Research Article

Gorenstein-Projective Modules over a Class of Morita Rings

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Let \( \Delta_{(0,0)} \) be a Morita ring such that the bimodule homomorphisms are zero. In this paper, we give sufficient conditions for a \( \Delta_{(0,0)} \)-module \((X, Y, f, g)\) to be Gorenstein-projective. As an application, we give sufficient conditions when the algebras \( A \) and \( B \) inherit the strongly CM-freeness of \( \Delta_{(0,0)} \).

1. Introduction

Gorenstein algebra and Gorenstein-projective modules are important topics of research in Gorenstein homological algebra. A fundamental problem in Gorenstein homological algebra is determining all the Gorenstein-projective \( A \)-modules for a given algebra \( A \). The class of Gorenstein-projective modules is a key component of relative homological algebra and has received a great deal of attention in the study of representation theory (e.g., [1–6, 8–13, 16–18, 20, 23–27]).

For algebras \( A \) and \( B \), bimodules \( B_M A \) and \( A_N B \), and a \( B-B \)-bimodule map \( \phi: M \otimes_A N \rightarrow B \), and an \( A-A \)-bimodule map \( \psi: N \otimes_B M \rightarrow A \) satisfying some special conditions. Bass [7] introduced Morita algebra \( \Delta_{(\phi,\psi)} = \begin{pmatrix} A & AN_B \\ BM_A & B \end{pmatrix} \), where the special conditions for \( \phi \) and \( \psi \) are to guarantee that the multiplication of \( \Delta_{(\phi,\psi)} \) has the associativity. Morita algebras \( \Delta_{(\phi,\psi)} \) give a very large class of algebras, and many important algebras can be realized as Morita algebras. For example, the \( 2 \times 2 \) matrix algebra \( M_2(A) = \begin{pmatrix} A & A \\ A & A \end{pmatrix} \) over \( A \), the algebra \( \Delta_{(0,0)} = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix} \), the upper triangular matrix algebra \( \begin{pmatrix} A & AN_B \\ 0 & B \end{pmatrix} \), the algebras defined by finite quivers and relations. Thus, researching Morita rings is pivotal.

Asefa [1] obtained sufficient conditions for Gorenstein-projective module \((X, Y, f, g)\) over \( \Delta_{(\phi,\psi)} \), implying that \( X \) is a Gorenstein-projective \( A \)-module and \( Y \) is a Gorenstein-projective \( B \)-module. Gao and Psaroudakis [13] constructed Gorenstein-projective modules over a Morita ring \( \Delta_{(0,0)} \). They stated that [13], [Theorem 3.10] does not give sufficient conditions for a module \((X, Y, f, g)\) Gorenstein-projective \(([13], \text{Remark 3.13})\). As a result, it is natural to ask, "When is a module \((X, Y, f, g)\) Gorenstein-projective?". This paper is motivated to answer this question. In the following main result, we give sufficient conditions for \((X, Y, f, g)\) to be a Gorenstein-projective module over a Morita ring \( \Delta_{(0,0)} \).

**Theorem 1.** Let \( \Delta_{(0,0)} \) be a Morita ring. Assume that

(i) \( M_A \) and \( N_B \) have finite flat dimensions.
(ii) \( BM \) and \( AN \) have finite projective dimensions.

Then, if each of the following conditions holds, a \( \Delta_{(0,0)} \)-module \((X, Y, f, g)\) is Gorenstein-projective.

(1) \( \text{Coker} g \) is Gorenstein-projective \( A \)-module;
(2) \( \text{Coker} f \) is Gorenstein-projective \( B \)-module; and
(3) \( M \otimes_A \text{Coker} g \cong \text{Im} f \) and \( N \otimes_B \text{Coker} f \cong \text{Im} g \).
Lastly, we give sufficient conditions when the algebras \( A \) and \( B \) inherit the strongly CM-freeness of \( \Delta_{(0,0)} \).

2. Preliminaries

This section discusses some basic definitions and facts that will be used throughout the paper.

Throughout, rings mean a ring with unity and an \( R \)-module mean a left \( R \)-module. Let \( R \) be a ring. Let \( M \) be an \( R \)-module, then the projective(injective and flat) dimension of \( M \) will be denoted by \( \text{projdim} \) \( M \) \( (\text{injdim} \) \( M \) \( \) and \( \text{flatdim} \) \( M \) \). The class of modules isomorphic to direct sums of copies of \( M \) is denoted by \( \text{Add}(M) \).

An \( R \)-module \( M \) is Gorenstein-projective if there exists an exact sequence of projective \( R \)-modules
\[
\mathcal{P}^* = \cdots \longrightarrow P^{-1} \longrightarrow p^0 \xrightarrow{d^0} P^1 \longrightarrow \cdots
\]
such that \( \text{Hom}_R(\mathcal{P}^*, Q) \) is exact for an arbitrary projective \( R \)-module \( Q \) and \( M \cong \text{Ker} d^0 \). The class of Gorenstein-projective \( R \)-modules will be denoted by \( \text{GProj}_R \).

Let \( A \) and \( B \) be rings, \( ANB \) and \( BMA \) bimodules, and \( \phi: ANA \longrightarrow B \) and \( \psi: NBB \longrightarrow A \) bimodules homomorphism. This class of Gorenstein-projective \( R \)-modules will be denoted by \( \text{GProj}_R \).

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2.2. We Now Recall Functors Given in [16]

(1) The functor \( T_A: A\text{-Mod} \rightarrow \Delta_{(0,0)}\text{-Mod} \) is given by \( T_A(X) = (X, M \otimes_A X, 1, 0) \) for any object \( X \) in \( A\text{-Mod} \).

2.1. Modules over \( \Delta_{(0,0)} \): A left module over \( \Delta_{(0,0)} \) is given as \( (X, Y, f, g) \), where \( X \) is an \( A \)-module, \( Y \) is a \( B \)-module, and
\[
\begin{align*}
f &: M \otimes_A X \longrightarrow Y, \\
g &: N \otimes_B Y \longrightarrow X,
\end{align*}
\]
where \( g \) is an \( A \)-module map and \( f \) is a \( B \)-map.

A \( \Delta_{(0,0)} \)-module morphism is given by \( (a, b): (X, Y, f, g) \longrightarrow (X', Y', f', g') \), where \( a: X \longrightarrow X' \) is a homomorphism in \( A\text{-Mod} \) and \( b: Y \longrightarrow Y' \) is a homomorphism in \( B\text{-Mod} \) such that the following diagrams are commutative.

\[
\begin{array}{ccc}
M \otimes_A X & \xrightarrow{f} & Y \\
\downarrow \text{Id}_M \otimes a & & \downarrow b \\
M \otimes_A X' & \xrightarrow{f'} & Y'
\end{array}
\]

\[
\begin{array}{ccc}
N \otimes_B Y & \xrightarrow{g} & X \\
\downarrow \text{Id}_N \otimes b & & \downarrow a \\
N \otimes_B Y' & \xrightarrow{g'} & X'
\end{array}
\]

Lemma 1 (see [13]). Let \( \Delta_{(\phi, \psi)} = \left( \begin{array}{c} A \\ B \end{array} \right) \) be a Morita ring.

1. A sequence \( 0 \longrightarrow (X'', Y'', f'', g'') \longrightarrow (X', Y', f', g') \longrightarrow (X, Y, f, g) \) is exact in \( \Delta\text{-Mod} \) if and only if the sequence \( 0 \longrightarrow X'' \longrightarrow X \longrightarrow X' \longrightarrow 0 \) is exact in \( A\text{-Mod} \) and the sequence \( 0 \longrightarrow Y'' \longrightarrow Y \longrightarrow Y' \longrightarrow 0 \) is exact in \( B\text{-Mod} \).

2. Let \( (a, b): (X, Y, f, g) \longrightarrow (X_1, Y_1, f_1, g_1) \) a morphism in \( \Delta\text{-Mod} \) and consider the maps \( \alpha: \text{Ker} a \longrightarrow X \) and \( \text{Ker} \beta \longrightarrow Y \).

The case \( \phi = 0 = \psi \) is a subclass of the general Morita rings(e.g., [7, 13–15, 21].)

Similarly, the Cokernel of \( (a, b) \) can be described.

2.2. We Now Recall Functors Given in [16]

(1) The functor \( T_A: A\text{-Mod} \rightarrow \Delta_{(0,0)}\text{-Mod} \) is given by \( T_A(X) = (X, M \otimes_A X, 1, 0) \) for any object \( X \) in \( A\text{-Mod} \).

(2) The functor \( T_B: B\text{-Mod} \rightarrow \Delta_{(0,0)}\text{-Mod} \) is given by \( T_B(Y) = (N \otimes_B Y, 0, 1) \) for any object \( Y \) in \( B\text{-Mod} \).

(3) The functor \( U_A: \Delta_{(0,0)}\text{-Mod} \rightarrow A\text{-Mod} \) is given by \( U_A(X, Y, f, g) = X \) for any \( (X, Y, f, g) \) in \( \Delta_{(0,0)}\text{-Mod} \).
(4) The functor $U_B: \Delta_{(0,0)}\text{-Mod} \to B\text{-Mod}$ is given by
$U_B(X, Y, f, g) := Y$ for any object $(X, Y, f, g)$ in
$\Delta_{(0,0)}\text{-Mod}$.

(5) Let $X \in A$ be any object in $\text{Mod}$, then we denote by
$\epsilon_X: N \otimes_B \text{Hom}_A(N, X) \to X$ the map $A$-module
given by involution. The functor $H_A: A\text{-Mod} 
\to \Delta_{(0,0)}\text{-Mod}$ is given by
$H_A(X) := (X, \text{Hom}_A(N, X), 0, \epsilon_X)$ for any object $X$ in
$A\text{-Mod}$.

(6) Let $Y$ be any object in $B\text{-Mod}$, the we denote by
$\epsilon_Y: M \otimes_A \text{Hom}_B(M, Y) \to Y$ the map $B$-module
given by involution. The functor $H_B: B\text{-Mod} 
\to \Delta_{(0,0)}\text{-Mod}$ is given by
$H_B(Y) := (\text{Hom}_B(M, Y), Y, \epsilon_Y, 0)$ for any object $Y$ in
$B\text{-Mod}$.

(7) The functor $Z_A: A\text{-Mod} \to \Delta_{(0,0)}\text{-Mod}$ is defined by
$Z_A(X) := (X, 0, 0, 0)$ for any object $X$ in $A\text{-Mod}$. The functor
$Z_B: B\text{-Mod} \to \Delta_{(0,0)}\text{-Mod}$ can be similarly defined.

More information about the functors given above can be found in the following result.

**Proposition 1** ([16], Proposition 2.4]). Let $\Delta_{(0,0)}$ be Morita ring. Then,

1. The functors $H_A, H_B, T_A,$ and $T_B,$ are fully faithful.
2. The pairs $(U_A, H_A), (U_B, H_B), (T_A, U_A),$ and $(T_B, U_B)$ are adjoint pairs.
3. The functors $U_A$ and $U_B$ are exact.

**Lemma 2.** Let $\Delta_{(0,0)}$ be Morita ring.

1. ([19], [Theorem 7.3]) A left $\Delta_{(0,0)}$-module $(P, Q, f, g)$ is projective if and only if
$(P, Q, f, g) = T_A(X) \oplus T_B(Y) = (X, M \otimes_A X, 1, 0) \oplus (Y, N \otimes_B Y, Y, 0, 1)$ for
some projective left $A$-module $X$ and projective left $B$-module $Y$.

2. ([22], [Corollary 2.2]) A left $\Delta_{(0,0)}$-module $(I, J, f, g)$ is injective if and only if
$(I, J, f, g) = H_A(X) \oplus H_B(Y) = (X, \text{Hom}_A(N, X), 0, \epsilon_X) \oplus (\text{Hom}_B(M, Y), Y, \epsilon_Y, 0)$ for some injective left $A$-module $X$ and injective left $B$-module $Y$.

**3. Gorenstein-Projective Modules over $\Delta_{(0,0)}$**

This section aims to construct Gorenstein-projective modules over $\Delta_{(0,0)}$.

The following lemmas are required in order to prove the main theorems of this paper.

**Lemma 3.** Let $A$ be a ring and $M$ a $B$-$A$-bimodule with finite flat dimension. If a complex of flat $A$-modules $F^\bullet$ is exact, then, the sequence $M \otimes_A F^\bullet$ is also exact.

**Proof.** Assume that $F^\bullet$ is an exact complex of flat $A$-modules. Because $M$ has a finite flat dimension, we have the following flat resolution of $M$.

$0 \to F^0 \to F^{n-1} \to \cdots \to F^0 \to M_A \to 0. \ (7)$

We obtain the following exact sequence of complexes because all terms in the complex $F^\bullet$ are flat.

$0 \to F^0 \otimes_A F^\bullet \to F^{n-1} \otimes_A F^\bullet \to \cdots \to F^0 \otimes_A F^\bullet \to M_A \otimes_A F^\bullet \to 0. \ (8)$

Since the complexes $F^i \otimes_A F^\bullet$ are exact for all $i$, so is $M \otimes_A F^\bullet$. □

**Lemma 4.** Let $B$ be a ring. If a $B$-module $N$ has finite injective dimension and the complex of projective $B$-modules,

$Q^\bullet := \cdots \to Q^{n+1} \to Q^n \to Q^{n+1} \to \cdots, \ (9)$

is exact, then so is $\text{Hom}_B(Q^\bullet, N)$.

**Lemma 5.** Let $\Delta_{(0,0)}$ be a Morita ring with zero bimodule homomorphisms. Then

1. ([13], [Lemma 3.8]) For each $X \in A$-$\text{Mod}$ and each $Y \in B$-$\text{Mod}$ we have the following exact sequences in $\Delta_{(0,0)}$-$\text{Mod}$.

$0 \to Z_A(M \otimes_A X) \to T_A(X) \to Z_A(X) \to 0. \ (10)$

and

$0 \to Z_A(N \otimes_B Y) \to T_B(Y) \to Z_B(Y) \to 0. \ (11)$

2. ([13], [Lemma 3.9]) For all $X, X' \in A$-$\text{Mod}$ and $Y, Y' \in B$-$\text{Mod}$, we have the following isomorphisms:

$\text{Hom}_A(X, X') \equiv \text{Hom}_A(Y, Y'). \ (12)$

and

$\text{Hom}_B(Y, Y') \equiv \text{Hom}_B(Y, Y'). \ (13)$

The following result provides sufficient conditions for the functor $T_A: A$-$\text{Mod} \to \Delta_{(0,0)}$-$\text{Mod}$ and the functor $T_B: B$-$\text{Mod} \to \Delta_{(0,0)}$-$\text{Mod}$ to preserve Gorenstein-projective modules.

**Proposition 2**

1. Assume that $M_A$ has a finite flat dimension and that $A_N$ has a finite projective dimension. $T_A(X)$ is a Gorenstein-projective $\Delta_{(0,0)}$-module if $X$ is a Gorenstein-projective $A$-module.

2. Assume that $N_B$ has a finite flat dimension and that $B_M$ has a finite projective dimension. $T_B(Y)$ is a Gorenstein-projective $\Delta_{(0,0)}$-module if $Y$ is a Gorenstein-projective $B$-module.
Proof. We show (1) and (2) can be proved in a similar manner. Since an $A$-module $X$ is a Gorenstein-projective, there is an exact sequence of projective $A$-modules,
\[
\varphi^* : \cdots \rightarrow P^{-1} \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots ,
\]
(14)
such that $T_A(X) \equiv \text{Ker}(d^0, 1 \otimes d^0)$. Now, it is left to show that $\text{Hom}_{A_{(0)}}(T_A(\varphi^*), (X', Y', f', g'))$ is exact for any projective $\Delta_{(0,0)}$-module $(X', Y', f', g')$. By Lemma 2, this can be proved by showing the exactness of $\text{Hom}_{A_{(0)}}(T_A(\varphi^*), T_A(\varphi^*))$ and $\text{Hom}_{A_{(0)}}(T_A(\varphi^*), T_B(Q))$ for any projective $A$-module $P$ and any projective $B$-module $Q$. By Proposition 1 the functor $T_A$ is fully faithful. Thus, $\text{Hom}_{A_{(0)}}(T_A(\varphi^*), T_A(\varphi^*)) \cong \text{Hom}_A(\varphi^*, \varphi^*)$. Hence $\text{Hom}_{A_{(0)}}(T_A(\varphi^*), T_A(\varphi^*))$ because $\text{Hom}_A(\varphi^*, \varphi^*)$ is exact. Since $(T_A, U_A)$ are adjoint pairs, we have the following equation:
\[
\text{Hom}_{A_{(0)}}(T_A(\varphi^*), T_B(Q)) \cong \text{Hom}_A(\varphi^*, N \otimes_B Q).
\]
(16)  

A module $N \otimes_B Q$ has finite projective dimension because it is isomorphic to a direct summand of direct sums of copies of $N$. Since $\varphi^*$ is a complete $A$-projective resolution, the complex $\text{Hom}_A(\varphi^*, N \otimes_B Q)$ is exact (see [18], Proposition 2). Thus, $\text{Hom}_{A_{(0)}}(T_A(\varphi^*), T_B(Q))$ is exact. Hence $\text{Hom}_{A_{(0)}}(T_A(\varphi^*), (X', Y', f', g'))$ is exact for any projective $\Delta_{(0,0)}$-module $(X', Y', f', g')$. Therefore, $T_A(X)$ is a Gorenstein-projective $\Delta_{(0,0)}$-module. □

In the following result, we give sufficient conditions for a $\Delta_{(0,0)}$-module $(X, Y, f, g)$ to be Gorenstein-projective.

Theorem 2. Let $\Delta_{(0,0)}$ be a Morita ring. Assume that
(i) $M_A$ and $N_B$ have finite flat dimensions.
(ii) $B$ and $A$ have finite projective dimensions.

Then, if each of the following conditions holds, a $\Delta_{(0,0)}$-module $(X, Y, f, g)$ is Gorenstein-projective.
(1) $\text{Coker} g$ is a Gorenstein-projective $A$-module;
(2) $\text{Coker} f$ is a Gorenstein-projective $B$-module; and
(3) $M \otimes_A \text{Coker} g \cong \text{Im} f$ and $N \otimes_B \text{Coker} f \cong \text{Im} g$.

Proof. Suppose that conditions (1)–(3) are true. Since $\text{Coker} f$ is a Gorenstein-projective $B$-module, there exists an exact complex of projective $B$-modules,
\[
\varphi^* : \cdots \rightarrow Q^{-1} \rightarrow Q^0 \rightarrow Q^1 \rightarrow \cdots ,
\]
(17)
such that $X \equiv \text{Ker} d^0$, and $\text{Hom}_A(\varphi^*, Q)$ exact for any projective $A$-module $Q$. Lemma 3 states that the assumption that $M_A$ has finite flat dimension implies the sequence $M \otimes_A \varphi^*$ is exact. Hence we get the exact sequence of projective $\Delta_{(0,0)}$-modules,
\[
\varphi^* : \cdots \rightarrow P^{-1} \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots ,
\]
(15)
such that $\text{Coker} f \equiv \text{Ker} d^0$ and $\text{Hom}_B(\varphi^*, Q)$ is exact for each projective $B$-module $Q$. Thus, we get the following exact sequence,
\[
0 \rightarrow N \otimes_B \text{Coker} f \rightarrow N \otimes_B Q \rightarrow \text{Id} \otimes d^0 \rightarrow N \otimes_B Q^1 \rightarrow \cdots ,
\]
(18)
for any projective $B$-module $Q$. Since $\text{Coker} g$ is a Gorenstein-projective $A$-module, there exists a complete projective resolution,
\[
\varphi^* : \cdots \rightarrow P^{-1} \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots ,
\]
(19)
of $A$-modules such that $\text{Coker} g \equiv \text{Ker} d^0$.

Let $\pi_1 : X \rightarrow \text{Coker} g$ and $\pi_2 : Y \rightarrow \text{Coker} f$. Consider the following commutative diagram of $A$-modules.
\[
\begin{array}{ccc}
0 & \rightarrow & N \otimes_B \text{Coker} f \\
\downarrow & & \downarrow \\
0 & \rightarrow & N \otimes_B Q \rightarrow \text{Id} \otimes d^0 \rightarrow N \otimes_B Q^1 \rightarrow \cdots \\
\end{array}
\]
(20)

Since $\psi = 0$, the above equation implies that there exists an $A$-map $i_1 : N \otimes_B \text{Coker} f \rightarrow X$ that is unique and $g = i_1 \circ (\text{Id}_N \otimes \pi_2)$. Thus, from $\text{Im} g \cong N \otimes_B \text{Coker} f$ it follows that $i_1$ is an injective $A$-map. Thus, we get the exact sequence as follows:
\[
0 \rightarrow N \otimes_B \text{Coker} f \stackrel{i_1}{\rightarrow} X \rightarrow \text{Coker} g \rightarrow 0.
\]
(21)

Similarly, the sequence
\[
0 \rightarrow M \otimes_A \text{Coker} g \stackrel{i_2}{\rightarrow} Y \rightarrow \text{Coker} f \rightarrow 0,
\](22)
is exact.

Since each $N \otimes_B Q^i$ has finite projective dimension, and since each $\text{Ker} d^i$ is a Gorenstein-projective $A$-module, we have that $\text{Ext}_A^i(\text{Ker} d^i, N \otimes_B Q^j) = 0, \forall i \geq 0$. Applying generalized Horseshoe Lemma([26], Lemma 1.6 (ii)) to the exact sequences (18) and (21), we obtain an exact sequence as follows:
\[
0 \rightarrow X \rightarrow P^0 \otimes (N \otimes_B Q^0) \rightarrow P^1 \otimes (N \otimes_B Q^1) \rightarrow \cdots .
\]
(23)

with $a_i = \begin{pmatrix} d_i & 0 \\ \gamma_i & \text{Id}_N \otimes d_i \\ \end{pmatrix}, \gamma_i : p^i \rightarrow N \otimes_B Q^{i+1}, \forall i \in \mathbb{Z}$, such that the following diagram
is commutative. The dual argument obtains the commutative diagram with exact rows shown below.

\[ \cdots \longrightarrow N \otimes_B Q^{-2} \overset{1d \otimes d^{-2}}\longrightarrow N \otimes_B Q^{-1} \overset{1d \otimes d^{-1}}\longrightarrow N \otimes_B \text{Coker} f \longrightarrow 0 \]
\[ \cdots \longrightarrow P^{-2} \oplus N \otimes_B Q^{-2} \overset{\alpha^{-2}}\longrightarrow P^{-1} \oplus N \otimes_B Q^{-1} \longrightarrow X \longrightarrow 0 \]

When we combine (24) and (25), we get the exact sequence shown below.

\[ \cdots \longrightarrow P^{-2} \oplus N \otimes_B Q^{-2} \overset{\alpha^{-2}}\longrightarrow P^{-1} \oplus N \otimes_B Q^{-1} \longrightarrow Y \longrightarrow 0 \]

with \( \text{Ker} \alpha^0 = X \).

We now construct an exact sequence similar to (26) for a left \( B \)-module \( Y \). Since each \( M \otimes_A P^i \) has finite projective dimension as \( B \)-module by assumption on \( M \), and \( \text{Ker} d^i \) is a Gorenstein-projective \( B \)-module, it follows that \( \text{Ext}_B^1(\text{Ker} d^i, M \otimes_A P^i) = 0 \). Thus, by ([26], Lemma 1.6 (ii)) again, we obtain the exact sequence as follows:

\[ \cdots \longrightarrow 0 \longrightarrow Y \longrightarrow (M \otimes_A P_0) \otimes Q^0 \overset{\beta^0}\longrightarrow (M \otimes_A P_1) \otimes Q^1 \longrightarrow \cdots \]

with \( \beta^0 = \begin{pmatrix} 1d_M \otimes d^{-1} & \sigma^i \\ 0 & d^i \end{pmatrix} \), and \( \sigma^i : Q^i \longrightarrow M \otimes_A P^{i+1} \), \( \forall i \in \mathbb{Z} \), such that the diagram

\[ \cdots \longrightarrow M \otimes_A P^{-2} \overset{1d \otimes d^{-2}}\longrightarrow M \otimes_A P^{-1} \overset{1d \otimes d^{-1}}\longrightarrow M \otimes_A \text{Coker} g \longrightarrow 0 \]
\[ \cdots \longrightarrow M \otimes_A P^{-2} \oplus Q^{-2} \overset{\beta^{-2}}\longrightarrow M \otimes_A P^{-1} \oplus Q^{-1} \longrightarrow Y \longrightarrow 0 \]

As a result, combining (28) and (29) yields the following exact sequence, which is similar to the following equation:
\[ \cdots \to M \otimes_A P^{-2} \otimes Q^{-2} \xrightarrow{\beta^{-2}} M \otimes_A P^{-1} \otimes Q^{-1} \xrightarrow{\beta^{-1}} M \otimes_A P^0 \otimes Q^0 \xrightarrow{\beta^0} M \otimes_A P^1 \otimes Q^1 \xrightarrow{\beta^1} \cdots \] (30)

with \( \ker \beta^0 = Y \).

Glue together the exact sequences (26) and (30) to obtain the following sequence:

\[ \mathcal{F}^* : \cdots \to T_A(P^{-1}) \otimes T_B(Q^{-1}) \xrightarrow{(a^{-1} \beta^{-1})} T_A(P^0) \otimes T_B(Q^0) \xrightarrow{(a^0 \beta^0)} \cdots \] (31)

with \( \ker (a^0 \beta^0) = (X, Y, f, g) \).

The morphism \((a^i \beta^i) \forall i \in \mathbb{Z}\) is a \( \Delta_{(0,0)} \)-map because

\[ \\
M \otimes_A (P^i \oplus N \otimes B Q^i) \xrightarrow{(\text{Id}_M \otimes P^i, 0)} M \otimes_A P^i \oplus Q^i \\
\xrightarrow{(\text{Id}_M \otimes (d^i, 0), \text{Id}_N \otimes d^i)} M \otimes_A (P^{i+1} \oplus N \otimes B Q^{i+1}) \xrightarrow{(\text{Id}_M \otimes P^{i+1}, 0)} M \otimes_A P^{i+1} \oplus Q^{i+1} \\
\] (32)

and

\[ \\
N \otimes_B (M \otimes_A P^i \oplus Q^i) \xrightarrow{(0, \text{Id}_N \otimes d^i)} P^i \oplus N \otimes_B Q^i \\
\xrightarrow{(0, \text{Id}_N \otimes (d^i, 0), \text{Id}_N \otimes \sigma^i)} N \otimes_B (M \otimes_A P^{i+1} \oplus Q^{i+1}) \xrightarrow{(0, \text{Id}_N \otimes d^i, 0)} P^{i+1} \oplus N \otimes_B Q^{i+1} \\
\] (33)

are commutative diagrams.

Since the complexes (26) and (30) are exact, it follows from Lemma 1 (1) that the sequence \( \mathcal{F}^* \) is exact. The object \((X, Y, f, g)\) arises as the kernel of the morphism \((a^0 \beta^0)\), and we see from Lemma 1 (2) that \( f = i_1^\circ (\text{Id}_M \otimes \pi_1) \) and \( g = i_1^\circ (\text{Id}_N \otimes \pi_2) \). However, based on the commutative diagram of \( A \)-modules shown below,

\[ N \otimes_B Y \xrightarrow{g} X \xrightarrow{\pi_1} \text{Coker } g \xrightarrow{} 0 \]

(34)

we know that \( g \) is uniquely determined by \( i_1^\circ (\text{Id}_N \otimes \pi_2) \). Similarly, \( f \) is uniquely determined by \( i_1^\circ (\text{Id}_M \otimes \pi_1) \).

We are now left with showing that \( \hom_{\Delta_{(0,0)}}(\mathcal{F}^*, (X', Y', f', g')) \) is exact for each projective \( \Delta_{(0,0)} \)-module \((X', Y', f', g')\). We can deduce from Lemma 2 that it is enough to show that \( \hom_{\Delta_{(0,0)}}(\mathcal{F}^*, T_A(P)) \) and \( \hom_{\Delta_{(0,0)}}(\mathcal{F}^*, T_B(Q)) \) are exact for each projective \( A \)-module \( P \) and for each projective \( B \)-module \( Q \). By Lemma 5 (1) the sequence \( 0 \to Z_B(M \otimes_A P) \xrightarrow{T_A(P)} Z_A(P) \to 0 \) is exact. Since each term in the complex \( \mathcal{F}^* \) is a projective \( \Delta_{(0,0)} \)-module, the sequence

\[ 0 \to \hom_{\Delta_{(0,0)}}(\mathcal{F}^*, Z_B(M \otimes_A P)) \to \hom_{\Delta_{(0,0)}}(\mathcal{F}^*, T_A(P)) \to \hom_{\Delta_{(0,0)}}(\mathcal{F}^*, Z_A(P)) \to 0, \] (35)
is exact. By Lemma 5 (2) we have the following equations,

$$\text{Hom}_{\Delta}(\mathcal{F}^*, Z_A(P)) \cong \text{Hom}_A(\mathcal{F}^*, P).$$

(36)

The complex $\text{Hom}_A(\mathcal{F}^*, P)$ is exact because $\mathcal{F}^*$ is a complete projective resolution. Thus, the complex $\text{Hom}_{\Delta}(\mathcal{F}^*, Z_A(P))$ is exact. Also, by Lemma 5 (2), we have

$$\text{Hom}_{\Delta}(\mathcal{F}^*, Z_B(M \otimes_A P)) \cong \text{Hom}_B(\mathcal{F}^*, M \otimes_A P).$$

(37)

To show the exactness of $\text{Hom}_B(\mathcal{F}^*, M \otimes_A P)$, we know that a $B$-module $M \otimes A P$ has finite projective dimension, since $M \otimes A P$ is isomorphic to a direct summand of a direct sum of copies of $M$. Thus, $\text{Hom}_B(\mathcal{F}^*, M \otimes_A P)$ is exact by [18], [Proposition 2.3], which implies $\text{Hom}_{\Delta}(\mathcal{F}^*, Z_B(M \otimes_A P))$ is exact. Hence from the exact sequence of complexes in (35) it follows that the complex $\text{Hom}_{\Delta}(\mathcal{F}^*, T_A(P))$ is exact. Similarly, the complex

$$\text{Hom}_{\Delta}(\mathcal{F}^*, T_B(Q)),$$

is exact. Thus, $\text{Hom}_{\Delta}(\mathcal{F}^*, (X', Y', f', g'))$ is exact for each projective $\Delta_{(0,0)}$-module $(X', Y', f', g')$. Therefore, a $\Delta_{(0,0)}$-module $(X, Y, f, g)$ is a Gorenstein-projective.

Proof

(1) Assume $AN$ is projective and $\Delta_{(0,0)}$ is a strongly CM-free. Let $X$ be a Gorenstein-projective $A$-module. Because $M_A$ has a finite projective dimension, Proposition 2 (1) asserts that $T_A(X) = (X, M \otimes_A X, \text{Id}_{M \otimes_A X}, 0)$ is a Gorenstein-projective $\Delta_{(0,0)}$-module. The assumption that $\Delta_{(0,0)}$ is a strongly CM-free implies that $T_A(X)$ is a projective $\Delta_{(0,0)}$-module. By Lemma 2 (1), $T_A(X) = T_A(P)$ for some projective $A$-module. For some projective $B$-module $Q$, hence $X = P$, or $X = N \otimes_A Q$. An $A$-module $N \otimes_A Q$ is projective because it is isomorphic to a direct summand of a direct sum of copies of $AN$ and $AN$ is projective. Thus, $X$ is a projective $A$-module. Therefore, $A$ is a strongly CM-free.

(2) Assume $BM$ is projective and $\Delta_{(0,0)}$ is a strongly CM-free. Let $Y$ be a Gorenstein-projective $B$-module. By similar argument as in (1), $Y$ is a projective $B$-module. Therefore, $B$ is a strongly CM-free.

As a consequence we have the following corollary.

Corollary 2. Let $\Delta_{(0,0)} = \left(\begin{array}{cc} A & A \\ A & A \end{array}\right)$ be Morita ring. If $\Delta_{(0,0)}$ is a strongly CM-free, then $A$ is a strongly CM-free.

Data Availability

No datasets were generated or analyzed during the study.

Conflicts of Interest

The author declares no conflicts of interest.

References


