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Research Article

Upper Bounds of Radio Number for Triangular Snake and Double Triangular Snake Graphs

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A radio labeling of a simple connected graph $G = (V, E)$ is a function $h: V \longrightarrow N$ such that $|h(x) - h(y)| \geq \text{diam}(G) + 1 - d(x, y)$, where diam (G) is the diameter of graph and $d(x, y)$ is the distance between the two vertices. The radio number of G, denoted by rn (G), is the minimum span of a radio labeling for G. In this study, the upper bounds for radio number of the triangular snake and the double triangular snake graphs are introduced. The computational results indicate that the presented upper bounds are better than the results of the mathematical model provided by Badr and Moussa in 2020. On the contrary, these proposed upper bounds are better than the results of algorithms presented by Saha and Panigrahi in 2012 and 2018.

1. Introduction

The field of graph theory assumes a crucial part in different fields. One of the significant regions in graph theory is graph labeling which is used in many applications such as coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, data base management, and channel assignment problem. The channel assignment problem is the problem of assigning channels (nonnegative integers) to the stations in an optimal way such as the interference is avoided. In [1], Badr and Moussa proposed a work on upper bound of radio k-chromatic number for a given graph against the other which is due to Saha and Panigrahi [2]. Badr and Moussa proposed a new mathematical model for finding the upper bound of a graph [1]. In [3], Saha and Panigrahi introduced another algorithm (with time complexity $O(n^4)$) for determining the upper bound of a graph. Ali et al. gave the upper bound for the radio number of generalized gear graph [4]. Fernandez et al. proved that the radio number of the n-gear is $4n + 2$ [5]. Yao et al. were defined as a new graph radio labeling on trees, and the properties of trees labeling were shown [6]. Smitha and Thirusangu determined the radio mean number of double triangular snake graph and alternate double triangular snake graph [7]. If $p\&q$ is prime numbers, the radio numbers of zero divisor graphs $\Gamma(Z_{P^2} \times Z_q)$ were investigated by Ahmad and Haider [8].

For more details about how to formulate a problem to a mathematical model, the reader can refer to [9–11]. On the contrary, for more details about other labeling that are related to radio labeling such as radio mean, radio mean square, and radio geometric. The reader is referred to [10, 11].

In this current work, the upper bounds for radio number of the triangular snake and the double triangular snake graphs are introduced. The computational results indicate that the presented upper bounds are better than the results of the mathematical model provided by Badr and Moussa [1].

On the contrary, these proposed upper bounds are better than the results of algorithms presented by Saha and Panigrahi [2, 3].

2. Materials and Methods

In this section, we introduce some basic definitions before we prove the theorems that determine the upper bounds' radio of the number for triangular snake and double triangular snake. On the contrary, we introduce the previous works which are related to the determining of the upper bound of radio number of a graph.

Definition 1 (see [12], diameter of graph). The diameter of G is the greatest eccentricity among all vertices of G and it is denoted by diam (G).

Definition 2 (see [13], triangular snake). A triangular snake (or Δ-snake) is a connected graph in which all blocks are triangles and the block-cut-point graph is a path.

Definition 3 (see [7], double triangular snake). A double triangular snake $D(Tn)$ is obtained from two triangular snakes with a common path.

In 2013, Algorithm 1 was introduced by Saha and Panigrahi [2] for determining the upper bound of the radio number of a given graph. Algorithm 1 has $O(n^3)$ time complexity such that n is the number of the vertices of G. In 2018, Saha and Panigrahi [2] proposed a new algorithm (Algorithm 2) for determining the upper bound of the radio number of a given graph. Algorithm 2 has $O(n^4)$ time complexity. On the contrary, in 2020, Badr and Moussa [1] proposed a novel mathematical model which finds the upper bound of the radio number of a given graph.

3. Results and Discussion

Here, we introduce two theorems which determine the upper bounds for radio number of triangular snake and double triangular snake. The presented upper bounds (by Theorems 1 and 2) are better than the results of the mathematical model provided by Badr and Moussa [1]. On the contrary, these proposed upper bounds are better than the results of algorithms presented by Saha and Panigrahi [2, 3].

Theorem 1. *Let G be a triangular snake graph $(\Delta_k - \text{snake})$ with k blocks and n vertices, where $d(x, y) \geq 1$; then, the upper bound of the radio number of $\Delta_k - \text{snake}$ is defined as follows:*

$$rn(\Delta_k - \text{snake}) \leq \begin{cases} k^2 + \dfrac{k}{2}, & \text{if } k \text{ is even}, \\ k^2 + k - \dfrac{k}{2}, & \text{if } k \text{ is odd}, \end{cases} \tag{1}$$

Proof. To prove this theorem, its suffices to give a distance labeling h of $\Delta_k - \text{snake}$.

Let $x_1, x_2, x_3, \ldots, x_n$ be a $\Delta_k - \text{snake}$ of length k, i.e., diameter of $(\Delta_k - \text{snake}) = k$.

Define a function h: $V(\Delta_k - \text{snake}) \longrightarrow N$ as the following cases. □

Case 1. k is odd:

$$h(x_i) = \begin{cases} h(x_{k+1}) = 0, & \\ h(x_{k+1-i}) = ki, & 1 \leq i \leq k, \\ h(x_{k+2+j}) = k^2 + \dfrac{k}{2} - jk, & 0 \leq j \leq k-1. \end{cases} \tag{2}$$

Now, we are in a position to prove that the function $h(x)$ is the distance labeling of $\Delta_k - \text{snake}$.

For each $(i, i+1)$,

$$\begin{aligned} |ki - k(i+1)| &\geq \text{diam} + 1 - d(x, y), \\ |ki - k(i+1)| &\geq k + 1 - d(x, y), \\ k &\geq k + 1 - d(x, y). \end{aligned} \tag{3}$$

Also, for each $(j, j+1)$,

$$\begin{aligned} \left| k^2 + \frac{k}{2} - jk - k^2 + \frac{k}{2} - (j+1)k \right| &\geq \text{diam} + 1 - d(x, y), \\ \left| k^2 + \frac{k}{2} - jk - \left(k^2 + \frac{k}{2} - jk - k \right) \right| &\geq k + 1 - d(x, y), \\ |k| &\geq k + 1 - d(x, y), \\ k &\geq k + 1 - d(x, y). \end{aligned} \tag{4}$$

Suppose that $1 \leq i \leq k$, $0 \leq j \leq k-1$.

If $i = j$,

$$\begin{aligned} \left| ki - \left(k^2 + \frac{k}{2} - jk \right) \right| &\geq k + 1 - d(x, y), \\ k^2 - 2k + \frac{k}{2} &\geq k + 1 - d(x, y), \end{aligned} \tag{5}$$

otherwise,

$$\left(k^2 - \frac{k}{2} - k \right) \geq k + 1 - d(x, y). \tag{6}$$

Case 2. k is even is similarly proved:

$$h(x_i) = \begin{cases} h(x_{k+1}) = 0, & \\ h(x_{k+1-i}) = ki, & 1 \leq i \leq k, \\ h(x_{k+2+j}) = k^2 + \dfrac{k}{2} - kj, & 0 \leq j \leq k-1. \end{cases} \tag{7}$$

We show that the function $h(x)$ is the distance labeling of $\Delta_k - \text{snake}$.

For each $(i, i+1)$,

$$\begin{aligned} |ki - k(i+1)| &\geq \text{diam} + 1 - d(x, y), \\ |ki - k(i+1)| &\geq k + 1 - d(x, y), \\ k &\geq k + 1 - d(x, y). \end{aligned} \quad (8)$$

Also, for each $(j, j+1)$,

$$\begin{aligned} \left|k^2 + \frac{k}{2} - kj - \left(k^2 + \frac{k}{2} - k(j+1)\right)\right| &\geq \text{diam} + 1 - d(x, y), \\ \left|k^2 + \frac{k}{2} - kj - \left(k^2 + \frac{k}{2} - kj - \text{k}\right)\right| &\geq k + 1 - d(x, y), \\ |k| &\geq k + 1 - d(x, y), \\ k &\geq k + 1 - d(x, y). \end{aligned} \quad (9)$$

Suppose that $1 \leq i \leq k$, $0 \leq j \leq k-1$.
If $i = j$,

$$\begin{aligned} \left|ki - \left(k^2 + \frac{k}{2} - jk\right)\right| &\geq k + 1 - d(x, y), \\ k^2 - \frac{3k}{2} &\geq k + 1 - d(x, y), \end{aligned} \quad (10)$$

otherwise,

$$\left(k^2 - \frac{1}{2}k\right) \geq k + 1 - d(x, y). \quad (11)$$

Example 1. Figure 1 presents the labeling Δ_3 (snake) according to Theorem 1.

Theorem 2. *Let G be a double triangular snake graph with k blocks and n vertices; then, the upper bound of the radio number of double Δ_k – snake is defined as follows:*

$$rn(\Delta_k(\text{snake}) \leq \begin{cases} 3, & \text{if } k = 1, \\ 7, & \text{if } k = 2, \\ 2k^2 - k + 3, & \text{if } k \text{ is odd}, \\ 2k^2 - k + 2, & \text{if } k \text{ is even}. \end{cases} \quad (12)$$

Proof. To prove this theorem, it suffices to give a distance labeling h of double Δ_k – snake. Notice that the diameter of double triangular snake is the same as the diameter of triangular snake graph. Let $x_1, x_2, x_3, \ldots, x_n$ be a $2\,\Delta_k$ – snake of length k, where the diameter of $2(\Delta_k - \text{snake}) = k$ and $n = 3k + 1$.

Define a function h: $V(\text{double}\Delta_k - \text{snake}) \longrightarrow N$ as the following cases:

For $k = 1$, let the sufficed labeling $h(x_1) = 0, h(x_2) = 2, h(x_3) = 5$, and $h(x_4) = 3$

For $k = 2$, let $h(x_1) = 4, h(x_2) = 2, h(x_3) = 0$, $h(x_4) = 5, h(x_5) = 3, h(x_6) = 6$, and $h(x_7) = 7$ □

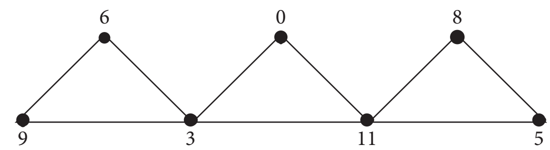

Figure 1: The radio number of Δ_3 (snake).

Case 3. k is even and $k > 2$,

$$h(x_i) = \begin{cases} h(x_{k+1}) = 0, & \\ h(x_{k-i+1}) = ki, & 1 \leq i \leq k, \\ h(x_{k+i+2}) = k^2 + \dfrac{k}{2} - ki, & 0 \leq i \leq k-1, \\ (x_{2k+2+i}) = k^2 + k + 1 + (k-1)i, & 0 \leq i \leq k-1. \end{cases} \quad (13)$$

We are in a position to prove that the function $h(x_i)$ are the distance labeling of double Δ_k – snake.

For each $(i, i+1)$,

$$\begin{aligned} |ki - (k(i+1))| &\geq \text{diam} + 1 - d(x, y), \\ k &\geq k + 1 - d(x, y). \end{aligned} \quad (14)$$

For each $(i, i+1)$,

$$\begin{aligned} \left|k^2 + \frac{k}{2} - ki - \left(k^2 + \frac{k}{2} - k(i+1)\right)\right| &\geq \text{diam} + 1 - d(x, y), \\ k &\geq k + 1 - d(x, y). \end{aligned} \quad (15)$$

Also, for each $(i, i+1)$,

$$\begin{aligned} &\left|k^2 + k + 1 + (k-1)i - \left(k^2 + k + 1 + (k-1)(i+1)\right)\right| \\ &\geq \text{diam} + 1 - d(x, y), \\ k - 1 &\geq k + 1 - d(x, y). \end{aligned} \quad (16)$$

Now, suppose that $1 \leq i \leq k$ and $0 \leq j \leq k-1$:

$$\begin{aligned} \left|ki - \left(k^2 + \frac{k}{2} - kj\right)\right| &\geq \text{diam} + 1 - d(x, y), \\ k^2 + \frac{3k}{2} &\geq k + 1 - d(x, y). \end{aligned} \quad (17)$$

Suppose that $1 \leq i \leq k$ and $0 \leq j \leq k-1$:

$$\begin{aligned} \left|ki - \left(k^2 + k + 1 + (k-1)j\right)\right| &\geq \text{diam} + 1 - d(x, y), \\ k^2 - k + 2 &\geq k + 1 - d(x, y). \end{aligned} \quad (18)$$

If $i = 1$ and $j = 0$,

$$k^2 + 1 \geq k + 1 - d(x, y), \quad (19)$$

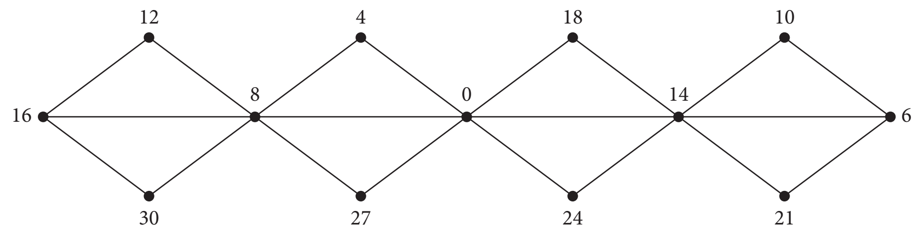

Figure 2: The radio number of double triangular Δ_3 (snake).

otherwise,

$$k+1 \geq k+1-d(x,y),$$

$$1 \leq i \leq k-1 \text{ and } 0 \leq j \leq k-1,$$

$$\left|k^2+\frac{k}{2}-ki-\left(k^2+k+1+(k-1)j\right)\right| \geq \operatorname{diam}+1-d(x,y),$$

$$2k^2-\frac{5}{2}k+2 \geq k+1-d(x,y). \tag{20}$$

If $i=0$ and $j=k-1$,

$$2k^2-k+2 \geq k+1-d(x,y), \tag{21}$$

otherwise,

$$k^2-\frac{k}{2}+1 \geq k+1-d(x,y). \tag{22}$$

Case 4. k is odd and $k>1$:

$$h(x_i)=\begin{cases} h(x_{k+1})=0, & \\ h(x_{k-i+1})=ki, & 1 \leq i \leq k, \\ h(x_{k+i+2})=k^2+\dfrac{k+1}{2}-ki, & 0 \leq i \leq k-1, \\ (x_{2k+2+i})=k^2+k+2-(k-1)i, & 0 \leq i \leq k-1. \end{cases} \tag{23}$$

For each $(i,i+1)$,

$$|ki-(k(i+1))| \geq \operatorname{diam}+1-d(x,y),$$
$$k \geq k+1-d(x,y). \tag{24}$$

For each $(i,i+1)$,

$$\left|k^2+\frac{k+1}{2}-ki-\left(k^2+\frac{k+1}{2}-k(i+1)\right)\right|$$
$$\geq \operatorname{diam}+1-d(x,y), \tag{25}$$
$$k \geq k+1-d(x,y).$$

Also, for each $(i,i+1)$,

$$\left|k^2+k+2-(k-1)i-\left(k^2+k+2-(k-1)(i+1)\right)\right|$$
$$\geq \operatorname{diam}+1-d(x,y),$$
$$k-1 \geq k+1-d(x,y). \tag{26}$$

Now, suppose that $1 \leq i \leq k$ and $0 \leq j \leq k-1$:

$$\left|ki-\left(k^2+\frac{k+1}{2}-kj\right)\right| \geq \operatorname{diam}+1-d(x,y),$$
$$k^2-\frac{3k}{2}-\frac{1}{2} \geq k+1-d(x,y), \tag{27}$$

otherwise,

$$\left|ki-\left(k^2+\frac{k+1}{2}-kj\right)\right| \geq \operatorname{diam}+1-d(x,y),$$
$$\frac{k}{2}+\frac{1}{2} \geq \operatorname{diam}+1-d(x,y). \tag{28}$$

Suppose that $1 \leq i \leq k$ and $0 \leq j \leq k-1$:

$$\left|ki-\left(k^2+k+2-(k-1)j\right)\right| \geq \operatorname{diam}+1-d(x,y),$$
$$k^2-3k-1 \geq k+1-d(x,y). \tag{29}$$

If $i=1$ and $j=k-1$,

$$2k-1 \geq k+1-d(x,y), \tag{30}$$

otherwise,

$$\left|ki-\left(k^2+k+2-(k-1)j\right)\right| \geq \operatorname{diam}+1-d(x,y),$$
$$k+2 \geq k+1-d(x,y). \tag{31}$$

Suppose that $0 \leq i \leq k-1$ and $0 \leq j \leq k-1$:

$$\left|k^2+\frac{k+1}{2}-ki-\left(k^2+k+2-(k-1)j\right)\right|$$
$$\geq \operatorname{diam}+1-d(x,y), \tag{32}$$
$$\frac{3}{2}k+\frac{1}{2} \geq k+1-d(x,y).$$

If $i=0$ and $j=k-1$,

```
Input: G be an n-vertex simple connected graph, k be a positive integer, and the adjacency matrix A[n][n] of G
Output: A radio k-coloring of G.
Begin
  Compute the distance matrix D[n][n] of G using Floyed–Warshall's algorithm and the adjacency matrix A[n][n] of G.
  RadioNumber = ∞;
  for l = 1 to n do
    for i = 1 to n do
      labeling [i] = 0;
    end
    for i = 1 to n do
        for j = 1 to n do
          c[i][j] = diam + 1 − D[i][j];
        end
        c[i][j] = ∞;
    end
    for i = 2 to n do
/*find the minimum value m of the column with position p*/
        [m, p] = min [c(l, :)];
        for j = 1 to n
            c[p][j] = c[p][j] + m
            if c[p][j] < c[l][j]
              c[p][j] = c[l][j]
            end
        end
        labeling [p] = m
        l = p
    end
      /* find the max value of the labeling */
        Max_Value = max (labeling))
        if RadioNumber > Max_Value
          RadioNumber = Max_Value
        end
  end
End
```

Algorithm 1: [2] Finding a radio k-coloring of a graph.

Input: G be an n-vertex graph, simple connected graph, and the diameter of (diam).
Output: an upper bound of radio number of G.
Begin
Step 1: choose a vertex u and $\mathrm{col}(u) = \mathrm{floor}(\sqrt{\mathrm{diam}})$.
Step 2: $S = \{u\}$.
Step 3: for all $v \in V(G) - S$, compute
$\mathrm{temp}(v) = \max_{t \in s}\left\{\mathrm{col}(t) + \max\left\{\sqrt{(D + 1 - d(u, v), 1}\right\}\right\}$.
Step 4: let $\min = \min_{v \in V(G)-S}\{\mathrm{temp}(v)\}$.
Step 5: choose a vertex $v \in V(G) - S$, such that $\mathrm{temp}(v) = \min$.
Step 6: give $\mathrm{col}(v) = \min$.
Step 7: $S = S \cup \{v\}$
Step 8: repeat Step 3 to Step 6 until all vertices are labeled.
Step 9: repeat Step 1 to Step 7 for every vertex $x \in V(G)$.
End

Algorithm 2: [3] Finding an upper bound of the radio number of a graph G.

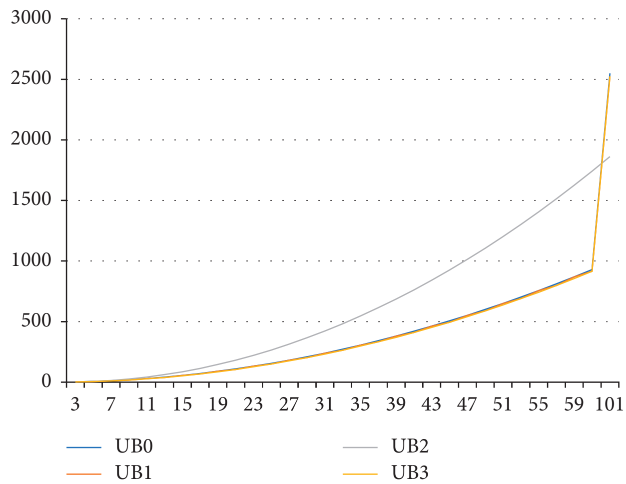

Figure 3: Comparison among UB0, Ub1, UB2, and UB3 for the radio number of triangular snake.

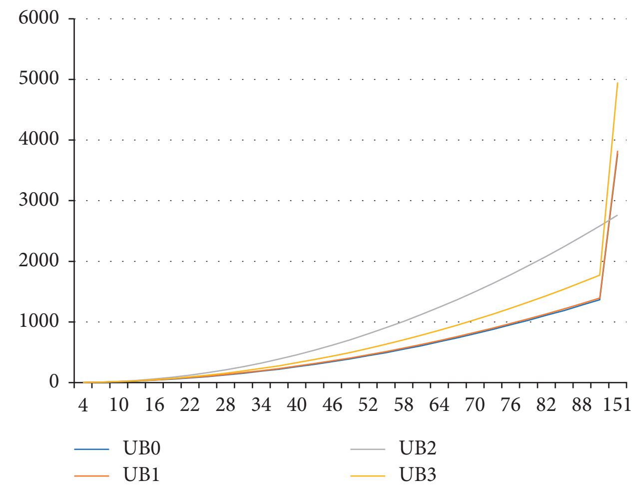

Figure 4: Comparison among UB0, Ub1, UB2, and UB3 for the radio number of double triangular snake.

Table 1: Comparison between standard radio number and the upper bound of radio number for the triangular snake graph.

k	n	UB0 [2]		UB1 [3]		UB2 [1]		UB3	
		rn	CPU time	rn	CPU time	rn	CPU time	rn	CPU time
1	3	2	0.007704	2	0.0040755	2	0.0017378	2	O(1)
2	5	6	0.004364	5	0.004255	6	0.002382	5	O(1)
3	7	12	0.00612	12	0.00555	14	0.005265	11	O(1)
4	9	20	0.024677	18	0.012055	26	0.005364	18	O(1)
5	11	30	0.051972	30	0.013155	42	0.005659	28	O(1)
6	13	42	0.230102	39	0.013621	62	0.005799	39	O(1)
7	15	56	0.307545	56	0.013755	86	0.006905	53	O(1)
8	17	72	0.308318	68	0.014298	114	0.007153	68	O(1)
9	19	90	0.468408	90	0.014749	146	0.007203	86	O(1)
10	21	110	0.751755	105	0.01493	182	0.00743	105	O(1)
11	23	132	0.924785	132	0.015201	222	0.007672	127	O(1)
12	25	156	1.219799	150	0.015988	266	0.008343	150	O(1)
13	27	182	1.674918	182	0.016105	314	0.008354	176	O(1)
14	29	210	2.689564	203	0.016197	366	0.008496	203	O(1)
15	31	240	3.224403	240	0.016201	422	0.008661	233	O(1)
16	33	272	3.567201	264	0.01711	482	0.009033	264	O(1)
17	35	306	4.447875	306	0.017625	546	0.009587	298	O(1)
18	37	342	5.561139	333	0.018556	614	0.009701	333	O(1)
19	39	380	6.933108	380	0.019099	686	0.009929	371	O(1)
20	41	420	8.345423	410	0.020327	762	0.010011	410	O(1)
21	43	462	12.868485	462	0.021287	842	0.010364	452	O(1)
22	45	506	13.787422	495	0.021983	926	0.011044	495	O(1)
23	47	552	15.946642	552	0.022126	1014	0.011472	541	O(1)
24	49	600	20.145523	588	0.029388	1106	0.011726	588	O(1)
25	51	650	22.931427	650	0.049946	1202	0.012145	638	O(1)
26	53	702	26.792638	689	0.053599	1302	0.013162	689	O(1)
27	55	756	30.007477	756	0.058895	1406	0.013417	743	O(1)
28	57	812	33.778689	798	0.060056	1514	0.013437	798	O(1)
29	59	870	39.570408	870	0.089137	1626	0.017043	856	O(1)
30	61	930	45.216577	915	0.148602	1742	0.026953	915	O(1)
50	101	2550	342.011401	2525	0.282964	1862	0.259496	2525	O(1)

Table 2: Comparison between standard radio number and for the upper bound of the radio number for the double triangular snake graph.

k	n	UB0 [2]		UB1 [3]		UB2 [1]		UB3	
		rn	CPU time	rn	CPU time	rn	CPU time	rn	CPU time
1	4	2	0.008905	3	0.013994	2	0.007179	3	O(1)
2	7	7	0.009302	8	0.014494	8	0.008093	7	O(1)
3	10	15	0.012654	17	0.015187	19	0.010171	18	O(1)
4	13	26	0.01432	29	0.016028	36	0.011224	30	O(1)
5	16	41	0.022551	44	0.026618	59	0.012734	48	O(1)
6	19	57	0.022565	62	0.031761	88	0.014698	68	O(1)
7	22	78	0.024324	83	0.097335	123	0.017663	94	O(1)
8	25	100	0.026327	107	0.099599	164	0.025045	122	O(1)
9	28	127	0.030762	134	0.168036	211	0.025465	156	O(1)
10	31	155	0.038029	164	0.217785	264	0.031522	192	O(1)
11	34	188	0.048976	197	0.312401	323	0.032649	234	O(1)
12	37	222	0.058516	233	0.476330	388	0.033239	278	O(1)
13	40	262	0.067316	272	0.648300	459	0.039245	328	O(1)
14	43	301	0.092605	314	0.831050	536	0.043670	380	O(1)
15	46	347	0.101886	359	1.087081	619	0.045682	438	O(1)
16	49	392	0.157635	407	1.389141	708	0.052569	498	O(1)
17	52	444	0.163901	458	1.740725	803	0.065006	564	O(1)
18	55	495	0.270554	512	2.151552	904	0.131148	632	O(1)
19	58	553	0.287547	569	2.625657	1011	0.156826	706	O(1)
20	61	610	0.325028	629	3.229021	1124	2.637903	782	O(1)
21	64	675	0.350093	692	3.867577	1243	2.655901	864	O(1)
22	67	737	0.355919	758	4.619990	1368	2.676703	948	O(1)
23	70	808	0.369294	827	5.625904	1499	2.698903	1038	O(1)
24	73	876	0.460056	899	6.570240	1636	3.174321	1130	O(1)
25	76	953	0.512382	974	7.552782	1779	3.321324	1228	O(1)
26	79	1027	0.553158	1052	9.093392	1928	3.592834	1328	O(1)
27	82	1110	0.625059	1133	10.404067	2083	3.720172	1434	O(1)
28	85	1190	0.740484	1217	12.811142	2244	4.019234	1542	O(1)
29	88	1280	0.77942	1304	13.942115	2411	4.892321	1656	O(1)
30	91	1365	0.974537	1394	16.356469	2584	5.109283	1772	O(1)
50	151	3775	4.497669	3824	130.59526	2763	9.981278	4952	O(1)

$$\left|k^2 + \frac{k+1}{2} - ki - \left(k^2 + k + 2 - (k-1)j\right)\right| \geq \text{diam} + 1 - d(x, y), \tag{33}$$

$$k^2 - \frac{5}{2}k - \frac{1}{2} \geq k + 1 - d(x, y),$$

otherwise,

$$k^2 - \frac{1}{2}k + \frac{3}{2} \geq k + 1 - d(x, y). \tag{34}$$

Example 2. Figure 2 presents the labeling double triangular Δ_3 (snake) according to Theorem 2.

4. Computational Study

In order to evaluate the proposed upper bounds presented by Theorems 1 and 2, we make a numerical experiment between the proposed results and the results of [1–3]. This experiment applies on two graphs (triangular snake and double triangular snake). The description of the environment is as follows: MATLAB R2016a with default options and all runs were carried out under MS Windows 7 Professional system, having Intel® Core™ i3-3217U CPU@ 1.80 GHz and 4 Gb RAM.

In Tables 1 and 2, the abbreviations Ub0, Ub1, Ub2, and Ub3 are used to denote upper bounds are due to the works of Saha and Panigrahi [2], Saha and Panigrahi [3], Badr and Moussa [1], and the proposed algorithm, respectively.

Table 1 and Figure 3 show that the proposed upper bound Ub3 overcomes the upper bound UB0 and UB2 which is due to the works of Saha and Panigrahi [2] and Badr and Moussa [1], respectively. On the contrary, the proposed upper bound Ub3 overcomes the upper bound UB1 (for k is odd only) which is due to the works of Saha and Panigrahi [2]. The upper bound (for k is even) of the UB3 and UB1 are equal.

Table 1 and Figure 4 explain that the proposed upper bounds outperform all results of UB0, UB1, and UB2 according to CPU time. On the contrary, the mathematical model UB2 [1] overcomes UB0 and UB1.

5. Conclusions

In this study, the upper bounds for the radio number of the triangular snake and the double triangular snake graphs are introduced. The computational results indicate that the presented upper bounds are better than the results of the mathematical model provided by Badr and Moussa [1]. On the contrary, these proposed upper bounds are better than the results of algorithms presented by Saha and Panigrahi [2, 3].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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