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Research Article

The Ordering of the Unicyclic Graphs with respect to Largest Matching Root with Given Matching Number

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The matching roots of a simple connected graph G are the roots of the matching polynomial which is defined as $M_G(x) = \sum_{k=0}^{n/2} (-1)^k m(G,k)x^{n-2k}$, where $m(G,k)$ is the number of the k matchings of G. Let $\lambda_1(G)$ denote the largest matching root of the graph G. In this paper, among the unicyclic graphs of order n, we present the ordering of the unicyclic graphs with matching number 2 according to the $\lambda_1(G)$ values for $n \geq 11$ and also determine the graphs with the first and second largest $\lambda_1(G)$ values with matching number 3.

1. Introduction

In this paper, all of the graphs considered are connected and finite. Let $G = (V(G), E(G))$ be a graph of order n, where $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. Let $N(u)$ and $d_G(u)$ be the neighbor set and the degree of the vertex $u \in V(G)$ in G, respectively. A k-edge matching of G is a set of k mutually independent edges, and the number of all the k-edge matchings of G is denoted by $m(G,k)$. Obviously, $m(G,k) = 0$ if $k > n/2$. For convenience, we set $m(G,0) = 1$.

The original definition of the matching polynomial was introduced in [9] as $\sum_k m(G,k)x^k$, which is now defined in [14] as

$$M_G(x) = \sum_{k=0}^{n/2} (-1)^k m(G,k)x^{n-2k}. \tag{1}$$

Each root of the matching polynomial $M_G(x)$ is called a matching root of G. The largest matching root of a graph G which is denoted by $\lambda_1(G)$ is the largest root of $M_G(x)$. In particular, $\lambda_1(G)$ has been proved to be positive real numbers excepting for the edgeless graphs in [3, 4].

The matching polynomial of a graph has been used in various branches of physics and chemistry. In analogy with the traditional graph energy, the matching energy of a graph has been conceived as the sum of the absolute values of the roots of the matching polynomial. This graph invariant has recently attracted much attention (see [2, 3, 5–11]). In [15], Gutman and Zhang have ordered the graphs by matching numbers. Zhang et al. [18] investigated the largest matching root $\lambda_1(G)$ for unicyclic graphs and characterized the extremal graph. Zhang and Chen [10] studied the largest matching roots of unicyclic graphs and trees with a given number of fixed matching numbers and characterized the extremal graph with respect to the largest matching roots. In [3], Liu et al. determined the graphs with the four largest and two smallest $\lambda_1(G)$ values.

Motivated by the previous research, in this paper, we focus on the ordering of the unicyclic graph with the fixed matching number with respect to the largest matching root $\lambda_1(G)$. Among the unicyclic graphs of order n, we order the unicyclic graphs with matching number 2 according to the $\lambda_1(G)$ values for $n \geq 11$ and also determine the graphs with the first and second largest $\lambda_1(G)$ values with matching number 3.

2. Preliminaries

Let H be a subgraph of G. If $V(H) = V(G)$, then H is said to be a spanning subgraph of G. The following lemma about the largest matching root is well known.

Lemma 1 (see [8]). *Let H be a spanning subgraph of G and $\lambda_1(G)$ the largest matching root of G. If $\kappa \geq \lambda_1(G)$, then $M_H(\kappa) \geq M_G(\kappa)$; if H is a proper subgraph of G and $\kappa > \lambda_1(G)$, then $M_H(\kappa) > M_G(\kappa)$.*

Definition 1. Let u, v denote two vertices of G. The Kelmans transformation of G is defined as follows (cf. Figure 1): delete all edges which are with one end v and another end in $N(v)/(N(u)\cup u)$ and add all edges between u and $N(v)\backslash(N(u)\cup u)$.

Let G' denote the graph obtained from G by a series of Kelmans transformation without referring to the vertices u, v. Obviously, G' and G have the same size.

The relationship between the largest matching roots of G' and G is given as follows.

Lemma 2 (see [7]). *Let G' be the graph obtained from G by some Kelmans transformation. Then, $\lambda_1(G') \geq \lambda_1(G)$.*

In [3], Liu et al. characterized the unicyclic graphs with a triangle as given in Figure 2 and obtained the following lemma.

Lemma 3 (see [3]). *ie graphs of order n D_k and D_{k-1} are shown in Figure 2. If $2 \leq k \leq n/2 - 1$, then $\lambda_1(D_k) \leq \lambda_1(D_{k-1})$, with equality holding if and only if $k = n/2 - 1$.*

Let S_n be the star graph of order n and S_n^+ the unicyclic graph obtained by adding a new edge to S_n. Let C_{n-1} be a cycle of length $n-1$ and C_n^2 the graph obtained from C_{n-1} by adding a new pendent edge. Let D_n^2 be the graph obtained by identifying one end vertex of the path P_{n-2} and one vertex of the triangle C_3.

Lemma 4 (see [3]). *Among all connected unicyclic graphs with $n\,(n \geq 8)$ vertices, the first four largest matching roots are $\lambda_1(S_n^+) > \lambda_1(S_n^2) > \lambda_1(S_n^3) > \lambda_1(S_n^4)$, and the last three largest matching roots are $\lambda_1(C_n^2)$ or $\lambda_1(D_n^2)$ and $\lambda_1(C_n)$, where S_n^2, S_n^3, and S_n^4 are shown in Figure 3.*

Lemma 5 (see [11]). *Let G be a graph of order n. Then, $\lambda_1^2(G) \geq 4|E(G)|/n - 1$, where $|E(G)|$ is the number of edges of G.*

Corollary 1. *Let G be a connected unicyclic graph with n vertices. Then, $\lambda_1(G) \geq \sqrt{3}$.*

3. Main Results

In this section, we first investigate the largest matching root of unicyclic graphs with matching number 2.

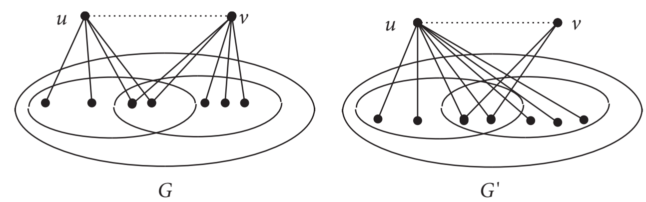

Figure 1: The Kelmans transformation.

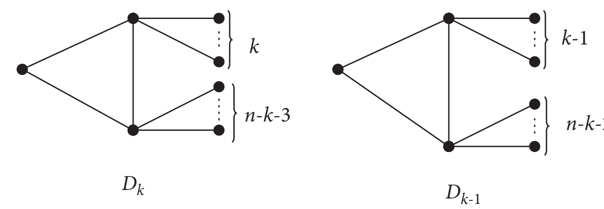

Figure 2: Two graphs D_k and D_{k-1}.

For $k \geq 2$, let S_k be a star graph with center u. Let v_1, v_2 be two vertices of S_{n-k+1}^+ which are of degree 1 and 2, respectively. We write $D_{n,k}\,(C_{n,k})$ denote the graph obtained by identifying u with $v_1\,(v_2)$. In fact, $C_{n,k}$ is the graph D_{k-1} in Figure 2. All the connected unicyclic graphs with matching number 2 are S_n^+, $C_{n,k}$, $D_{n,n-3}$, S_n^4, and T_n^k as shown in Figure 4.

Theorem 1. $\lambda_1(C_{n,k+1}) > \lambda_1(T_n^k) > \lambda_1(C_{n,k+2}) > \lambda_1(T_n^{k+1})$, $k < n - 4/3$.

Proof. We can easily get $m(T_n^k, 2) = n + kn - k^2 - 2k - 3$, $m(T_n^{k+1}, 2) = 2n + kn - k^2 - 4k - 6$ and $m(C_{n,k+1}, 2) = n + kn - k^2 - 3k - 3$, $m(C_{n,k+2}, 2) = 2n + kn - k^2 - 5k - 7$, then the matching polynomial of graphs T_n^k and T_n^{k+1} is

$$\begin{aligned} M_{T_n^{k+1}}(x) &= x^n - nx^{n-2} + \left(2n + kn - k^2 - 4k - 6\right)x^{n-4}, \\ M_{T_n^k}(x) &= x^n - nx^{n-2} + \left(n + kn - k^2 - 2k - 3\right)x^{n-4}, \\ M_{C_{n,k+1}}(x) &= x^n - nx^{n-2} + \left(n + kn - k^2 - 3k - 3\right)x^{n-4}, \\ M_{C_{n,k+2}}(x) &= x^n - nx^{n-2} + \left(2n + kn - k^2 - 5k - 7\right)x^{n-4}. \end{aligned} \tag{2}$$

Therefore,

$$\begin{aligned} M_{T_n^{k+1}}(x) - M_{T_n^k}(x) &= (n - 2k - 3)x^{n-4}, \\ M_{C_{n,k+1}}(x) - M_{T_n^k}(x) &= -kx^{n-4}, \\ M_{C_{n,k+2}}(x) - M_{T_n^k}(x) &= (n - 3k - 4)x^{n-4}. \end{aligned} \tag{3}$$

Then $M_{T_n^{k+1}}(\lambda_1(T_n^k)) > 0$, $M_{C_{n,k+1}}(\lambda_1(T_n^k)) < 0$, and $M_{C_{n,k+2}}(\lambda_1(T_n^k)) > 0$, so $\lambda_1(C_{n,k+1}) > \lambda_1(T_n^k) > \lambda_1(C_{n,k+2}) > \lambda_1(T_n^{k+1})$, since the $M_G(x)$ is strictly monotonically increasing for $x > 0$. □

Theorem 2. *Among all connected unicyclic graphs of order $n\,(n \geq 11)$ with matching number 2, $\lambda_1(S_n^+) > \lambda_1\ (C_{n,2}) > \lambda_1\ (S_n^4)\ > \lambda_1(C_{n,3}) > \lambda_1\ (D_{n,n-3}) = \lambda_1(T_n^2)\ > \lambda_1(C_{n,4}) > \cdots >$*

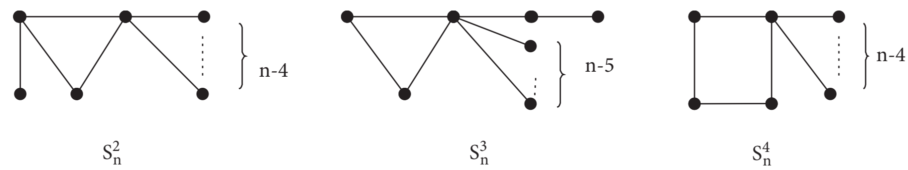

FIGURE 3: Graphs S_n^2, S_n^3, and S_n^4.

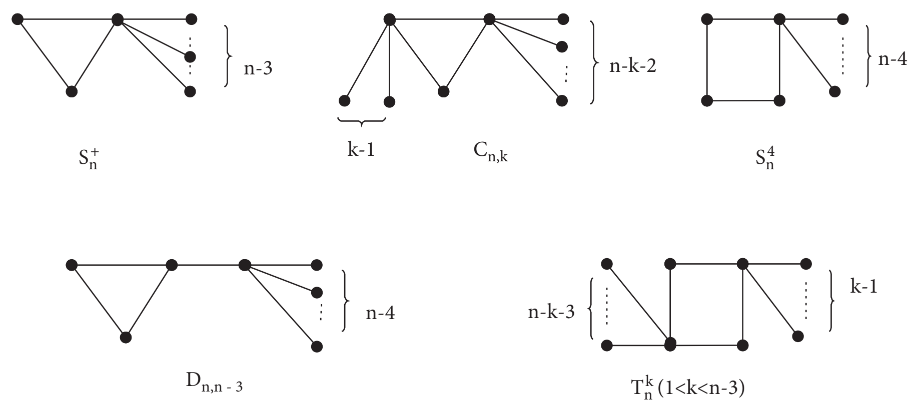

FIGURE 4: Graphs S_n^+, $C_{n,k}$, S_n^4, $D_{n,n-3}$, and T_n^k.

$\lambda_1(C_{n,k+1}) > \lambda_1(T_n^k) > \lambda_1(C_{n,k+2}) > \lambda_1(T_n^{k+1})$, *with* $k < n - 4/3$.

Proof. It is obvious that $m(G,1) = n$ for any connected unicyclic graph G of order n. By simple calculations, we have

$$m\left(C_{n,2},2\right) = 2n-7, m\left(S_n^4,2\right) = 2n-6, m\left(C_{n,3},2\right) = 3n-13,$$
$$m\left(D_{n,n-3},2\right) = 3n-11, m\left(C_{n,4},2\right) \tag{4}$$

Then,

$$\begin{aligned}
M_{C_{n,2}}(x) &= x^n - nx^{n-2} + (2n-7)x^{n-4},\\
M_{S_n^4}(x) &= x^n - nx^{n-2} + (2n-6)x^{n-4},\\
M_{C_{n,3}}(x) &= x^n - nx^{n-2} + (3n-13)x^{n-4},\\
M_{D_{n,n-3}}(x) &= M_{T_n^2}(x) = x^n - nx^{n-2} + (3n-11)x^{n-4},\\
M_{C_{n,4}}(x) &= x^n - nx^{n-2} + (4n-21)x^{n-4}.
\end{aligned} \tag{5}$$

Since $M_{S_n^4}(x) - M_{C_{n,2}}(x) = x^{n-4}$, then $M_{S_n^4}(\lambda_1(C_{n,2})) > 0$. Since the matching polynomial $M_G(x)$ is strictly monotonically increasing for $x > 0$, $\lambda_1(C_{n,2}) > \lambda_1(S_n^4)$. Similarly, since $M_{C_{n,3}}(x) - M_{S_n^4}(x) = (n-7)x^{n-4}$, then $M_{C_{n,3}}(\lambda_1(S_n^4)) > 0$ by putting $x = \lambda_1(S_n^4)$, and so $\lambda_1(S_n^4) > \lambda_1(C_{n,3})$. Similarly, we have

$$\begin{aligned}
M_{C_{n,3}}(x) - M_{D_{n,n-3}}(x) &= -2x^{n-4},\\
M_{D_{n,n-3}}(x) - M_{C_{n,4}}(x) &= (-n+10)x^{n-4},\\
M_{D_{n,n-3}}(x) - M_{T_n^2}(x) &= 0.
\end{aligned} \tag{6}$$

Putting $x = \lambda_1(C_{n,3})$ and $x = \lambda_1(D_{n,n-3})$ to the above two equations, respectively, we have

$$\lambda_1\left(C_{n,3}\right) > \lambda_1\left(D_{n,n-3}\right) = \lambda_1\left(T_n^2\right) > \lambda_1\left(C_{n,4}\right). \tag{7}$$

Since S_n^+ can be obtained from $C_{n,k}$ by Kelmans transformations, then $\lambda_1(S_n^+) > \lambda_1(C_{n,k})$ by Lemma 3. By Lemma 3, we have $\lambda_1(C_{n,k-1}) > \lambda_1(C_{n,k})$. Combining the discussions above and Theorem 1, the proof is completed. □

We now investigate the largest matching root of graphs with matching number 3.

Theorem 3. *Let the graphs* $D_{n,k+1}$ *and* $D_{n,k}$ *be defined above. Then,* $\lambda_1(D_{n,k+1}) < \lambda_1(D_{n,k})$ *if* $2 \le k \le n - 1/2$ *and* $\lambda_1(D_{n,k+1}) > \lambda_1(D_{n,k})$ *if* $k > n - 1/2$.

Proof. By simple calculations, it is obtained that

$$\begin{aligned} m\left(D_{n,k+1},1\right) &= n, m\left(D_{n,k+1},2\right) = kn-k^2-2k+n-3, m\left(D_{n,k+1},3\right) = kn-k^2-4k,\\ m\left(D_{n,k},1\right) &= n, m\left(D_{n,k},2\right) = kn-k^2-2, m\left(D_{n,k},3\right) = kn-k^2-2k-n+3. \end{aligned} \tag{8}$$

Then,

$$\begin{aligned} M_{D_{n,k+1}}(x) &= x^n - nx^{n-2} + \left(kn-k^2-2k+n-3\right)x^{n-4} - \left(kn-k^2-4k\right)x^{n-6},\\ M_{D_{n,k}}(x) &= x^n - nx^{n-2} + \left(kn-k^2-2\right)x^{n-4} - \left(kn-k^2-2k-n+3\right)x^{n-6}. \end{aligned} \tag{9}$$

If $k \neq n-1/2$, then

$$M_{D_{n,k+1}}(x) - M_{D_{n,k}}(x) = (n-2k-1)x^{n-6}\left(x^2 - \frac{n-2k-3}{n-2k-1}\right). \tag{10}$$

Putting $x = \lambda_1(D_{n,k+1})$. Since $D_{n,k+1}$ is a connected unicyclic graph, then $\lambda_1^2(D_{n,k+1}) \geq 3$ by Lemma 5. Thus, if $2 \leq k < n-1/2$, then $M_{D_{n,k}}(\lambda_1(D_{n,k+1})) < 0$ and so $\lambda_1(D_{n,k+1}) < \lambda_1(D_{n,k})$. If $k > n-1/2$, then $M_{D_{n,k}}(\lambda_1(D_{n,k+1})) > 0$ and so $\lambda_1(D_{n,k+1}) > \lambda_1(D_{n,k})$.

If $k = n-1/2$, then $M_{D_{n,k+1}}(x) - M_{D_{n,k}}(x) = 2x^{n-6}$ and so $\lambda_1(D_{n,k+1}) < \lambda_1(D_{n,k})$. □

Theorem 4. *Let L_k and L_{k-1} be the graphs as shown in Figure 5. If $2 \leq k \leq n-3/2$, then $\lambda_1(L_k) \leq \lambda_1(L_{k-1})$, with equality holding if and only if $k = n-3/2$.*

Proof. By simple calculations, it is obtained that

$$\begin{aligned} m\left(L_k,1\right) &= n, m\left(L_k,2\right) = kn-k^2-4k+2n-6, m\left(L_k,3\right) = kn-k^2-4k,\\ m\left(L_{k-1},1\right) &= n, m\left(L_{k-1},2\right) = kn-k^2-2k+n-3, m\left(L_{k-1},3\right) = kn-k^2-2k-n+3. \end{aligned} \tag{11}$$

Then,

$$\begin{aligned} M_{L_k}(x) &= x^n - nx^{n-2} + \left(kn-k^2-4k+2n-6\right)x^{n-4} - \left(kn-k^2-4k\right)x^{n-6},\\ M_{L_{k-1}}(x) &= x^n - nx^{n-2} + \left(kn-k^2-2k+n-3\right)x^{n-4} - \left(kn-k^2-2k-n+3\right)x^{n-6}. \end{aligned} \tag{12}$$

It follows that

$$M_{L_k}(x) - M_{L_{k-1}}(x) = (n-2k-3)x^{n-6}\left(x^2-1\right). \tag{13}$$

By Lemma 5, $\lambda_1^2(L_k) \geq 3$. Therefore, putting $x = \lambda_1(L_k)$, if $k \leq n-3/2$, then $M_{L_{k-1}}(\lambda_1(L_k)) \leq 0$ and so $\lambda_1(L_k) \leq \lambda_1(L_{k-1})$, with equality holding if and only if $k = n-3/2$. □

Theorem 5. *The graphs $G(a,b,c)$ and $G(a,b-1,c+1)$ with n vertices are shown in Figure 6, where $a+b+c = n-3$. If $a \geq n/3-1$ and $b \geq c$, then $\lambda_1(G(a,b,c)) < \lambda_1(G(a,b+1,c-1))$. If $a = 1$ and $b \geq c$, then $\lambda_1(G(a,b,c)) < \lambda_1(G(a,b+1,c-1))$.*

Proof. Let G_1, G_2 denote the graphs $G(a,b,c)$ and $G(a,b+1,c-1)$, respectively. Without loss of generality, assume that $a \geq n/3-1$ and $b \geq c$. It is easy to get that

$$\begin{aligned} m\left(G_1,1\right) &= n, m\left(G_1,2\right) = ab+ac+bc+a+b+c, m\left(G_1,3\right) = abc,\\ m\left(G_2,1\right) &= n, m\left(G_2,2\right) = ab+ac+bc+a+2c-1, m\left(G_2,3\right) = abc+ac-ab-a. \end{aligned} \tag{14}$$

Then,

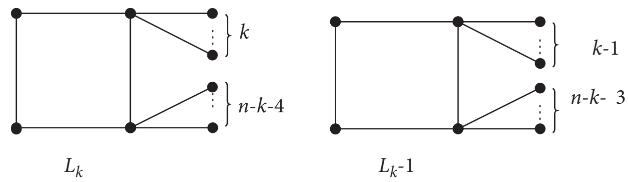

Figure 5: Graph transformation in Theorem 4.

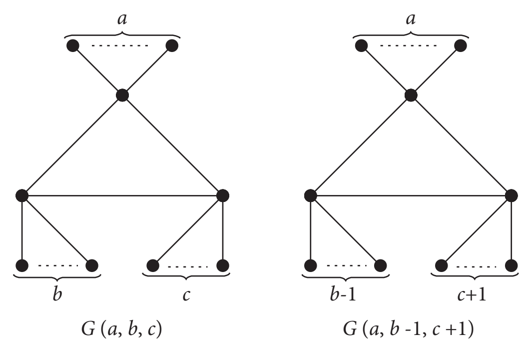

Figure 6: Graph in Theorem 5.

$$\begin{aligned} M_{G_1}(x) &= x^n - nx^{n-2} + (ab + ac + bc + a + b + c)x^{n-4} - abcx^{n-6}, \\ M_{G_2}(x) &= x^n - nx^{n-2} + (ab + ac + bc + a + 2c - 1)x^{n-4} - (abc + ac - ab - a)x^{n-6}. \end{aligned} \tag{15}$$

Putting $x = \sqrt{a+1}$ to $M_{G_1}(x)$, then

$$M_{G_1}(\sqrt{a+1}) = (a+1)^{n/2-3}\left[(a+1)^3 - n(a+1)^2 + (ab + ac + bc + a + b + c)(a+1) - abc\right]. \tag{16}$$

Since $a + b + c = n - 3$, then

$$M_{G_1}(\sqrt{a+1}) = (a+1)^{n/2-3}\left(-a^2 - 3a - 2 + bc\right). \tag{17}$$

Since $a \geq n/3 - 1$ and $a + b + c = n - 3$, then

$$bc \leq \left(\frac{b+c}{2}\right)^2 \leq \left(\frac{n-3}{3}\right)^2 < a^2 + 3a + 2. \tag{18}$$

Therefore, $M_{G_1}(\sqrt{a+1}) < 0$, and so $\lambda_1(G_1) > \sqrt{a+1}$. Moreover, $M_{G_1}(x) - M_{G_2}(x) = (b - c + 1)x^{n-6}(x^2 - a)$. Since $b \geq c$, then $M_{G_1}(x) - M_{G_2}(x) < 0$ for any $x > \sqrt{a+1}$. Thus, if $a \geq n/2 - 1$ and $b \geq c$, then

$$M_{G_1}(\lambda_1(G_1)) - M_{G_2}(\lambda_1(G_1)) > 0, \tag{19}$$

that is, $M_{G_2}(\lambda_1(G_1)) < 0$. Therefore, $\lambda_1(G_1) < \lambda_1(G_2)$.

We now discuss the case for $a = 1$. Without loss of generality, assume that $b \geq c$. Obviously, we have

$$\begin{aligned} M_{G_1}(x) &= x^n - nx^{n-2} + (2b + 2c + bc + 1)x^{n-4} - bcx^{n-6}, \\ M_{G_2}(x) &= x^n - nx^{n-2} + (b + 3c + bc)x^{n-4} - (bc + c - b - 1)x^{n-6}. \end{aligned} \tag{20}$$

Then,

$$M_{G_1}(x) - M_{G_2}(x) = (b - c + 1)x^{n-6}\left(x^2 - 1\right). \tag{21}$$

By Corollary 1, $\lambda_1^2(G_1) \geq 3$. Thus, if $b \geq c$, then $M_{G_2}(\lambda_1(G_1)) < 0$. Hence, $\lambda_1(G_1) < \lambda_1(G_2)$. □

Theorem 6. *The graphs $H_i (i = 1, 2, \ldots, 5)$ with $n(>8)$ vertices are shown in Figure 7. Then, $\lambda_1(H_1) > \lambda_1(H_2) > \lambda_1(H_3) = \lambda_1(H_4) > \lambda_1(H_5)$.*

Proof. Obviously, $m(H_i, 1) = n$ (for $i = 1, 2, 3, 4, 5$) and

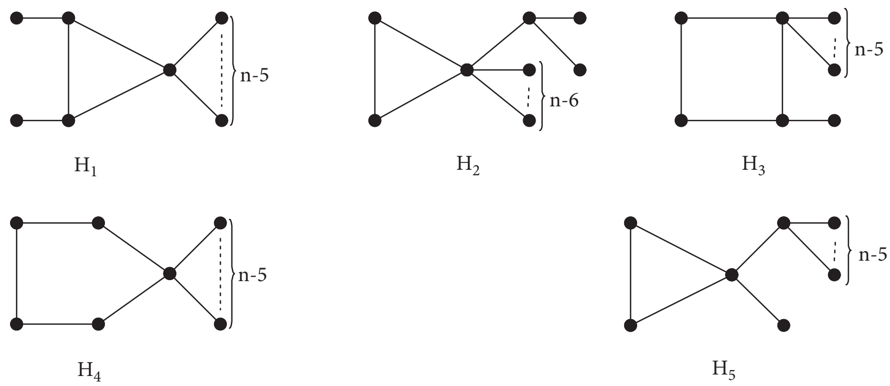

Figure 7: Five graphs in Theorem 6.

$$
\begin{aligned}
&m(H_1,2) = 3n-12, m(H_1,3) = n-5,\\
&m(H_2,2) = 3n-11, m(H_2,3) = 2n-12,\\
&m(H_3,2) = 3n-10, m(H_3,3) = n-5,\\
&m(H_4,2) = 3n-10, m(H_4,3) = n-5,\\
&m(H_5,2) = 4n-18, m(H_5,3) = n-5.
\end{aligned}
\tag{22}
$$

Then,

$$
\begin{aligned}
&M_{H_1}(x) = x^n - nx^{n-2} + (3n-12)x^{n-4} - (n-5)x^{n-6},\\
&M_{H_2}(x) = x^n - nx^{n-2} + (3n-11)x^{n-4} - (2n-12)x^{n-6},\\
&M_{H_3}(x) = x^n - nx^{n-2} + (3n-10)x^{n-4} - (n-5)x^{n-6},\\
&M_{H_4}(x) = x^n - nx^{n-2} + (3n-10)x^{n-4} - (n-5)x^{n-6},\\
&M_{H_5}(x) = x^n - nx^{n-2} + (4n-18)x^{n-4} - (n-5)x^{n-6}.
\end{aligned}
\tag{23}
$$

Putting $x = \sqrt{n-6}$ to matching polynomial of H_2, we have

$$
\begin{aligned}
M_{H_2}(\sqrt{n-6}) &= (n-6)^{n/2-3}\left[(n-6)^3 - n(n-6)^2 + (3n-11)(n-6) - 2(n-6)\right],\\
&= (n-6)^{n/2-3}\left(-3n^2 + 41n - 138\right).
\end{aligned}
\tag{24}
$$

Since $n > 8$, then $M_{H_2}(\sqrt{n-6}) < 0$ and so $\lambda_1(H_2) > \sqrt{n-6}$. Since $M_{H_2}(x) - M_{H_1}(x) = x^{n-6}[x^2 - (n-7)]$, then $M_{H_2}(x) - M_{H_1}(x) > 0$ for any $x > \sqrt{n-6}$. Therefore, $M_{H_1}(\lambda_1(H_2)) < 0$, and thus, $\lambda_1(H_1) > \lambda_1(H_2)$.

Since $M_{H_3}(x) - M_{H_2}(x) = x^{n-6}[x^2 + (n-7)]$, then $M_{H_3}(\lambda_1(H_3)) - M_{H_2}(\lambda_1(H_3)) > 0$, and thus, $\lambda_1(H_2) > \lambda_1(H_3)$. It is obvious that $\lambda_1(H_3) = \lambda_1(H_4)$. Since $M_{H_4}(x) - M_{H_5}(x) = (-n+8)x^{n-4}$ and $n > 8$, then $M_{H_4}(\lambda_1(H_4)) - M_{H_5}(\lambda_1(H_4)) < 0$ and so $\lambda_1(H_4) > \lambda_1(H_5)$. Therefore, $\lambda_1(H_1) > \lambda_1(H_2) > \lambda_1(H_3) = \lambda_1(H_4) > \lambda_1(H_5)$. □

Theorem 7. *Among all unicyclic graphs of order $n(>8)$ with matching number 3, the graphs with the first second largest matching root are $D_{n,2}$ and H_1.*

Proof. Since $D_{n,2} \cong S_n^3$, then the graph with the largest matching root among all unicyclic graphs with matching number 3 is $D_{n,2}$ by Lemma 4.

Let G be the unicyclic graph of order $n(\geq 8)$ with the second largest matching root among all graphs with matching number 3. Since G is a unicyclic graph of order n with matching number 3, then the girth g of G is possibly $3, 4$, or 5. □

3.1. Case 1: $g = 3$. Under the Kelmans transformation, G is $D_{n,k}$ or $G(a,b,c)$ in Figure 6. By Theorem 3, $\lambda_1(D_{n,k+1}) < \lambda_1(D_{n,k})$ if $2 \leq k < n - 1/2$ and $\lambda_1(D_{n,k+1}) > \lambda_1(D_{n,k})$ if $k > n - 1/2$. Therefore, among all the graphs of type $D_{n,k}$, G is $D_{n,3}$ or $D_{n,n-4}$. Since $D_{n,3} \cong H_2$ and $D_{n,n-4} \cong H_5$, then $\lambda_1(D_{n,3}) > \lambda_1(D_{n,n-4})$ by Theorem 6.

By Theorem 5, $G(1, n-5, 1)$ is the graph with the first largest matching root among the graphs of type $G(a,b,c)$. Moreover, $G(1, n-5, 1) \cong H_1$. Thus, by Theorem 6, the graph with the second largest matching root and girth 3 is H_1. □

3.2. Case 2: $g = 4$. In this case, G possibly is the graph T_i ($i = 1, 2, 3, 4, 5$) in Figure 8. Obviously, T_1 can be obtained by a series of Kelmans transformations from the graph T_i ($i = 2, 3, 4, 5$). Moreover, T_1 is the graph L_k in Theorem 4. By Theorem 4, the graph with the largest matching roots among graphs of type L_k is L_1 which is isomorphic to H_3. Therefore, the graph with the largest matching root among the graphs with matching number 3 and girth 4 is H_3.

3.3. Case 3: $g = 5$. In this case, the unicyclic graphs with matching number 3 isomorphism to the following four families graphs denoted by $C(n-5, 0, 0, 0, 0)$

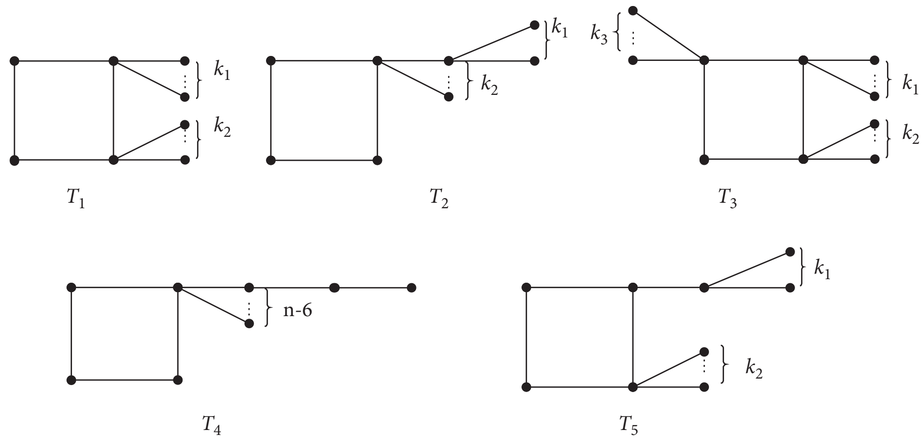

Figure 8: Five graphs in Case 2.

$, C(n-5-k, k, 0, 0, 0), C(n-5-k, 0, k, 0, 0)$, and $C(n-5-k_1-k_2, 0, k_1, 0, k_2)$, where $C(m_1, m_2, m_3, m_4, m_5)$ denote the unicyclic graphs with girth 5, such that there are m_i pendent edges attached to the vertex v_i of the cycle. Obviously, $C(n-5, 0, 0, 0, 0)$ is isomorphic to H_4 in Theorem 5, and the rest three families can arrive at graphs of Case 1 or 2 by a series of Kelmans transformations. Thus, G is H_4 in this case.

Therefore, by Theorem 6, the graph with the second largest matching root is H_1.

4. Further Remarks

In the introduction, we already talk about that the matching energies have been getting a lot of attention recently. The graphs with maximum and minimum values of matching roots are the graphs with minimum and maximum matching energy. Therefore, any result for the largest matching roots can reveal the structural dependence of the matching energy or at least provide guidance for its research. Recently, References [2,5,6] showed some new findings on the matching energy of unicyclic graphs. Therefore, the results of this paper have a direct application value [4, 16].

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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