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Research Article

Sum of the Hurwitz-Lerch Zeta Function over Natural Numbers: Derivation and Evaluation

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We consider a Hurwitz-Lerch zeta function $\Phi(s, z, a)$ sum over the natural numbers. We provide an analytically continued closed form solution for this sum in terms of the addition of Hurwitz-Lerch zeta functions. A new recurrence identity with consecutive neighbours and the product of trigonometric functions is derived.

1. Introduction

The Hurwitz-Lerch zeta functions have been published in the following papers [1–7]. The object of this paper is the derivation and evaluation of a new expression for the finite sum of the Hurwitz-Lerch zeta function $\Phi(s, z, a)$. This derivation is expressed in terms of the sum of two Hurwitz-Lerch zeta functions given by the following theorem.

Theorem 1. *For all $k, a, m \in \mathbb{C}, n \in \mathbb{N}^{+}$, then*

$$\begin{aligned}
&\sum_{p=0}^{n-1} e^{4i\pi p/n}\Phi\left(-e^{2i(m+2p\pi/n)}, -k, 1-\frac{1}{2}i\log(a)\right)\\
&= -\frac{1}{2}n(\mathrm{i}n)^{k}e^{1/2i(\pi(n-k)+2m(n-2))}\left(\left(1+e^{i\pi n}\right)\Phi\left(e^{in(2m+\pi)}, -k, \frac{n-i\log(a)}{2n}\right)\right.\\
&\left.+2e^{1/2i(2m+\pi)n}\Phi\left(e^{in(2m+\pi)}, -k, 1-\frac{i\log(a)}{2n}\right)\right).
\end{aligned} \tag{1}$$

Proof. Observe that the addition of the right-hand sides of equations (3) and (4) is equal to the addition of the right-hand sides of equations (3)–(8), so we may equate the left-hand sides and simplify the Gamma function to yield the stated result. □

The derivations follow the method used by us in [8]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^{k}}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_{C}\frac{e^{wy}}{w^{k+1}}\mathrm{d}w, \tag{2}$$

where $y, w \in \mathbb{C}$, and C is in general an open contour in the complex plane where the bilinear concomitant [8] has the same value at the end points of the contour. This method involves using a form of (2), then multiplies both sides by a function, and then takes the sum of both sides. This yields a sum in terms of a contour integral. Then, we multiply both sides of (2) by another function and take the infinite sum of both sides such that the contour integrals of both equations are the same.

2. The Lerch Function

See section 4 in [9].

3. Derivation of the Contour Integrals

3.1. Finite Sum of the Contour Integral. We use the method in [8]. The cut and contour are in the first quadrant of the complex $w+m$-plane with $0<\mathrm{Re}(2(w+m))+\pi$. The cut approaches the origin from the interior of the first quadrant and goes to infinity vertically and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula (2), we first replace $y \longrightarrow \log(a)+2i(y+1)$, then multiply both sides by $-2i(-1)^y e^{2i(y+1)(m+2\pi p/n)}$, and take the infinite sums over $y \in [0,\infty)$ and $p \in \{0,1,\ldots,n-1\}$ and simplify in terms of the Hurwitz-Lerch zeta function to get

$$\begin{aligned}
&-\sum_{p=0}^{n-1}\frac{(2i)^{k+1}e^{2im+4i\pi p/n}\Phi\left(-e^{2i(m+2p\pi/n)},-k,1-1/2i\log(a)\right)}{\Gamma(k+1)}\\
&=-\frac{1}{2\pi i}\sum_{y=0}^{\infty}\sum_{p=0}^{n-1}\int_C 2i(-1)^y a^w w^{-k-1}e^{2i(y+1)(n(m+w)+2\pi p)/n}dw\\
&=-\frac{1}{2\pi i}\int_C\sum_{p=0}^{n-1}\sum_{y=0}^{\infty}2i(-1)^y a^w w^{-k-1}e^{2i(y+1)(n(m+w)+2\pi p)/n}\,dw\\
&=\frac{1}{2\pi i}\int_C\sum_{p=0}^{n-1}\left(a^w w^{-k-1}\tan\left(m+\frac{2\pi p}{n}+w\right)-ia^w w^{-k-1}\right)dw\\
&=-\frac{1}{2\pi i}\int_C\frac{1}{2}na^w w^{-k-1}\left(2\cot\left(\frac{1}{2}n(2(m+w)+\pi)\right)+(-1)^n\csc\left(\frac{1}{2}n(2(m+w)+\pi)\right)\right.\\
&\quad\left.+\csc\left(\frac{1}{2}n(2(m+w)+\pi)\right)\right)dw,
\end{aligned}\tag{3}$$

from [10] and equation (2) in [11] where $\mathrm{Re}(2(m+w)+\pi)>0$, $\mathrm{Im}(2(m+w)+\pi)>0$, $n\in\mathbb{N}^+$ in order for the sums to converge. We apply Tonelli's theorem for multiple sums, see page 189 in [12] as the summand is of bounded measure over the space $\mathbb{C}\times\{0,\ldots,n-1\}\times[0,\infty)$.

3.1.1. The Additional Contour Integral. Using a generalization of Cauchy's integral formula (2), we first replace $y \longrightarrow \log(a)$ and multiply both sides by$-in$ simplified to get

$$-\frac{in\log^k(a)}{\Gamma(k+1)}=-\frac{in}{2\pi i}\int_C \mathrm{i}na^w w^{-k-1}\,dw.\tag{4}$$

3.2. Infinite Sum of the Contour Integral

3.2.1. Derivation of the First Contour Integral. We use the method in [8]. Using a generalization of Cauchy's integral formula (2), we first replace $y \longrightarrow \log(a)+2\mathrm{i}n(y+1)$, then multiply both sides by $2\mathrm{i}ne^{2i(m+\pi/2)n(y+1)}$, and take the infinite sums over $y\in[0,\infty)$ and simplify in terms of the Hurwitz-Lerch zeta function to get

$$\begin{aligned}
&\frac{2^{k+1}(in)^{k+1}e^{i(2m+\pi)n}\Phi\left(e^{in(2m+\pi)},-k,1-i\log(a)/2n\right)}{\Gamma(k+1)}\\
&=\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C 2ina^w w^{-k-1}e^{in(y+1)(2m+2w+\pi)}\,dw
\end{aligned}$$

$$= \frac{1}{2\pi i}\int_C \sum_{y=0}^{\infty} 2ina^w w^{-k-1} e^{in(y+1)(2m+2w+\pi)}\,dw$$

$$= -\frac{1}{2\pi i}\int_C na^w w^{-k-1}\cot\left(\frac{1}{2}n(2m+2w+\pi)\right) - ina^w w^{-k-1}\,dw. \tag{5}$$

From equation (2) in [11], $\operatorname{Im}(2(m+w)+\pi) > 0$ in order for the sum to converge.

3.2.2. The Additional Contour Integral. Using a generalization of Cauchy's integral formula (2), we first replace $y \longrightarrow \log(a)$ and multiply both sides by $-i$ simplified to get

$$-\frac{i\log^k(a)}{\Gamma(k+1)} = -\frac{1}{2\pi i}\int_C ia^w w^{-k-1}\,dw. \tag{6}$$

3.2.3. Derivation of the Second Contour Integral. We use the method in [8]. Using a generalization of Cauchy's integral formula (2), we first replace $y \longrightarrow \log(a) + 2\mathrm{in}(y+1)$, then multiply both sides by $\mathrm{in}e^{i(m+\pi/2)n(2y+1)}$, and take the infinite sums over $y \in [0,\infty)$ and simplify in terms of the Hurwitz-Lerch zeta function to get

$$\frac{2^k(\mathrm{in})^{k+1}e^{1/2i(2m+\pi)n}\Phi\left(e^{in(2m+\pi)}, -k, n - i\log(a)/2n\right)}{\Gamma(k+1)}$$
$$= \frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C ina^w w^{-k-1}e^{1/2in(2y+1)(2m+2w+\pi)}\,dw$$
$$= \frac{1}{2\pi i}\int_C \sum_{y=0}^{\infty} ina^w w^{-k-1}e^{1/2in(2y+1)(2m+2w+\pi)}\,dw$$
$$= -\frac{1}{2\pi i}\int_C \frac{1}{2}na^w w^{-k-1}\csc\left(\frac{1}{2}n(2m+2w+\pi)\right)dw. \tag{7}$$

From equation (2) in [11], $\operatorname{Im}(2(m+w)+\pi) > 0$ in order for the sum to converge.

3.2.4. Derivation of the Third Contour Integral. We use the method in [8]. Using a generalization of Cauchy's integral formula (2), we first replace $y \longrightarrow \log(a) + 2\mathrm{in}(y+1)$, then multiply both sides by $(-1)^n\mathrm{in}e^{i(m+\pi/2)n(2y+1)}$, and take the infinite sums over $y \in [0,\infty)$ and simplify in terms of the Hurwitz-Lerch zeta function to get

$$\frac{2^k(in)^{k+1}e^{1/2i(2m+3\pi)n}\Phi\left(e^{in(2m+\pi)}, -k, n - i\log(a)/2n\right)}{\Gamma(k+1)}$$
$$= \frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C i(-1)^n na^w w^{-k-1}e^{1/2in(2y+1)(2m+2w+\pi)}\,dw$$
$$= \frac{1}{2\pi i}\int_C \sum_{y=0}^{\infty} i(-1)^n na^w w^{-k-1}e^{1/2in(2y+1)(2m+2w+\pi)}\,dw$$
$$= -\frac{1}{2\pi i}\int_C \frac{1}{2}e^{i\pi n}na^w w^{-k-1}\csc\left(\frac{1}{2}n(2m+2w+\pi)\right)dw. \tag{8}$$

From equation (2) in [11], $\mathrm{Im}(2(m+w)+\pi)>0$ in order for the sum to converge.

4. Table of Finite Products and Recursive Identities

In this section, we will evaluate (1) for various ranges of the parameters involved and apply the method in [13] to derive recursive identities and finite products involving trigonometric functions.

Example 1. The degenerate case.

$$\begin{aligned}&\sum_{p=0}^{n-1}\tan\left(m+\frac{2\pi p}{n}\right)\\&=-\frac{1}{2}\csc\left(mn+\frac{\pi n}{2}\right)\left(2n\cos\left(mn+\frac{\pi n}{2}\right)+n+in\sin(\pi n)+n\cos(\pi n)\right).\end{aligned}\tag{9}$$

Proof. Use equation (1) and set $k=0$ and simplify using entry (4) below (64 : 12 : 7) in [14]. ☐

Example 2. A recurrence identity with consecutive neighbours:

$$\Phi(-z,s,a)=2^{-s}\left(\Phi\left(z^2,s,\frac{a}{2}\right)-z\Phi\left(z^2,s,\frac{a+1}{2}\right)\right).\tag{10}$$

Proof. Use equation (1) and set $n=2, a=e^{2i(a-1)}, k=-s, m=\log(2)/(2i)$ and simplify. ☐

Example 3.

$$\begin{aligned}&\prod_{p=0}^{n-1}\cos\left(\frac{2p\pi}{n}+\frac{x}{2}\right)\sec\left(\frac{2p\pi}{n}+x\right)\exp\left(\sec\left(\frac{2p\pi}{n}+\frac{x}{2}\right)\sec\left(\frac{2p\pi}{n}+x\right)\sin\left(\frac{x}{2}\right)\right)\\&=\exp\left(\frac{1}{2}n\left(i\left(\frac{2e^{1/2in(3\pi+x)}}{-1+e^{in(\pi+x)}}+x\right)-\frac{4ie^{in(\pi+x)}\cos(n\pi/2)}{-1+e^{in(\pi+2x)}}+\csc\left(\frac{1}{2}n(\pi+x)\right)\right)\right)\\&\cdot\left(\frac{-1+e^{1/2in(\pi+x)}}{-1+e^{1/2in(\pi+2x)}}\right)^{3/2}\sqrt{\frac{1+e^{1/2in(\pi+x)}}{1+e^{1/2in(\pi+2x)}}}\exp\left(n\csc\left(\frac{1}{2}n(\pi+x)\right)\csc\left(\frac{1}{2}n(\pi+2x)\right)\sin\left(\frac{nx}{2}\right)\right)\\&\cdot\left(\cot\left(\frac{1}{4}n(\pi+2x)\right)\tan\left(\frac{1}{4}n(\pi+x)\right)\right)^{1/2e^{in\pi}}.\end{aligned}\tag{11}$$

Proof. Use equation (1) and set $k=1, a=e, m=x$ and apply the method in section 8 in [13]. ☐

Example 4.

$$\begin{aligned}&\prod_{p=0}^{n-1}e^{2\tan(2\pi p/n+x)-2\tan(2\pi p/n+x/2)}\left(\cos\left(\frac{2\pi p}{n}+\frac{x}{2}\right)\sec\left(\frac{2\pi p}{n}+x\right)\right)^{2i\pi}\\&=\left(1-e^{1/2in(x+\pi)}\right)^{i\pi\left(3+e^{i\pi n}\right)}\left(1+e^{1/2in(x+\pi)}\right)^{-i\pi\left(-1+e^{i\pi n}\right)}\\&\cdot\left(1-e^{1/2in(2x+\pi)}\right)^{-i\pi\left(3+e^{i\pi n}\right)}\left(1+e^{1/2in(2x+\pi)}\right)^{i\pi\left(-1+e^{i\pi n}\right)}\\&\cdot\exp\left(n\left(\frac{2i\left(e^{1/2in(x+\pi)}+e^{1/2in(x+3\pi)}+2\right)}{-1+e^{in(x+\pi)}}-\frac{4i\left(1+e^{in(x+\pi)}\cos(\pi n/2)\right)}{-1+e^{in(2x+\pi)}}-\pi x\right)\right).\end{aligned}\tag{12}$$

Proof. Use equation (1) and set $k = 1, a = -1, m = x$ and apply the method in section (8) in [13]. □

Example 5.

$$\prod_{p=0}^{n-1} \frac{\cos^4(2\pi p/n + x/8)\cos^{7/2}(2\pi p/n + x/2)\sec^7(2\pi p/n + x/4)}{\sqrt{\cos(2\pi p/n + x)}}$$
$$= \left(1 - e^{1/2in(x+\pi)}\right)^{\left(1+e^{i\pi n}\right)\left(-3e+7e^{i\pi n}-1\right)/4\left(-1+e^{i\pi n}\right)} \left(1 + e^{1/2in(x+\pi)}\right)^{-\left(1+e^{i\pi n}\right)\left(3e+7e^{i\pi n}-13\right)/4\left(-1+e^{i\pi n}\right)}$$
$$\cdot \left(1 - e^{1/4in(x+2\pi)}\right)^{-\left(1+e^{i\pi n}\right)\left(-e^{i\pi n/2}+7e^{i\pi n}-5\right)/2\left(-1+e^{i\pi n}\right)} \left(1 - e^{1/8in(x+4\pi)}\right)^{2\left(3+e^{i\pi n}\right)} \left(1 + e^{1/8in(x+4\pi)}\right)^{2-2e^{i\pi n}}$$
$$\cdot \left(1 - e^{in(2x+\pi)}\right)^{-e^{i\pi n/2}+e^{3i\pi n/2}-4/4\left(-1+e^{i\pi n}\right)} \left(1 - e^{1/2in(x+2\pi)}\right)^{8-e^{i\pi n}(\cos(\pi n/2)+6)/-1+e^{i\pi n}} \tag{13}$$
$$\cdot \left(1 + e^{1/4in(x+2\pi)}\right)^{e^{i\pi n}(8i\sin(\pi n)+\cos(\pi n/2)-\cos(\pi n)-1)/-1+e^{i\pi n}} \left(1 + e^{1/2in(2x+\pi)}\right)^{e^{i\pi n}(2i\sin(\pi n)+\cos(\pi n/2)-\cos(\pi n)-1)/2\left(-1+e^{i\pi n}\right)}$$
$$\cdot \left(1 - e^{1/2in(2x+\pi)}\right)^{1/4i\cos(\pi n/2)(2\cos(\pi n/2)-1)(\cot(\pi n/2)+i)}$$
$$\cdot \left(1 - e^{in(x+\pi)}\right)^{1/8(6i\sin(\pi n/2)+6\cos(\pi n/2)-9i\tan(\pi n/2)+3i\cot(\pi n/2)+28)}.$$

Proof. Use equation (1) and set $k = 2, a = 1, m = x$ and apply the method in section (8) in [13]. □

Example 6.

$$\prod_{p=0}^{n-1} \cos\left(\frac{2\pi p}{n} + \frac{x}{2}\right)\sec\left(\frac{2\pi p}{n} + x\right)\left(e^{8\sin(x/2)/\cos(4\pi p/n+3x/2)+\cos(x/2)}\right)^{-i/2/\pi}$$
$$= \left(\tan\left(\frac{1}{4}n(x+\pi)\right)\cot\left(\frac{1}{4}n(2x+\pi)\right)\right)^{1/2e^{i\pi n}}$$
$$\cdot \exp\left(\frac{in\left(\pi x - 4\sin(nx/4)\left(2\cos(nx/4) + \left(1 + e^{i\pi n}\right)\cos(1/4n(3x+2\pi))\right)\csc(1/2n(x+\pi))\csc(1/2n(2x+\pi))\right)}{2\pi}\right) \tag{14}$$
$$\cdot \sqrt{e^{-inx}\sin^3\left(\frac{1}{4}n(x+\pi)\right)\cos\left(\frac{1}{4}n(x+\pi)\right)\csc^3\left(\frac{1}{4}n(2x+\pi)\right)\sec\left(\frac{1}{4}n(2x+\pi)\right)}.$$

Proof. Use equation (1) and set $k = 1, a = i, m = x$ and apply the method in section (8) in [13]. □

Example 7.

$$\prod_{p=0}^{n-1} \left(1 - ie^{i(2\pi p/n+x)}\right)^{-ie^{i(2\pi p/n+x)}} \left(1 + ie^{i(i(2\pi p/n+x))}\right)^{ie^{i(2\pi p/n+x)}}$$
$$= \exp\left(e^{in(2x+\pi)}\,\Phi\left(e^{in(2x+\pi)}, 1, 1 - \frac{1}{2n}\right) + \frac{1}{2}\left(1 + e^{i\pi n}\right)e^{1/2\mathrm{in}(2x+\pi)}\Phi\left(e^{\mathrm{in}(2x+\pi)}, 1, \frac{n-1}{2n}\right)\right). \tag{15}$$

Proof. Use equation (1) and set $k=-1, a=e^{-i}, m=x$ and apply the method in section (8) in [13]. □

Example 8.

$$\prod_{p=0}^{n-1}\left(1+e^{2i(2\pi p/n+x)}\right)^{e^{-2i(2\pi p/n+x)}} = \exp\left(\frac{1}{2}\left(\left(1+e^{i\pi n}\right)e^{1/2\mathrm{in}(2x+\pi)}\Phi\left(e^{\mathrm{in}(2x+\pi)},1,\frac{1}{2}+\frac{1}{n}\right)+2e^{\mathrm{in}(2x+\pi)}\Phi\left(e^{\mathrm{in}(2x+\pi)},1,1+\frac{1}{n}\right)+2n\right)\right). \tag{16}$$

Proof. Use equation (1) and set $k=-1, a=e^{2i}, m=x$ and apply the method in section (8) in [13]. □

Example 9. A recurrence identity with consecutive neighbours.

$$\Phi(z,s,a) = 3^{1-s}z^2\Phi\left(z^3,s,\frac{a+2}{3}\right)+\frac{1}{2}(1+i\sqrt{3})\Phi\left(-\frac{1}{2}(1+i\sqrt{3})z,s,a\right) - \frac{1}{2}i(\sqrt{3}+i)\Phi\left(\frac{1}{2}i(i+\sqrt{3})z,s,a\right). \tag{17}$$

Proof. Use equation (1) and set $n=3, a=e^{2i(a-1)}$, $k=-s, m=\log(iz^{1/2})/(i)$ and simplify. □

Example 10.

$$\prod_{p=0}^{n-1}\cos\left(m+\frac{2\pi p}{n}\right)\sec\left(\frac{2\pi p}{n}+q\right) = \sin\left(\frac{1}{2}(2m+\pi)n\right)\csc\left(\frac{1}{2}n(2q+\pi)\right)\left(\cot\left(\frac{1}{4}(2m+\pi)n\right)\tan\left(\frac{1}{4}n(2q+\pi)\right)\right)^{1/2\left(-1-e^{i\pi n}\right)}. \tag{18}$$

Proof. Use equation (1) and form a second equation by replacing m by q and take their difference and set $k=-1, a=1$ and simplify using entry (3) below (64 : 12 : 7) in [14]. □

5. Conclusion

In this paper, we have presented a novel method for deriving a new finite sum of the Hurwitz-Lerch zeta function over the natural numbers along with some interesting products of trigonometric functions and a new recurrence identity using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

Data Availability

No data were used to support this study.

Disclosure

This paper is available as a preprint on arXiv with reference number 2204.03821.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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