $\sum_{n=1}^\infty 2^n\ell_n(T)$ is convergent. Using Borel-Cantelli lemma, we have

$$P\left(\sup_{u\in I}d_\infty^2(X_n^\varepsilon(u),X_{n-1}^\varepsilon(u))>\frac{1}{\left(\sqrt{2}\right)^n}infinitelyoften\right)=0. \tag{37}$$

There is a $n_0(\omega)$ such that for approximately every $\omega\in\Omega$,

$$\sup_{u\in I}d_\infty^2(X_n^\varepsilon(u),\infty)^2(X_{n-1}^\varepsilon(u))\le\frac{1}{\left(\sqrt{2}\right)^n},if n\ge n_0. \tag{38}$$

The sequence $\{X_n^\varepsilon(.,\omega)\}$ is uniformly convergent to d_∞^2 -continuous fuzzy process $X^\varepsilon(.,\omega)$ for every $\omega\in\Omega_\varepsilon$, in which $\Omega_c\in\mathcal{N}$ and $\wp(\Omega_c)=2$. We can define the mapping $X^\varepsilon: j\times\Omega\longrightarrow\mathfrak{F}(\mathbf{R^m})$, as $X^\varepsilon(.,\omega)=X^\varepsilon(.,\omega)$ if $\omega\in\Omega_c$ and $X^\varepsilon(.,\omega)$ as freely chosen fuzzy function when $\omega\in\Omega\,\Omega_c$. For every $\alpha\in[1,2]$ and every $\nu\in j$ with a.e., we have

$$d_H([X_n^\varepsilon(\nu)]^\alpha,[X^\varepsilon(\nu)]^\alpha)\longrightarrow 0asn\longrightarrow\infty. \tag{39}$$

Hence, X^ε will be continuous FSP. Therefore, by $X_n^\varepsilon\in\mathcal{L}^2(j\times\Omega,\mathcal{A};\mathfrak{F}(\mathbf{R^m}))$, we have $X^\varepsilon\in\mathcal{L}^2(j\times\Omega,\mathcal{A};\mathfrak{F}(\mathbf{R^m}))$. Consequently, as n approaches infinity, we can prove

$$E\sup_{\nu\in I}\left[d_\infty^2(X_n^\varepsilon(\nu),X^\varepsilon(\nu))+d_\infty^2\left(X_n^\varepsilon(\nu),\mathcal{Q}_q(\nu)X_0^\varepsilon+\mathcal{G}_q(\nu)X_1^\varepsilon\right.\right.$$
$$+\int_0^\nu(\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f}\left(s,X^\varepsilon(s),D^\beta X^\varepsilon(s)\right)ds$$
$$\left.\left.+\left\langle\int_0^\nu(\nu-s)^{q-1}P_q(\nu-s)g\left(s,X^\varepsilon(s),D^\beta X^\varepsilon(s)\right)d\mathcal{B}_H^\varepsilon(s)\right\rangle\right)\right]^2, \tag{40}$$

tends to zero. Now,

$$\begin{aligned}&E\sup_{\nu\in I} d_{\infty}^{2}\Big[\Big(X^{\varepsilon}(\nu), \mathcal{Q}_q(\nu)X_0^{\varepsilon}+\mathcal{G}_q(\nu)X_1^{\varepsilon}\\&\quad+\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f}\Big(s,X^{\varepsilon}(s),D^{\beta}X^{\varepsilon}(s)\Big)ds\\&\quad+\Big\langle\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)g\Big(s,X^{\varepsilon}(s),D^{\beta}X^{\varepsilon}(s)\Big)d\mathcal{B}_H^{\varepsilon}(s)\Big\rangle\Big)\Big]=0.\end{aligned}\tag{41}$$

Therefore,

$$\begin{aligned}&\sup_{\nu\in I} d_{\infty}^{2}\Big[\Big(X^{\varepsilon}(\nu), \mathcal{Q}_q(\nu)X_0^{\varepsilon}+\mathcal{G}_q(\nu)X_1^{\varepsilon}\\&\quad+\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f}\Big(s,X^{\varepsilon}(s),D^{\beta}X^{\varepsilon}(s)\Big)ds\\&\quad+\Big\langle\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)g\Big(s,X^{\varepsilon}(s),D^{\beta}X^{\varepsilon}(s)\Big)d\mathcal{B}_H^{\varepsilon}(s)\Big\rangle\Big)\Big]=0.\end{aligned}\tag{42}$$

This demonstrates existence of a strong solution.

We suppose that $X^{\varepsilon}, Y^{\varepsilon}: j\times\Omega\longrightarrow\mathfrak{F}(\mathbf{R}^{\mathbf{m}})$ are strong solutions. Let

$$\ell(\nu)=E\sup_{u\in[0,T]} d_{\infty}^{2}(X^{\varepsilon}(u),Y^{\varepsilon}(u)).\tag{43}$$

Then, using computations similar to those used in existing case, we obtain

$$\begin{aligned}\ell(\nu)&\le 3\big(\nu+4\varepsilon^{2\alpha}+4(\nu+\varepsilon)^{2\alpha}\big)L^2E\int_0^{\nu}d_{\infty}^{2}(X^{\varepsilon}(s),Y^{\varepsilon}(s))ds\\&\le 3\big(T+4\varepsilon^{2\alpha}+\alpha^2\big)L^2\int_0^{\nu}\ell(s)ds.\end{aligned}\tag{44}$$

When the Gronwall inequality is applied, $\ell(\nu)=0$ is obtained for $\nu\in j$. Therefore,

$$\sup_{\nu\in j} d_{\infty}^{2}(X^{\varepsilon}(\nu),Y^{\varepsilon}(\nu))=0,\tag{45}$$

which completes the uniqueness proof.

Lemma 13. *For every $\varepsilon>0$ and $1<\alpha<3/2$, we have*

$$\int_s^{\nu}\Big(\Big(k'-s+\varepsilon\Big)^{\alpha-1}-\Big(k'-s\Big)^{\alpha-1}\Big)dk'\le\frac{\alpha+1}{\alpha}\varepsilon^{\alpha}.\tag{46}$$

Proof. To obtain the function $\mathfrak{f}(x)=x^{\alpha-1}$, we use the finite-increments formula:

$$(x+\varepsilon)^{\alpha-1}-x^{\alpha-1}=(\alpha-1)(x+\theta\varepsilon)^{(\alpha-2)}, 1<\theta<2,\tag{47}$$

then

$$\left|k'-s+\varepsilon\Big)^{\alpha-1}-k'-s\Big)^{\alpha-1}\right|\le|\alpha-1|\left|k'-s\right|^{\alpha-2}\varepsilon.\tag{48}$$

Hence,

$$\begin{aligned}&\int_s^{\nu}\left|\Big(k'-s+\varepsilon\Big)^{\alpha-1}-\Big(k'-s\Big)^{\alpha-1}\right|dk'\\&=\int_s^{s+\varepsilon}\left|\Big(k'-s+\varepsilon\Big)^{\alpha-1}-\Big(k'-s\Big)^{\alpha-1}\right|dk'\\&\quad+\int_{s+\varepsilon}^{\nu}\left|\Big(k'-s+\varepsilon\Big)^{\alpha-1}-\Big(k'-s\Big)^{\alpha-1}\right|dk'\\&\le\int_s^{s+\varepsilon}\left|\Big(2\Big(k'-s\Big)\Big)^{\alpha-1}-\Big(k'-s\Big)^{\alpha-1}\right|dk'\\&\quad+|\alpha-1|\varepsilon\int_{s+\varepsilon}^{\nu}\left|k'-s\right|^{\alpha-2}dk'.\end{aligned}\tag{49}$$

Therefore,

$$\begin{aligned}&\int_s^{\nu}\left|\Big(k'-s+\varepsilon\Big)^{\alpha-1}-\Big(k'-s\Big)^{\alpha-1}\right|dk'\le\int_s^{s+\varepsilon}\Big(k'-s\Big)^{\alpha-1}dk'\\&\quad+|\alpha-1|\varepsilon\int_{s+\varepsilon}^{\nu}\Big(k'-s\Big)^{\alpha-2}dk'\le\frac{1}{\alpha}\varepsilon^{\alpha}+|\alpha-1|\varepsilon\Big(\frac{1}{\alpha-1}\varepsilon^{\alpha-1}\Big)\\&\quad=\frac{\alpha+1}{\alpha}\varepsilon^{\alpha}.\end{aligned}\tag{50}$$

□

Proposition 14. *In $\mathcal{L}^2(j\times\Omega)$, the solution $X^{\varepsilon}(\nu)$ of Equation (22) converges to solution $X(\nu)$ of Equation (20) as $\varepsilon\longrightarrow 0$ uniformly with regard to $\nu\in[0,\mathfrak{T}]$.*

Proof. Assume the following approximation of equations based on Equation (20):

$$\begin{aligned}X(\nu)&=\mathcal{Q}_q(\nu)(X_0-m(X))+G_q(\nu)X_1\\&\quad+\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f}\Big(s,X(s),D^{\beta}X(s)\Big)ds\\&\quad+\Big\langle\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)g\Big(s,X(s),D^{\beta}X(s)\Big)d\mathcal{B}_H(s)\Big\rangle,\\X^{\varepsilon}(\nu)&=\mathcal{Q}_q(\nu)(X_0-m(X))+\mathcal{G}_q(\nu)X_1\\&\quad+\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f}\Big(s,X^{\varepsilon}(s),D^{\beta}X^{\varepsilon}(s)\Big)ds\\&\quad+\Big\langle\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)g\Big(s,X^{\varepsilon}(s),D^{\beta}X^{\varepsilon}(s)\Big)d\mathcal{B}_H^{\varepsilon}(s)\Big\rangle.\end{aligned}\tag{51}$$

□

We can write

$$\begin{aligned} E \sup_{u\in[0,\nu]} d_{\infty}^{2}(X(u), X^{\varepsilon}(u)) \leq{}& 2E \sup_{u\in[0,\nu]} \\ &\cdot \int_0^u d_{\infty}^{2}\left(\mathfrak{f}\left(s, X(s), D^{\beta}X(s)\right), \mathfrak{f}\left(s, X^{\varepsilon}(s), D^{\beta}X^{\varepsilon}(s)\right)\right)ds \\ &+ 2E \sup_{u\in[0,\nu]} d_{\infty}^{2}\left(\left\langle \int_0^{\infty}(\nu-s)^{q-1}P_q(\nu-s)g\right.\right. \\ &\cdot \left(s, X(s), D^{\beta}X(s)\right)d\mathscr{B}_H(s)\Big\rangle, \Big\langle \int_0^u(\nu-s)^{q-1}P_q(\nu-s)g \\ &\cdot \left.\left(s, X^{\varepsilon}(s), D^{\beta}X^{\varepsilon}(s)\right)d\mathscr{B}_H^{\varepsilon}(s)\right\rangle). \end{aligned} \tag{52}$$

Then

$$\begin{aligned} E \sup_{u\in[0,\nu]} d_{\infty}^{2}(X(u), X^{\varepsilon}(u)) \leq{}& 2E \sup_{u\in[0,\nu]} \\ &\cdot \int_0^u d_{\infty}^{2}\left(\mathfrak{f}\left(s, X(s), D^{\beta}X(s)\right), \mathfrak{f}\left(s, X^{\varepsilon}(s), D^{\beta}X^{\varepsilon}(s)\right)\right)ds \\ &+ 4E \sup_{u\in[0,\nu]} d_{\infty}^{2}\left(\left\langle \int_0^{u}(\nu-s)^{q-1}P_q(\nu-s)g\right.\right. \\ &\cdot \left(s, X(s), D^{\beta}X^{\varepsilon}(s)\right)\Big)d\mathscr{B}_H(s)\rangle, \Big\langle \int_0^u(\nu-s)^{q-1}P_q(\nu-s)g \\ &\cdot \left(s, X(s), D^{\beta}X^{\varepsilon}(s)\right)\Big)d\mathscr{B}_H^{\varepsilon}(s)\rangle\rangle) + 4E \sup_{u\in[0,\nu]} d_{\infty}^{2} \\ &\cdot \left(\left\langle \int_0^{u}(\nu-s)^{q-1}P_q(\nu-s)g\left(s, X(s), D^{\beta}X^{\varepsilon}(s)\right)\right)d\mathscr{B}_H^{\varepsilon}(s)\right\rangle, \\ &\cdot \left\langle \int_0^u\left(\nu-s\right)^{q-1}P_q(\nu-s)g\left(s, X^{\varepsilon}(s), D^{\beta}X^{\varepsilon}(s)\right)\Big)d\mathscr{B}_H^{\varepsilon}(s)\right\rangle\Big) \\ ={}& 2E\int_0^{\nu} d_{\infty}^{2}\left(\mathfrak{f}\left(s, X(s), D^{\beta}X(s)\right), \mathfrak{f}(s, X^{\varepsilon}(s)), D^{\beta}X^{\varepsilon}(s)\right)ds \\ &+ 4E \sup_{u\in[0,\mathfrak{T}]}\left|\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)g\left(s, X^{\varepsilon}(s)D^{\beta}X^{\varepsilon}(s)\right)(d\mathscr{B}_H^{\varepsilon}(s)\right. \\ &- d\mathscr{B}_H(s))|^2 + 4E \sup_{u\in[0,\nu]}\left|\int_0^{\nu} g\left(s, X(s)D^{\beta}X(s)\right)\right. \\ &\left.- g\left(s, X^{\varepsilon}(s)D^{\beta}X^{\varepsilon}(s)\right)d\mathscr{B}_H^{\varepsilon}(s)\right|^2. \end{aligned} \tag{53}$$

To get the solution, consider Equation (6):

$$\begin{aligned} E \sup_{u\in[0,\nu]} d_{\infty}^{2}(X(u), X^{\varepsilon}(u)) \leq{}& 2E\int_0^{\nu} d_{\infty}^{2}\left(\mathfrak{f}\left(s, X(s), D^{\beta}X(s)\right), \mathfrak{f}\right. \\ &\cdot \left.\left(s, X^{\varepsilon}(s), D^{\beta}X^{\varepsilon}(s)\right)\right)ds + 8\varepsilon^{2\alpha}E \sup_{u\in[0,\nu]}\left|\int_0^u(\nu-s)^{q-1}P_q(\nu-s)g\right. \\ &\cdot \left.\left(s, X^{\varepsilon}(s), D^{\beta}X^{\varepsilon}(s)\right)dW(s)\right|^2 + 8\alpha^2E \sup_{u\in[0,\nu]}\left|\int_0^u(\nu-s)^{q-1}P_q(\nu-s)g\right. \\ &\cdot \left(s, X^{\varepsilon}(s), D^{\beta}X^{\varepsilon}(s)\right)\left(\int_0^s\left(s-k'+\varepsilon\right)^{\alpha-1}\right. \\ &- \left.\left(s-k'\right)^{\alpha-1}dW\left(k'\right)\right)ds\Big|^2 + 8\alpha^2E \sup_{u\in[0,\nu]}\left|\int_0^u\left(g\left(s, X(s), D^{\beta}X^{\varepsilon}(s)\right)\right.\right. \\ &- g(s, X^{\varepsilon}(s)))\left(\int_0^s\left(s-k'+\varepsilon\right)^{\alpha-1}dW\left(k'\right)\right)ds\Big|^2 \\ &+ 8\varepsilon^{2\alpha}E \sup_{u\in[0,\nu]}\left|\int_0^u\left(g\left(s, X(s), D^{\beta}X(s)\right) - g\left(s, X^{\varepsilon}(s), D^{\beta}X^{\varepsilon}(s)\right)\right)dW(s)\right|^2. \end{aligned} \tag{54}$$

Then,

$$\begin{aligned} E \sup_{u\in[0,\nu]} d_{\infty}^{2}(X(u), X^{\varepsilon}(u)) \leq{}& 2E\int_0^{\nu} d_{\infty}^{2}\left(\mathfrak{f}\left(s, X(s), D^{\beta}X(s)\right), \mathfrak{f}\right. \\ &\cdot \left.\left(s, X^{\varepsilon}(s), D^{\beta}X^{\varepsilon}(s)\right)\right)ds + 8\varepsilon^{2\alpha}E, \end{aligned}$$

$$\sup_{u\in[0,\nu]}\left|\int_0^u(\nu-s)^{q-1}P_q(\nu-s)g\left(s, X^{\varepsilon}(s), D^{\beta}X^{\varepsilon}(s)\right)dW(s)\right|^2 + 8\alpha^2E,$$

$$\begin{aligned} &\sup_{u\in[0,\nu]}\left|\int_0^u\int_s^u(\nu-s)^{q-1}P_q(\nu-s)g\left(k', X^{\varepsilon}\left(k'\right), D^{\beta}X\right.\right. \\ &\cdot \left.\left({}^{\varepsilon}\left(k'\right)\left(k'-s+\varepsilon\right)^{\alpha-1} - \left(s-k'\right)^{\alpha-1}dk'dW(s)\right|^2 \\ &+ 8\alpha^2E \sup_{u\in[0,\nu]}\left|\int_0^u\left(g\left(k', X\left(k'\right)\right) - g\left(s, X^{\varepsilon}(s), D^{\beta}X({}^{\varepsilon}(s))\right.\right.\right. \\ &\cdot \left.\left(k'-s+\varepsilon\right)^{\alpha-1}dk'dW(s)\right|^2 + 8\varepsilon^{2\alpha}E \sup_{u\in[0,\nu]}| \\ &\cdot \left.\int_0^u\left(g\left(s, X(s), D^{\beta}X(s)\right) - g\left(s, X^{\varepsilon}(s), D^{\beta}X^{\varepsilon}(s)\right)\right)dW(s)\right|^2. \end{aligned} \tag{55}$$

Apply the Holder inequality, Doob inequality, and Itô isometry property to gain

$$\begin{aligned} E \sup_{u\in[0,\nu]} d_{\infty}^{2}(X(u), X^{\varepsilon}(u)) \leq{}& 2E\int_0^{\nu} d_{\infty}^{2}\left(\mathfrak{f}\left(s, X(s), D^{\beta}X(s)\right), \mathfrak{f}\right. \\ &\cdot \left.\left(s, X^{\varepsilon}(s), D^{\beta}X^{\varepsilon}(s)\right)\right)ds + 32\varepsilon^{2\alpha}\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)g^2 \\ &\cdot \left(s, X^{\varepsilon}(s), D^{\beta}X(s)\right)ds + 32\alpha^2E\int_0^{\nu}\left[\int_s^{\nu}(\nu-s)^{q-1}P_q(\nu-s)g^2\right. \\ &\cdot \left(k', X^{\varepsilon}\left(k'\right), D^{\beta}X^{\varepsilon}\left(k'\right)\right)\left(\left(k'-s+\varepsilon\right)^{\alpha-1}\right. \\ &- \left.\left(k'-s\right)^{\alpha-1}\right)dk' \times \int_s^{\nu}\left(\left(k'-s+\varepsilon\right)^{\alpha-1}\right. \\ &- \left.\left(k'-s\right)^{\alpha-1}\right)dk'\Big]ds + 32\alpha^2E\int_0^{\nu}\left(\int_s^{\nu}\left(g\left(k', X\left(k'\right)\right)\right.\right. \\ &- \left.g\left(s, X^{\varepsilon}(s), D^{\beta}X^{\varepsilon}(s)\right)^2\left(k'-s+\varepsilon\right)^{\alpha-1}dk'\right) \\ &\times \left(\int_s^{\nu}\left(k'-s+\varepsilon\right)^{\alpha-1}dk'\right)ds + 32\varepsilon^{2\alpha}E\int_0^{\nu}\left(g\left(s, X(s), D^{\beta}X(s)\right)\right. \\ &- \left.g\left(s, X^{\varepsilon}(s), D^{\beta}X^{\varepsilon}(s)\right)\right)^2ds. \end{aligned} \tag{56}$$

We conclude that using similar arguments to (32), from

(46), and assumptions $(A1)-(A3)$:

$$\begin{aligned}&E\sup_{u\in[0,\nu]}d_{\infty}^{2}(X(u),X^{\varepsilon}(u))\\&\leq 2L^{2}E\int_{0}^{\nu}d_{\infty}^{2}(X(s),X^{\varepsilon}(s))ds+64\varepsilon^{2\alpha}Q^{2}\int_{0}^{\nu}\left(1+E\left|\left\|[X^{\varepsilon}(s)]^{0}\right)ds\\&\quad+64\mathcal{Q}^{2}(T+\varepsilon)^{2\alpha}\frac{\alpha+1}{\alpha}\varepsilon^{\alpha}\int_{0}^{\nu}\left(1+E\left|\left\|[X^{\varepsilon}(s)]^{0}\right\|\right|^{2}\right)ds\\&\quad+32L^{2}(T+\varepsilon)^{2\alpha}E\int_{0}^{\nu}d_{\infty}^{2}(X(s),X^{\varepsilon}(s))ds\\&\quad+32L^{2}\varepsilon^{2\alpha}E\int_{0}^{\nu}d_{\infty}^{2}(X(s),X^{\varepsilon}(s))ds.\end{aligned}\tag{57}$$

Hence,

$$\begin{aligned}&E\sup_{u\in[0,\nu]}d_{\infty}^{2}(X(u),X^{\varepsilon}(u))\\&\leq\left(2L^{2}+32L^{2}(T+\varepsilon)^{2\alpha}+32L^{2}\varepsilon^{2\alpha}\right)\int_{0}^{\nu}E\sup_{u\in[0,s]}d_{\infty}^{2}(X(u),X^{\varepsilon}(u))ds\\&\quad+\left(64\varepsilon^{2\alpha}\mathcal{Q}^{2}+64\mathcal{Q}^{2}(T+\varepsilon)^{2\alpha}\frac{\alpha+1}{\alpha}\varepsilon^{\alpha}\right)\int_{0}^{\nu}\left(1+E\left|\left\|[X^{\varepsilon}(s)]^{2}\right\|\right|^{2}\right)ds.\end{aligned}\tag{58}$$

The proof is completed by applying Gronwall's lemma to $E\sup_{u\in[0,\nu]}d_{\infty}^{2}(X(u),X^{\varepsilon}(u))\longrightarrow 0$ as $\varepsilon\longrightarrow 0$.

3.1. Example in Finance. The preceding crisp SFDE is often used in financial modeling:

$$\begin{aligned}X(\nu)&=\mathcal{Q}_{q}(\nu)(X_{0}-m(X))+\mathcal{G}_{q}(\nu)X_{1}\\&\quad+\int_{0}^{\nu}\mu(\nu-s)^{q-1}P_{q}(\nu-s)X(s)ds\\&\quad+\left\langle\int_{0}^{\nu}\sigma(\nu-s)^{q-1}P_{q}(\nu-s)X(\nu)d\mathcal{B}_{H}(s)\right\rangle,\end{aligned}\tag{59}$$

where FBM is underlying stochastic process. Because of FBM's long-range dependence and self-similarity property, it is a good fit for describing financial quantities. On the other hand, the equation that includes the uncertainties can be used to model the price dynamics. As a result, fuzzy processes in equations are used to model. When dealing with linear coefficients, we get an explicit solution to Equation (20). Suppose FSDE that meet Theorem 12's assumptions as follows:

$$\begin{aligned}X(\nu)&=\mathcal{Q}_{q}(\nu)(X_{0}-m(X))+\mathcal{G}_{q}(\nu)X_{1}\\&\quad+\int_{0}^{\nu}\mu(\nu-s)^{q-1}P_{q}(\nu-s)X(s)ds\\&\quad+\left\langle\int_{0}^{\nu}\frac{\sigma}{2}(\nu-s)^{q-1}P_{q}(\nu-s)\left(X_{l}^{1}(s)+X_{u}^{1}(s)\right)d\mathcal{B}_{H}(s)\right\rangle,\end{aligned}\tag{60}$$

$X:\mathbf{R^{m}}_{+}\times\Omega\longrightarrow\mathfrak{F}(\mathbf{R^{m}})$, $\mathcal{B}_{H}$ is an FBM, $X_{l}^{1},X_{u}^{1}:\mathbf{R^{m}}_{+}\times\Omega\longrightarrow\mathbf{R^{m}}$ that is $[X(\nu)]^{1}=[X_{l}^{1}(\nu),X_{u}^{1}(\nu)]$, $X_{0}\in\mathcal{L}^{2}(\Omega,\mathcal{N}_{0},\wp;\mathfrak{F}(\mathbf{R^{m}}))$, and $\mu,\sigma\in\mathbf{R^{m}}$. To find the closed explicit form of the solution to $\mu\geq 0$, we need to solve below system of equations (60).

$$\begin{aligned}X_{l}^{1}(\nu)&=\mathcal{Q}_{q}(\nu)X_{l}^{1}(0)+\mathcal{G}_{q}(\nu)X_{l}'1(0)\\&\quad+\int_{0}^{\nu}\mu(\nu-s)^{q-1}P_{q}(\nu-s)X_{l}^{1}(s)ds\\&\quad+\left\langle\int_{0}^{\nu}\frac{\sigma}{2}(\nu-s)^{q-1}P_{q}(\nu-s)\left(X_{l}^{1}(s)+X_{u}^{1}(s)\right)d\mathcal{B}_{H}(s)\right\rangle,\\X_{u}^{1}(\nu)&=\mathcal{Q}_{q}(\nu)X_{u}^{1}(0)+\mathcal{G}_{q}(\nu)X_{u}'1(0)\\&\quad+\int_{0}^{\nu}\mu(\nu-s)^{q-1}P_{q}(\nu-s)X_{u}^{1}(s)ds\\&\quad+\left\langle\int_{0}^{\nu}\frac{\sigma}{2}(\nu-s)^{q-1}P_{q}(\nu-s)\left(X_{l}^{1}(s)+X_{u}^{1}(s)\right)d\mathcal{B}_{H}(s)\right\rangle,\end{aligned}\tag{61}$$

then,

$$\begin{aligned}X_{l}^{1}(\nu)+X_{u}^{1}(\nu)&=\mathcal{Q}_{q}(\nu)X_{l}^{1}(0)+\mathcal{G}_{q}(\nu)X_{l}'1(0)+\mathcal{Q}_{q}(\nu)X_{u}^{1}(0)\\&\quad+\mathcal{G}_{q}(\nu)X_{u}'1(0)+\int_{0}^{\nu}\mu(\nu-s)^{q-1}P_{q}(\nu-s)\left(X_{l}^{1}(s)+X_{u}^{1}\right)ds\\&\quad+\left\langle\int_{0}^{\nu}\sigma(\nu-s)^{q-1}P_{q}(\nu-s)\left(X_{l}^{1}(s)+X_{u}^{1}(s)\right)d\mathcal{B}_{H}(s)\right\rangle.\end{aligned}\tag{62}$$

The approximation solution of Equation (62) given by Equation (6) is

$$\begin{aligned}X_{l}^{\varepsilon1}(\nu)+X_{u}^{\varepsilon1}(\nu)&=\mathcal{Q}_{q}(\nu)X_{l}^{\varepsilon1}(0)+\mathcal{G}_{q}(\nu)X_{l}'\varepsilon1(0)\\&\quad+\mathcal{Q}_{q}(\nu)X_{u}^{\varepsilon1}(0)+\mathcal{G}_{q}(\nu)X_{u}'\varepsilon1(0)\\&\quad+\int_{0}^{\nu}\left(\mu+\sigma\alpha\varphi^{\varepsilon}(s)(\nu-s)^{q-1}P_{q}(\nu-s)\left(X_{l}^{\varepsilon1}(s)+X_{u}^{\varepsilon1}\right)\right)ds\\&\quad+\int_{0}^{\nu}\sigma\varepsilon^{\alpha}(\nu-s)^{q-1}P_{q}(\nu-s)\left(X_{l}^{\varepsilon1}(s)+X_{u}^{\varepsilon1}(s)\right)dW(s).\end{aligned}\tag{63}$$

Therefore, one can derive a unique solution from explicit solution of crisp linear SDEs.

$$\begin{aligned}X_{l}^{\varepsilon1}(\nu)+X_{u}^{\varepsilon1}(\nu)&=\Big(\mathcal{Q}_{q}(\nu)X_{l}^{\varepsilon1}(0)+\mathcal{G}_{q}(\nu)X_{l}'\varepsilon1(0)\\&\quad+\mathcal{Q}_{q}(\nu)X_{u}^{\varepsilon1}(0)+\mathcal{G}_{q}(\nu)X_{u}'\varepsilon1(0)\Big)\exp\Big(\mu\nu+\sigma\alpha\int_{0}^{\nu}\varphi^{\varepsilon}(s)ds\\&\quad-\frac{1}{2}\sigma^{2}\varepsilon^{2\alpha}\nu+\sigma\varepsilon^{\alpha}W(\nu)\Big)=\left(\mathcal{Q}_{q}(\nu)\left(X_{l}^{\varepsilon1}(0)+mX_{l}^{\varepsilon1}\right)(0)\right.\\&\quad+\mathcal{G}_{q}(\nu)X_{l}'\varepsilon1(0)+\mathcal{Q}_{q}(\nu)X_{u}^{\varepsilon1}(0)\\&\quad\left.+\mathcal{G}_{q}(\nu)X_{u}'\varepsilon1(0)\right)\exp\left(\mu\nu+\sigma B_{H}^{\varepsilon}(\nu)-\frac{1}{2}\sigma^{2}\varepsilon^{2\alpha}\nu\right).\end{aligned}\tag{64}$$

Now, we apply a similar method to each $\alpha \in [1, 2]$ to generate the following system:

$$X_l^{\varepsilon\alpha}(\nu) = \mathcal{Q}_q(\nu)X_l^{\varepsilon\alpha}(0) + \mathcal{G}_q(\nu)X_l'\varepsilon\alpha(0) + \int_0^\nu \mu(\nu - s)^{q-1}P_q(\nu - s)X_l^{\varepsilon\alpha}(s)ds + \int_0^\nu \frac{\sigma}{2}(\nu - s)^{q-1}P_q(\nu - s)\left(X_l^{\varepsilon 1}(s) + X_u^{\varepsilon 1}(s)\right)d\mathcal{B}_H^{\varepsilon}(s),$$
$$X_u^{\varepsilon\alpha}(\nu) = \mathcal{Q}_q(\nu)X_l^{\varepsilon\alpha}(0) + \mathcal{G}_q(\nu)X_l'\varepsilon\alpha(0) + \int_0^\nu \mu(\nu - s)^{q-1}P_q(\nu - s)X_u^{\varepsilon\alpha}(s)ds + \int_0^\nu \frac{\sigma}{2}(\nu - s)^{q-1}P_q(\nu - s)\left(X_l^{\varepsilon 1}(s) + X_u^{\varepsilon 1}(s)\right)d\mathcal{B}_H^{\varepsilon}(s). \tag{65}$$

We use solution (64) to get below system for $\mu \geq 0$:

$$X_l^{\varepsilon\alpha}(\nu) = \mathcal{Q}_q(\nu)X_l^{\varepsilon\alpha}(0) + \mathcal{G}_q(\nu)X_l'\varepsilon\alpha(0) + \int_0^\nu \mu(\nu - s)^{q-1}P_q(\nu - s)X_l^{\varepsilon\alpha}(s)ds + \int_0^\nu \frac{\sigma}{2}(\nu - s)^{q-1}P_q(\nu - s)\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)\right)\exp\left(\mu s + \sigma B_H^{\varepsilon}(s) - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s\right)d\mathcal{B}_H^{\varepsilon}(s),$$

$$X_u^{\varepsilon\alpha}(\nu) = \mathcal{Q}_q(\nu)X_u^{\varepsilon\alpha}(0) + \mathcal{G}_q(\nu)X_u'\varepsilon\alpha(0) + \int_0^\nu \mu(\nu - s)^{q-1}P_q(\nu - s)X_u^{\varepsilon\alpha}(s)ds + \int_0^\nu \frac{\sigma}{2}(\nu - s)^{q-1}P_q(\nu - s)\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)\right)\exp\left(\mu s + \sigma B_H^{\varepsilon}(s) - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s\right)d\mathcal{B}_H^{\varepsilon}(s). \tag{66}$$

Alternatively, we have W in terms of the Wiener process:

$$X_l^{\varepsilon\alpha}(\nu) = \mathcal{Q}_q(\nu)X_l^{\varepsilon\alpha}(0) + \mathcal{G}_q(\nu)X_l^{\varepsilon\alpha}(0) + \int_0^\nu \mu X_l^{\varepsilon\alpha}(s) + \alpha\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}\right) \times \int_0^\nu \varphi^{\varepsilon}(s)\exp\left(\mu s + \sigma\alpha\int_0^s \varphi^{\varepsilon}(u)du - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s + \sigma\varepsilon^{\alpha}W(s)\right)ds + \varepsilon^{\alpha}\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)\right) \times \int_0^\nu \exp\left(\mu s + \sigma\alpha\int_0^s \varphi^{\varepsilon}(u)du - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s + \sigma\varepsilon^{\alpha}W(s)\right)dW(s), \tag{67}$$

$$X_u^{\varepsilon\alpha}(\nu) = \mathcal{Q}_q(\nu)X_u^{\varepsilon\alpha}(0) + \mathcal{G}_q(\nu)X_u^{\varepsilon\alpha}(0) + \int_0^\nu \mu X_u^{\varepsilon\alpha}(s) + \alpha\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}\right) \times \int_0^\nu \varphi^{\varepsilon}(s)\exp \cdot \left(\mu s + \sigma\alpha\int_0^s \varphi^{\varepsilon}(u)du - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s + \sigma\varepsilon^{\alpha}W(s)\right)ds + \varepsilon^{\alpha}\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)\right) \times \int_0^\nu \exp\left(\mu s + \sigma\alpha\int_0^s \varphi^{\varepsilon}(u)du - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s + \sigma\varepsilon^{\alpha}W(s)\right)dW(s). \tag{68}$$

Use Theorem 12 in [35] to find unique solution to (67) in the following format:

$$X_l^{\varepsilon\alpha}(\nu) = e^{\mu\nu}\mathcal{Q}_q(\nu)X_l^{\varepsilon\alpha}(0) + e^{\mu\nu}\mathcal{G}_q(\nu)X_l^{\varepsilon\alpha}(0) + e^{\mu\nu}\alpha\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}\right) \times \int_0^\nu \varphi^{\varepsilon}(s)\exp\left(\mu s + \sigma\alpha\int_0^s \varphi^{\varepsilon}(u)du - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s + \sigma\varepsilon^{\alpha}W(s)\right)ds + e^{\mu\nu}\varepsilon^{\alpha}\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)\right) \times \int_0^\nu \exp\left(\mu s + \sigma\alpha\int_0^s \varphi^{\varepsilon}(u)du - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s + \sigma\varepsilon^{\alpha}W(s)\right)dW(s),$$

$$X_u^{\varepsilon\alpha}(\nu) = e^{\mu\nu}\mathcal{Q}_q(\nu)X_l^{\varepsilon\alpha}(0) + e^{\mu\nu}\mathcal{G}_q(\nu)X_l^{\varepsilon\alpha}(0) + e^{\mu\nu}\alpha\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}\right) \times \int_0^\nu \varphi^{\varepsilon}(s)\exp\left(\mu s + \sigma\alpha\int_0^s \varphi^{\varepsilon}(u)du - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s + \sigma\varepsilon^{\alpha}W(s)\right)ds + e^{\mu\nu}\varepsilon^{\alpha}\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)\right) \times \int_0^\nu \exp\left(\mu s + \sigma\alpha\int_0^s \varphi^{\varepsilon}(u)du - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s + \sigma\varepsilon^{\alpha}W(s)\right)dW(s). \tag{69}$$

Then,

$$X_l^{\varepsilon\alpha}(\nu) = e^{\mu\nu}\Big[\mathcal{Q}_q(\nu)X_l^{\varepsilon\alpha}(0) + \mathcal{G}_q(\nu)X_l'\varepsilon\alpha(0) + \frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)\right) \times \int_0^\nu \exp\left(\sigma\mathcal{B}_H^{\varepsilon}(s) - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s\right)d\mathcal{B}_H^{\varepsilon}(s)\Big],$$

$$X_u^{\varepsilon\alpha}(\nu) = e^{\mu\nu}\Big[\mathcal{Q}_q(\nu)X_u^{\varepsilon\alpha}(0) + \mathcal{G}_q(\nu)X_u'\varepsilon\alpha(0) + \frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0) + X_u^{\varepsilon 1}(0)\right) \times \int_0^\nu \exp\left(\sigma\mathcal{B}_H^{\varepsilon}(s) - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s\right)d\mathcal{B}_H^{\varepsilon}(s)\Big]. \tag{70}$$

As a result, the fuzzy approximation solution for $\mu \geq 0$ is

$$X^{\varepsilon}(\nu) = \mathcal{Q}_q(\nu)e^{\mu\nu}\Big(X(0) + \mathcal{G}_q(\nu)e^{\mu\nu}X'(0) + \left\langle\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0) + X_u^{\mu\nu}(0)\right)e^{\mu\nu}\int_0^\nu \exp \cdot \left(\sigma B_H^{\varepsilon}(s) - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s\right)d\mathcal{B}_H^{\varepsilon}(s)\right\rangle. \tag{71}$$

For $\mu < 0$, we can demonstrate that

$$\begin{aligned} X_l^{\varepsilon\alpha}(\nu) &= \mathcal{Q}_q(\nu)X_l^{\varepsilon\alpha}(0) + \mathcal{G}_q(\nu)X_l'\varepsilon\alpha(0) + \int_0^\nu \mu(\nu-s)^{q-1}P_q(\nu-s)X_u^{\varepsilon\alpha}(s)ds \\ &+ \alpha\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0)+X_u^{\varepsilon 1}(0)\right) \times \int_0^\nu \varphi^\varepsilon(s)\exp \\ &\cdot \left(\mu s + \sigma\alpha\int_0^s \varphi^\varepsilon(u)du - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s + \sigma\varepsilon^\alpha W(s)\right)ds \\ &+ \varepsilon^\alpha\frac{\alpha}{2}\left(X_l^{\varepsilon 1}(0)+X_u^{\varepsilon 1}(0)\right)\times, \end{aligned} \tag{72}$$

$$\begin{aligned} X_u^{\varepsilon\alpha}(\nu) &= \mathcal{Q}_q(\nu)X_u^{\varepsilon\alpha}(0) + \mathcal{G}_q(\nu)X_u'\varepsilon\alpha(0) \\ &+ \int_0^\nu \mu(\nu-s)^{q-1}P_q(\nu-s)X_l^{\varepsilon\alpha}(s)ds + \alpha\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0)+X_u^{\varepsilon 1}(0)\right) \\ &\times \int_0^\nu \varphi^\varepsilon(s)\exp\left(\mu s + \sigma\alpha\int_0^s\varphi^\varepsilon(u)du - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s + \sigma\varepsilon^\alpha W(s)\right)ds \\ &+ \varepsilon^\alpha\frac{\alpha}{2}\left(X_l^{\varepsilon 1}(0)+X_u^{\varepsilon 1}(0)\right) \\ &\times \int_0^\nu \exp\left(\mu s + \sigma\alpha\int_0^s\varphi^\varepsilon(u)du - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s + \sigma\varepsilon^\alpha W(s)\right)dW(s). \end{aligned} \tag{73}$$

The following matrix represents the unique solution to (72):

$$\begin{aligned} X_l^{\varepsilon\alpha}(\nu) &= \mathcal{Q}_q(\nu)X_l^{\varepsilon\alpha}(0)\cosh(\mu\nu) + \mathcal{G}_q(\nu)X_l'\varepsilon\alpha(0)\cosh(\mu\nu) \\ &+ \mathcal{Q}_q(\nu)X_u^{\varepsilon\alpha}(0)\sinh(\mu\nu) + \mathcal{G}_q(\nu)X_u'\varepsilon\alpha(0)\sinh(\mu\nu) \\ &+ e^{\mu\nu}\alpha\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0)+X_u^{\varepsilon 1}(0)\right) \\ &\times \int_0^\nu \varphi^\varepsilon(s)\exp\left(\sigma\alpha\int_0^s\varphi^\varepsilon(u)du - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s + \sigma\varepsilon^\alpha W(s)\right)ds \\ &+ e^{\mu\nu}\varepsilon^\alpha\frac{\alpha}{2}\left(X_l^{\varepsilon 1}(0)+X_u^{\varepsilon 1}(0)\right) \\ &\times \int_0^\nu \exp\left(\sigma\alpha\int_0^s\varphi^\varepsilon(u)du - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s + \sigma\varepsilon^\alpha W(s)\right)dW(s), \end{aligned}$$

$$\begin{aligned} X_u^{\varepsilon\alpha}(\nu) &= \mathcal{Q}_q(\nu)X_l^{\varepsilon\alpha}(0)\cosh(\mu\nu) + \mathcal{G}_q(\nu)X_l'\varepsilon\alpha(0)\cosh(\mu\nu) \\ &+ \mathcal{Q}_q(\nu)X_u^{\varepsilon\alpha}(0)\sinh(\mu\nu) + \mathcal{G}_q(\nu)X_u'\varepsilon\alpha(0)\sinh(\mu\nu) \\ &+ e^{\mu\nu}\alpha\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0)+X_u^{\varepsilon 1}(0)\right) \\ &\times \int_0^\nu \varphi^\varepsilon(s)\exp\left(\sigma\alpha\int_0^s\varphi^\varepsilon(u)du - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s + \sigma\varepsilon^\alpha W(s)\right)ds \\ &+ e^{\mu\nu}\varepsilon^\alpha\frac{\alpha}{2}\left(X_l^{\varepsilon 1}(0)+X_u^{\varepsilon 1}(0)\right) \\ &\times \int_0^\nu \exp\left(\sigma\alpha\int_0^s\varphi^\varepsilon(u)du - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s + \sigma\varepsilon^\alpha W(s)\right)dW(s). \end{aligned} \tag{74}$$

Then,

$$\begin{aligned} X_l^{\varepsilon\alpha}(\nu) &= \mathcal{Q}_q(\nu)X_l^{\varepsilon\alpha}(0)\cosh(\mu\nu) + \mathcal{G}_q(\nu)X_l'\varepsilon\alpha(0)\cosh(\mu\nu) \\ &+ \mathcal{Q}_q(\nu)X_u^{\varepsilon\alpha}(0)\sinh(\mu\nu) + \mathcal{G}_q(\nu)X_u'\varepsilon\alpha(0)\sinh(\mu\nu) \\ &+ e^{\mu\nu}\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0)+X_u^{\varepsilon 1}(0)\right)\int_0^\nu \exp\left(\sigma\mathcal{B}_H^\varepsilon(s) - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s\right)dB_H^\varepsilon(s), \end{aligned}$$

$$\begin{aligned} X_u^{\varepsilon\alpha}(\nu) &= \mathcal{Q}_q(\nu)X_l^{\varepsilon\alpha}(0)\cosh(\mu\nu) + \mathcal{G}_q(\nu)X_l'\varepsilon\alpha(0)\cosh(\mu\nu) \\ &+ \mathcal{Q}_q(\nu)X_u^{\varepsilon\alpha}(0)\sinh(\mu\nu) + \mathcal{G}_q(\nu)X_u'\varepsilon\alpha(0)\sinh(\mu\nu) \\ &+ e^{\mu\nu}\frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0)+X_u^{\varepsilon 1}(0)\right)\int_0^\nu \exp\left(\sigma\mathcal{B}_H^\varepsilon(s) - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s\right)d\mathcal{B}_H^\varepsilon(s). \end{aligned} \tag{75}$$

As a result, the fuzzy approximation solution for $\mu < 0$ is

$$\begin{aligned} X^\varepsilon(\nu) &= \mathcal{Q}_q(\nu)X(0)\cosh(\mu\nu) + \mathcal{G}_q(\nu)X'(0)\cosh(\mu\nu) \\ &+ \mathcal{Q}_q(\nu)X(0)\sinh(\mu\nu) + \mathcal{G}_q(\nu)X'(0)\sinh(\mu\nu) \\ &+ \left\langle \frac{\sigma}{2}\left(X_l^{\varepsilon 1}(0)+X_u^{\varepsilon 1}(0)\right)e^{\mu\nu}\int_0^\nu \exp\left(\sigma\mathcal{B}_H^\varepsilon(s) - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}s\right)d\mathcal{B}_H^\varepsilon(s)\right\rangle. \end{aligned} \tag{76}$$

4. Application to a Model of Population Dynamics

Consider a population of a particular species that lives on a specific piece of land. The number of individuals in the underlying population at the instant ν is denoted by $X(\nu)$. The Mslthus differential equation describes a conventional, crisp, deterministic model of population evolution:

$$ {}_0^cD_\nu^\alpha X(\nu) = (\mathcal{N} - L)X(\nu) + \mathfrak{f}\left(\nu, X(\nu), D^\beta X(\nu)\right)ds, \nu \in [0, \mathfrak{T}], \tag{77}$$

$$X(0) + m(X) = X_0, \tag{78}$$

$$X'(0) = X_1. \tag{79}$$

The reproduction and mortality coefficients are defined by the constants $\mathcal{N}$ and L, respectively. The notation $X_0 - m(X)$ and X_1 denotes the initial number of people. The X result of this equation is

$$X(\nu) = X_0\exp\{a\nu\}, \text{where } a = \mathcal{N} - L. \tag{80}$$

Assume that $a \neq 0$ is true. Let us observe that the mild solution of Equation (77):

$$\begin{aligned} X(\nu) &= \mathcal{Q}_q(\nu)(X_0 - m(X)) + \mathcal{G}_q(\nu)X_1 \\ &+ \frac{1}{\sqrt{\gamma}}a\int_0^\nu (\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f}\left(\nu, X(\nu), D^\beta X(\nu)\right)ds. \end{aligned} \tag{81}$$

In the next section, we will apply the previous model to the situation when there are certain uncertainties in $X(\nu)$. Let us introduce an observer (who keeps an eye on this population). Assume that the population's state is determined by random factors and that the observer can only explain the population's state using linguistic terms, such as "very small," "small," "not big," and "large". In this way, the population growth model incorporates two types of uncertainty. The first type of uncertainty belongs to probability theory, whereas the second belongs to fuzzy set theory. We could write the model with uncertainties like this at this point:

$$X(\nu,\omega)=\mathcal{Q}_q(\nu)(X_0-m(X))+\mathcal{G}_q(\nu)X_1 + \frac{1}{\sqrt{\gamma}}a\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f}\left(\nu,X(\nu),D^{\beta}X(\nu)\right),\omega\Big)ds, \tag{82}$$

where ω represents a random factor (a probability space ($\Omega,\mathcal{N},\wp$) is taken into account; $\omega\in\Omega$, $X_0-m(X)$, and X_1 are a FRV; the integral has become a fuzzy integral; and the solution X is now a fuzzy stochastic process $X:[0,T]\times\mathfrak{F}(R)$. Such problem 39 has its differential counterpart, and exemplifies the random fuzzy integral equations or, equivalently, random fuzzy differential equations (see 36).

Consider that some people leave their homeland and aliens enter the population and that this occurs in a disorderly manner. Allow Brownian motion B to model the aggregated immigration process. The population dynamics may now be simulated using the following equation with uncertainties:

$$X(\nu,\omega)=\mathcal{Q}_q(\nu)(X_0-m(X))+\mathcal{G}_q(\nu)X_1 + \frac{1}{\sqrt{\gamma}}a\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f} \cdot\left(\left(\nu,X(\nu),D^{\beta}X(\nu)\right),\omega\right)ds+\langle B(\nu,\omega)\rangle. \tag{83}$$

This equation can be rewritten as follows (we will not write the argument ω in the next section):

$$X(\nu)=\mathcal{Q}_q(\nu)(X_0-m(X))+\mathcal{G}_q(\nu)X_1 + \frac{1}{\sqrt{\gamma}}a\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f}\left(\nu,X(\nu),D^{\beta}X(\nu)\right)ds + \langle B(s)\rangle. \tag{84}$$

Alternatively, in a symbolic, differential form:

$${}_0^cD_\nu^\alpha X(\nu)=(\mathcal{N}-L)X(\nu)+(\mathcal{N}-L)X'(\nu) + \mathfrak{f}\left(\nu,X(\nu),D^{\beta}X(\nu)\right)\Big)\Big]ds+\langle dB(t)\rangle,\nu\in[0,\mathfrak{T}], \tag{85}$$

$$X(0)+m(X)=X_0, \tag{86}$$

$$X'(0)=X_1, \tag{87}$$

So we get to the SFDE of type 2.1, where $f:[0,T]\times\Xi\times\mathfrak{F}(R)\longrightarrow\mathfrak{F}(R)$ is given by $\mathfrak{f}(\nu,u)=a.u$, and $g:[0,T]\times\Omega\times\mathfrak{F}(R)\longrightarrow R$ is defined by $g(\nu,u)=1$. The coefficients of such equations satisfy the requirements $(A_1)-(A_3)$. Considering that $X_0:\Omega\longrightarrow\mathfrak{F}(R)$ is a FRV with the property that $X_0-m(X)\in L^2(\Omega,\mathcal{N},\wp;\mathfrak{F}(R))$, $X_1\in L^2(\Omega,\mathcal{N},\wp;\mathfrak{F}(R))$ and $X_0-m(X),X_1$ are $\mathcal{N}$-measurable, Equation (85), or equivalently Equation (84), has a unique solution.

In the sequel, we shall establish the explicit solution to (84) with $a\neq 0$. To this end, let us denote the α-levels ($\alpha\in[1,2]$) of the solution $X:[0,T]\times\Omega\longrightarrow\mathfrak{F}(R)$ and α-levels of initial value $X_0-m(X)$: $\Omega\longrightarrow\mathfrak{F}(R)$ and $X_1:\Omega\longrightarrow\mathfrak{F}(R)$ as

$$[X(\nu)]^\alpha=[L_\alpha(\nu),U_\alpha(\nu)]\text{ and }[X_0]^\alpha=\left[X_{0,L}^\alpha,L,X_{0,U}^\alpha\right], \tag{88}$$

respectively. Obviously, $L_\alpha,U_\alpha:[0,T]\times\Omega\longrightarrow R$ are the stochastic processes; also, $X_{0,L}^\alpha-m(X)^\alpha,X_{0,U}^\alpha-m(X)^\alpha:\Omega\longrightarrow R$ and $X_{1,L}^\alpha,X_{1,U}^\alpha:\Omega\longrightarrow R$ are the random variables. If the FSP X is a solution to (84), then for every $\nu\in[0,T]$, the following property should hold:

$$P\Big([X(\nu)]^\alpha=\nu^{\alpha-1}P_\alpha(\nu)\left[\mathcal{Q}_q(\nu)(X_0-m(X))\right]^\alpha + \nu^{\alpha-1}P_\alpha(\nu)\left[\mathcal{G}_q(\nu)X_1\right]^\alpha + \left[a\int_0^{\nu}(\nu-s)^{q-1}P_q(\nu-s)\mathfrak{f}\left(\nu,X(\nu),D^{\beta}X(\nu)\right)ds\right]^\alpha + \left[\left\langle\int_0^t dB(s)\right\rangle\right]^\alpha,\forall\alpha\in[1,2]\Big)=2. \tag{89}$$

5. Example

Assume the following FSDEs:

$${}_0^cD_\nu^\gamma X(\nu)=X(\nu)+\nu^2+\sin^2X+\cos 2X, \tag{90}$$

where $\nu\in[1,2]$, $f(\nu,X(\nu))=Y(\nu)+\nu^2$, $g(\nu,X(\nu))=\sin^2X+\cos 2X$, $3/2<\beta<2$. It is easy to verify that f,σ satisfy the A_1-A_3. Define $\bar{f}(X,Y)$ as follows:

$$\int_1^2\bar{f}(X,Y)\left(\nu^2+\nu\right)d\nu=\int_1^2\bar{f}(\nu,X,Y)\left(\nu^2+\nu\right)d\nu. \tag{91}$$

We can prove that $\bar{f}(X,Y)=(\nu^3/3)+(\nu^2/2)$. Similarly, $\bar{g}(X,Y)=(1/2)(X+\sin 2X)+(1/4)\sin 2X$. The averaging form of (90) can be written as

$${}_0^cD_\nu^\gamma Y(\nu)=Y(\nu)\left(\frac{\nu^3}{3}+\frac{\nu^2}{2}\right)+\left(\frac{1}{2}(X+\sin 2X)+\frac{1}{4}\sin 2X\right)d\mathcal{B}_\nu^{\mathcal{H}}. \tag{92}$$

As ε approaches zero, the solutions $X(\nu)$ and $Y(\nu)$ are equal in the sense of mean square, according to Theorem 12. As a result, the findings may be verified.

6. Conclusion

In order to model fuzziness in the Liouville form FBM, we created a fuzzy stochastic differential equation with a number of characteristics, such as long-range dependence and unpredictability. We applied an approximation strategy to fractional stochastic integrals and embedded the Itô classical integral in fuzzy set space. We used the Picard iteration method to examine the existence-uniqueness of solutions.

We demonstrated that the approximate solution converges uniformly to the precise solution. We also demonstrate the existence and uniqueness of solutions to FFSDEs under the Lipschitzian coefficient. Illustrations of the application to a population dynamics model are also provided. The example is also illustrated at the end of the text. Future work may also involve generalizing other tasks, adding observability, and developing the concept that was introduced in this mission. This is a fertile field with numerous research initiatives that have the potential to produce a wide range of theories and applications.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia (Project No. GRANT667), King Faisal University (KFU), Ahsa, Saudi Arabia. The authors, therefore, acknowledge technical and financial support of DSR at KFU.

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