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Research Article

Gorenstein-Projective Modules over a Class of Morita Rings

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Let $\Delta_{(0,0)} = \begin{bmatrix} A & AN_B \\ BM_A & B \end{bmatrix}$ be a Morita ring such that the bimodule homomorphisms are zero. In this paper, we give sufficient conditions for a $\Delta_{(0,0)}$-module (X, Y, f, g) to be Gorenstein-projective. As an application, we give sufficient conditions when the algebras A and B inherit the strongly CM-freeness of $\Delta_{(0,0)}$.

1. Introduction

Gorenstein algebra and Gorenstein-projective modules are important topics of research in Gorenstein homological algebra. A fundamental problem in Gorenstein homological algebra is determining all the Gorenstein-projective A-modules for a given algebra A. The class of Gorenstein-projective modules is a key component of relative homological algebra and has received a great deal of attention in the study of representation theory (e.g., [1–6, 8–13, 16–18, 20, 23–27]).

For algebras A and B, bimodules ${}_BM_A$ and ${}_AN_B$, and a B-B-bimodule map $\phi\colon M \otimes_A N \longrightarrow B$, and an A-A-bimodule map $\psi\colon N \otimes_B M \longrightarrow A$ satisfying some special conditions. Bass [7] introduced Morita algebra $\Delta_{(\phi,\psi)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$, where the special conditions for ϕ and ψ are to guarantee that the multiplication of $\Delta_{(\phi,\psi)}$ has the associativity. Morita algebras $\Delta_{(\phi,\psi)}$ give a very large class of algebras, and many important algebras can be realized as Morita algebras. For example, the 2×2 matrix algebra $M_2(A) = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$ over A, the algebra $\Delta_{(0,0)} = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$, the upper triangular matrix algebra $\begin{pmatrix} A & {}_AN_B \\ 0 & B \end{pmatrix}$, the algebras defined by finite quivers and relations. Thus, researching Morita rings is pivotal.

Asefa [1] obtained sufficient conditions for Gorenstein-projective module (X, Y, f, g) over $\Delta_{(\phi,\psi)}$, implying that X is a Gorenstein-projective A-module and Y is a Gorenstein-projective B-module. Gao and Psaroudakis [13] constructed Gorenstein-projective modules over a Morita ring $\Delta_{(0,0)}$. They stated that [13], [Theorem 3.10] does not give sufficient conditions for a module (X, Y, f, g) Gorenstein-projective ([13], Remark 3.13). As a result, it is natural to ask, "When is a module (X, Y, f, g) Gorenstein-projective?". This paper is motivated to answer this question. In the following main result, we give sufficient conditions for (X, Y, f, g) to be a Gorenstein-projective module over a Morita ring $\Delta_{(0,0)}$.

Theorem 1. *Let $\Delta_{(0,0)}$ be a Morita ring. Assume that*

(i) M_A and N_B have finite flat dimensions.

(ii) ${}_BM$ and ${}_AN$ have finite projective dimensions.

Then, if each of the following conditions holds, a $\Delta_{(0,0)}$-module (X, Y, f, g) is Gorenstein-projective.

(1) $\operatorname{Coker} g$ is Gorenstein-projective A-module;

(2) $\operatorname{Coker} f$ is Gorenstein-projective B-module; and

(3) $M \otimes_A \operatorname{Coker} g \cong \operatorname{Im} f$ and $N \otimes_B \operatorname{Coker} f \cong \operatorname{Im} g$.

Lastly, we give sufficient conditions when the algebras A and B inherit the strongly CM-freeness of $\Delta_{(0,0)}$.

2. Preliminaries

This section discusses some basic definitions and facts that will be used throughout the paper.

Throughout, rings mean a ring with unity and an R-module mean a left R-module. Let R be a ring. Let M be an R-module, then the projective(injective and flat) dimension of M will be denoted by projdim$M$ (injdim$M$ and flatdim$M$). The class of modules isomorphic to direct summands of direct sums of copies of M is denoted by Add(M).

An R-module M is Gorenstein-projective if there exists an exact sequence of projective R-modules

$$\mathscr{P}^{\bullet} := \cdots \longrightarrow P^{-1} \longrightarrow P^{0} \xrightarrow{d^0} P^{1} \longrightarrow \cdots. \tag{1}$$

such that $\mathrm{Hom}_R(\mathscr{P}^{\bullet}, Q)$ is exact for an arbitrary projective R-module Q and that $M \cong \mathrm{Ker}d^0$. The class of Gorenstein-projective R-modules will be denoted by GProj$R$.

Let A and B be rings, ${}_AN_B$ and ${}_BM_A$ bimodules, and $\phi\colon M\otimes_A N \longrightarrow B$ and $\psi\colon N\otimes_B M \longrightarrow A$ bimodules homomorphism. This paper focuses on the case of $\phi = 0 = \psi$. Then,

$$\Delta_{(0,0)} := \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix} = \left\{ \begin{pmatrix} a & n \\ m & b \end{pmatrix} \mid a \in A, b \in B, m \in M, n \in N \right\}, \tag{2}$$

is a Morita ring, where the addition is that of a matrix, and multiplication of this Morita ring is given as follows:

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} . \begin{pmatrix} a' & n' \\ m' & b' \end{pmatrix} = \begin{pmatrix} aa' & an' + nb' \\ ma' + bm' & bb' \end{pmatrix}. \tag{3}$$

The case $\phi = 0 = \psi$ is a subclass of the general Morita rings(e.g., [7, 13–15, 21].)

2.1. Modules over $\Delta_{(0,0)}$. A left module over $\Delta(0.0)$ is given as (X, Y, f, g), where X is an A-module, Y is a B-module, and

$$\begin{aligned} f\colon M\otimes_A X &\longrightarrow Y, \\ g\colon N\otimes_B Y &\longrightarrow X, \end{aligned} \tag{4}$$

where g is an A-module map and f is a B-map.

A $\Delta_{(0,0)}$-module morphism is given by $(a, b)\colon (X, Y, f, g) \longrightarrow (X', Y', f', g')$, where $a\colon X \longrightarrow X'$ is a homomorphism in $A-$Mod and $b\colon Y \longrightarrow Y'$ is a homomorphism in $B-$Mod such that the following diagrams are commutative.

$$\begin{array}{ccc} M\otimes_A X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{Id}_M\otimes a}\downarrow & & \downarrow{\scriptstyle b} \\ M\otimes_A X' & \xrightarrow{f'} & Y' \\ N\otimes_B Y & \xrightarrow{g} & X \\ {\scriptstyle \mathrm{Id}_N\otimes b}\downarrow & & \downarrow{\scriptstyle a} \\ N\otimes_B Y' & \xrightarrow{g'} & X' \end{array} \tag{5}$$

Lemma 1 (see [13]). *Let* $\Delta_{(\phi,\psi)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$ *be a Morita ring.*

(1) A sequence $0 \longrightarrow (X'', Y'', f'', g'') \longrightarrow (X, Y, f, g) \longrightarrow (X', Y', f', g') \longrightarrow 0$ *is exact in* Δ*-Mod if and only if the sequence* $0 \longrightarrow X'' \longrightarrow X \longrightarrow X' \longrightarrow 0$ *is exact in* A*-Mod and the sequence* $0 \longrightarrow Y'' \longrightarrow Y \longrightarrow Y' \longrightarrow 0$ *is* B*-Mod.*

(2) Let $(\alpha, \beta)\colon (X, Y, f, g) \longrightarrow (X_1, Y_1, f_1, g_1)$ *a morphism in* Δ*-module and consider the maps* $\sigma\colon \mathrm{Ker}\alpha \longrightarrow X$ *and* $\gamma\colon \mathrm{Ker}\beta \longrightarrow Y$*.*
Then, $\mathrm{Ker}(\alpha, \beta)$ *is given by* $(\mathrm{Ker}\alpha, \mathrm{Ker}\beta, t, s)$ *where the maps* t *and* s *are induced from the commutative diagrams given below.*

$$\begin{array}{ccccc} M\otimes_A \mathrm{Ker}\,\alpha & \xrightarrow{Id_M\otimes\sigma} & M\otimes_A X & \xrightarrow{Id_M\otimes\alpha} & M\otimes_A X_1 \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} \\ \mathrm{Ker}\,\beta & \xrightarrow{\gamma} & Y & \xrightarrow{\beta} & Y_1 \end{array} \quad \begin{array}{ccccc} N\otimes_B \mathrm{Ker}\,\beta & \xrightarrow{Id_N\otimes\gamma} & N\otimes_B Y & \xrightarrow{Id_N\otimes\beta} & N\otimes_B Y_1 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_1} \\ \mathrm{Ker}\,\alpha & \xrightarrow{\sigma} & X & \xrightarrow{\alpha} & X_1 \end{array}, \tag{6}$$

Similarly, the Cokernel of (a, b)can be described.

2.2. We Now Recall Functors Given in [16]

(1) The functor $\mathrm{T}_A\colon A$-Mod $\longrightarrow \Delta_{(0,0)}$-Mod is given by $\mathrm{T}_A(X) := (X, M\otimes_A X, 1, 0)$ for any object X in A-Mod.

(2) The functor $\mathrm{T}_B\colon B$-Mod $\longrightarrow \Delta_{(0,0)}$-Mod is given by $\mathrm{T}_B(Y) := (N\otimes_B Y, Y, 0, 1)$ for any object Y in B-Mod.

(3) The functor $\mathrm{U}_A\colon \Delta_{(0,0)}$-Mod $\longrightarrow A$-Mod is given by $\mathrm{U}_A(X, Y, f, g) := X$ for any object (X, Y, f, g) in $\Delta_{(0,0)}$-Mod.

(4) The functor U_B: $\Delta_{(0,0)}$-Mod $\longrightarrow B$-Mod is given by $U_B(X,Y,f,g) := Y$ for any object (X,Y,f,g) in $\Delta_{(0,0)}$-Mod.

(5) Let $X \in A$ be any object in Mod, then we denote by ϵ_X: $N\otimes_B \mathrm{Hom}_A(N,X) \longrightarrow X$ the map A-module given by involution. The functor H_A: A-Mod $\longrightarrow \Delta_{(0,0)}$-Mod is given by $\mathrm{H}_A(X) := (X, \mathrm{Hom}_A(N,X), 0, \epsilon_X)$ for any object X in A-Mod.

(6) Let Y be any object in B-Mod, the we denote by ϵ_Y: $M\otimes_A \mathrm{Hom}_B(M,Y) \longrightarrow Y$ the map B-module given by involution. The functor H_B: B-Mod $\longrightarrow \Delta_{(0,0)}$-Mod is given by $\mathrm{H}_B(Y) := (\mathrm{Hom}_B(M,Y), Y, \epsilon_Y, 0)$ for any object Y in B-Mod.

(7) The functor Z_A: A-Mod $\longrightarrow \Delta_{(0,0)}$-Mod is defied by $Z_A(X) := (X,0,0,0)$ for any object X in A-Mod. The functor Z_B: B-Mod $\longrightarrow \Delta_{(0,0)}$-Mod can be similarly defined.

More information about the functors given above can be found in the following result.

Proposition 1 ([16], Proposition 2.4]), *Let$\Delta_{(0,0)}$be Morita ring. Then,*

(1) The functorsH_A,H_B,T_A, andT_B, are fully faithful.

(2) The pairs(U_A, H_A), (U_B, H_B),(T_A, U_A), and$(T_B, U_B)$are adjoint pairs.

(3) The functors$U_A$and$U_B$are exact.

Lemma 2. *Let $\Delta_{(0,0)}$be Morita ring.*

(1) [19], [Theorem 7.3] A left$\Delta_{(0,0)}$-module$(P,Q,f,g)$is projective if and only if$(P,Q,f,g) = T_A(X) \oplus T_B(Y) = (X, M\otimes_A X, 1, 0) \oplus (Y, N\otimes_B Y, Y, 0, 1)$for some projective left$A$-module$X$and projective left$B$-module$Y$.

(2) [22], [Corollary 2.2] A left$\Delta_{(0,0)}$-module$(I,J,f,g)$is injective if and only if$(I,J,f,g) = H_A(X) \oplus H_B(Y) = (X, Hom_A(N, X), 0, \epsilon_X) \oplus (Hom_B(M,Y), Y, \epsilon_Y, 0)$for some injective left$A$-module$X$and injective left$B$-module$Y$.

3. Gorenstein-Projective Modules over $\Delta_{(0,0)}$

This section aims to construct Gorenstein-projective modules over $\Delta_{(0,0)}$.

The following lemmas are required in order to prove the main theorems of this paper.

Lemma 3. *Let A be a ring and M a B-A-bimodule with finite flat dimension. If a complex of flat$A$-modules$\mathscr{F}^\bullet$ is exact, then, the sequence$M\otimes_A \mathscr{F}^\bullet$ is also exact.*

Proof. Assume that $\mathscr{F}^\bullet$ is an exact complex of flat A-modules. Because M has a finite flat dimension, we have the following flat resolution of M.

$$0 \longrightarrow F^n \longrightarrow F^{n-1} \longrightarrow \cdots \longrightarrow F^0 \longrightarrow M_A \longrightarrow 0. \quad (7)$$

We obtain the following exact sequence of complexes because all terms in the complex $\mathscr{F}^\bullet$ are flat.

$$\begin{aligned} 0 \longrightarrow F^n\otimes_A \mathscr{F}^\bullet \longrightarrow F^{n-1}\otimes_A \mathscr{F}^\bullet \longrightarrow \cdots \\ \longrightarrow F^0\otimes_A \mathscr{F}^\bullet \longrightarrow M_A\otimes_A \mathscr{F}^\bullet \longrightarrow 0. \end{aligned} \quad (8)$$

Since the complexes$F^i\otimes_A \mathscr{F}^\bullet$ are exact for all i, so is $M\otimes_A \mathscr{F}^\bullet$. □ □

Lemma 4. *Let Bbe a ring. If a B-module Nhas finite injective dimension and the complex of projective B-modules,*

$$\mathscr{Q}^\bullet := \cdots \longrightarrow Q^{n-1} \longrightarrow Q^n \longrightarrow Q^{n+1} \longrightarrow \cdots, \quad (9)$$

is exact, then so is $Hom_B(\mathscr{Q}^\bullet, N)$.

Lemma 5. *Let $\Delta_{(0,0)}$be a Morita ring with zero bimodule homomorphisms. Then*

(1) [13], [Lemma 3.8] For each$X \in A$-Mod and each$Y \in B$-Mod we have the following exact sequences in$\Delta_{(0,0)}$-Mod.

$$0 \longrightarrow Z_B(M\otimes_A X) \longrightarrow T_A(X) \longrightarrow Z_A(X) \longrightarrow 0. \quad (10)$$

and

$$0 \longrightarrow Z_A(N\otimes_B Y) \longrightarrow T_B(Y) \longrightarrow Z_B(Y) \longrightarrow 0. \quad (11)$$

(2) [13], [Lemma 3.9] For all$X, X' \in A$-Mod and$Y, Y' \in B$-Mod, we have the following isomorphisms:

$$\mathrm{Hom}_{\Delta_{(0,0)}}(T_A(X)\oplus T_B(Y), Z_A(X')) \cong \mathrm{Hom}_A(X, X'). \quad (12)$$

and

$$\mathrm{Hom}_{\Delta_{(0,0)}}(T_A(X)\oplus T_B(Y), Z_B(Y')) \cong \mathrm{Hom}_B(Y, Y'). \quad (13)$$

The following result provides sufficient conditions for the functor T_A: A − Mod $\longrightarrow \Delta_{(0,0)}$ − Mod and the functor T_B: B − Mod $\longrightarrow \Delta_{(0,0)}$ − Modto preserve Gorenstein-projective modules.

Proposition 2

(1) Assume that M_A has a finite flat dimension and that ${}_AN$ has a finite projective dimension. $T_A(X)$ is a Gorenstein-projective $\Delta_{(0,0)}$-module if X is a Gorenstein-projective A-module.

(2) Assume that N_B has a finite flat dimension and that ${}_BM$ has a finite projective dimension. $T_B(Y)$ is a Gorenstein-projective $\Delta_{(0,0)}$-module if Y is a Gorenstein-projective B-module.

Proof. We show (1) and (2) can be proved in a similar manner. Since an A-module X is a Gorenstein-projective, there is an exact sequence of projective A-modules,

$$\mathscr{P}^{\bullet}\colon \cdots \longrightarrow P^{-1} \longrightarrow P^{0} \xrightarrow{d^{0}} P^{1} \longrightarrow \cdots, \quad (14)$$

such that $X \cong \mathrm{Ker} d^{0}$, and $\mathrm{Hom}_{A}(\mathscr{P}^{\bullet}, Q)$ exact for any projective A-module Q. Lemma 3 states that the assumption that M_{A} has finite flat dimension implies the sequence $M \otimes_{A} \mathscr{P}^{\bullet}$ is exact. Hence we get the exact sequence of projective $\Delta_{(0,0)}$-modules,

$$\mathrm{T}_{A}(\mathscr{P}^{\bullet})\colon \cdots \longrightarrow \mathrm{T}_{A}\left(P^{-1}\right) \longrightarrow \mathrm{T}_{A}\left(P^{0}\right) \overset{\left(d^{0}, 1 \otimes d^{0}\right)}{\longrightarrow} \mathrm{T}_{A}\left(P^{1}\right) \longrightarrow \cdots, \quad (15)$$

such that $\mathrm{T}_{A}(X) \cong \mathrm{Ker}(d^{0}, 1 \otimes d^{0})$. Now, it is left to show that $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_{A}(\mathscr{P}^{\bullet}), (X', Y', f', g'))$ is exact for any projective $\Delta_{(0,0)}$-module (X', Y', f', g'). By Lemma 2, this can be proved by showing the exactness of $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_{A}(\mathscr{P}^{\bullet}), \mathrm{T}_{A}(P))$ and $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_{A}(\mathscr{P}^{\bullet}), \mathrm{T}_{B}(Q))$ for any projective A-module P, and any projective B-module Q. By Proposition 1 the functor T_{A} is fully faithful. Thus, $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_{A}(\mathscr{P}^{\bullet}), \mathrm{T}_{A}(P)) \cong \mathrm{Hom}_{A}(\mathscr{P}^{\bullet}, P)$. Hence $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_{A}(\mathscr{P}^{\bullet}), \mathrm{T}_{A}(P))$ because $\mathrm{Hom}_{A}(\mathscr{P}^{\bullet}, P)$ is exact. Since (T_{A}, U_{A}) are adjoint pairs, we have the following equation:

$$\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_{A}(\mathscr{P}^{\bullet}), \mathrm{T}_{B}(Q)) \cong \mathrm{Hom}_{A}(\mathscr{P}^{\bullet}, N \otimes_{B} Q). \quad (16)$$

A module $N \otimes_{B} Q$ has finite projective dimension because it is isomorphic to a direct summand of direct sums of copies of N. Since $\mathscr{P}^{\bullet}$ is a complete A-projective resolution, the complex $\mathrm{Hom}_{A}(\mathscr{P}^{\bullet}, N \otimes_{B} Q)$ is exact(see [18], [Proposition 2]). Thus, $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_{A}(\mathscr{P}^{\bullet}), \mathrm{T}_{B}(Q))$ is exact. Hence $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_{A}(\mathscr{P}^{\bullet}), (X', Y', f', g'))$ is exact for any projective $\Delta_{(0,0)}$-module (X', Y', f', g'). Therefore, $\mathrm{T}_{A}(X)$ is a Gorenstein-projective $\Delta_{(0,0)}$-module. □

In the following result, we give sufficient conditions for a $\Delta_{(0,0)}$-module (X, Y, f, g) to be Gorenstein-projective.

Theorem 2. *Let $\Delta_{(0,0)}$be a Morita ring. Assume that*

(i) $M_{A}$and N_{B}have finite flat dimensions.

(ii) ${}_{B}M$and ${}_{A}N$have finite projective dimensions.

Then, if each of the following conditions holds, a $\Delta_{(0,0)}$-module $(X, Y, f, g)$is Gorenstein-projective.

(1) Coker$g$is a Gorenstein-projective A-module;

(2) Coker$f$is a Gorenstein-projective B-module; and

(3) $M \otimes_{A}$Coker$g \cong$ Im$f$and $N \otimes_{B}$Coker$f \cong$ Img.

Proof. Suppose that conditions (1)–(3) are true. Since Coker$f$ is a Gorenstein-projective B-module, there exists an exact complex of projective B-modules,

$$\mathscr{Q}^{\bullet}\colon \cdots \longrightarrow Q^{-1} \longrightarrow Q^{0} \xrightarrow{d'^{0}} Q^{1} \longrightarrow \cdots. \quad (17)$$

such that $\mathrm{Coker} f \cong \mathrm{Ker} d'^{0}$ and $\mathrm{Hom}_{B}(\mathscr{Q}^{\bullet}, Q)$ is exact for each projective B-module Q. Thus, we get the following exact sequence,

$$0 \longrightarrow N \otimes_{B} \mathrm{Coker} f \longrightarrow N \otimes_{B} Q^{0} \overset{\mathrm{Id} \otimes d'^{0}}{\longrightarrow} N \otimes_{B} Q^{1} \longrightarrow \cdots, \quad (18)$$

because N_{B} has a finite flat dimension. Since Coker$g$ is a Gorenstein-projective A-module, there exists a complete projective resolutions,

$$\mathscr{P}^{\bullet}\colon \cdots \longrightarrow P^{-1} \longrightarrow P^{0} \xrightarrow{d^{0}} P^{1} \longrightarrow \cdots, \quad (19)$$

of A-modules such that $\mathrm{Coker} g \cong \mathrm{Ker} d^{0}$.

Let $\pi_{1}\colon X \longrightarrow \mathrm{Coker} g$ and $\pi_{2}\colon Y \longrightarrow \mathrm{Coker} f$. Consider the following commutative diagram of A-modules.

$$, \quad (20)$$

Since $\psi = 0$, the above equation implies that there exists an A-map $i_{1}\colon N \otimes_{B} \mathrm{Coker} f \longrightarrow X$ that is unique and $g = i_{1} \circ (\mathrm{Id}_{N} \otimes \pi_{2})$. Thus, from $\mathrm{Im} g \cong N \otimes_{B} \mathrm{Coker} f$ it follows that i_{1} is an injective A-map. Thus, we get the exact sequence as follows:

$$0 \longrightarrow N \otimes_{B} \mathrm{Coker} f \xrightarrow{i_{1}} X \xrightarrow{\pi_{1}} \mathrm{Coker} g \longrightarrow 0. \quad (21)$$

Similarly, the sequence

$$0 \longrightarrow M \otimes_{A} \mathrm{Coker} g \xrightarrow{i_{2}} Y \xrightarrow{\pi_{2}} \mathrm{Coker} f \longrightarrow 0, \quad (22)$$

is exact.

Since each $N \otimes_{A} Q^{i}$ has finite projective dimension, and since each $\mathrm{Ker} d^{i}$ is a Gorenstein-projective A-module, we have that $Ext_{A}^{1}(\mathrm{Ker} d^{i}, N \otimes_{B} Q^{i}) = 0$, $\forall i \geq 0$. Applying generalized Horseshoe Lemma([26], Lemma 1.6 (ii)) to the exact sequences (18) and (21), we obtain an exact sequence as follows:

$$0 \longrightarrow X \longrightarrow P^{0} \oplus \left(N \otimes_{B} Q^{0}\right) \xrightarrow{\alpha^{0}} P^{1} \oplus \left(N \otimes_{B} Q^{1}\right) \xrightarrow{\alpha^{1}} \cdots. \quad (23)$$

with $\alpha^{i} = \begin{pmatrix} d^{i} & 0 \\ \gamma^{i} & \mathrm{Id}_{N} \otimes d'^{i} \end{pmatrix}$, $\gamma^{i}\colon P^{i} \longrightarrow N \otimes_{B} Q^{i+1}$, $\forall i \in \mathbb{Z}$, such that the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N\otimes_B \mathrm{Coker} f & \longrightarrow & N\otimes_B Q^0 & \xrightarrow{\mathrm{Id}\otimes d'^0} & N\otimes_B Q^1 & \xrightarrow{\mathrm{Id}\otimes d'^1} & \cdots \\ & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle \binom{0}{\mathrm{Id}}} & & \downarrow{\scriptstyle \binom{0}{\mathrm{Id}}} & & \\ 0 & \longrightarrow & X & \longrightarrow & P^0\oplus N\otimes_B Q^0 & \xrightarrow{\alpha^0} & P^1\oplus N\otimes_B Q^1 & \xrightarrow{\alpha^1} & \cdots \end{array}, \quad (24)$$

is commutative. The dual argument obtains the commutative diagram with exact rows shown below.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & N\otimes_B Q^{-2} & \xrightarrow{\mathrm{Id}\otimes d'^{-2}} & N\otimes_B Q^{-1} & \xrightarrow{\mathrm{Id}\otimes d'^{-1}} & N\otimes_B \mathrm{Coker} f & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \binom{0}{\mathrm{Id}}} & & \downarrow{\scriptstyle \binom{0}{\mathrm{Id}}} & & \downarrow{\scriptstyle i_1} & & \\ \cdots & \longrightarrow & P^{-2}\oplus N\otimes_B Q^{-2} & \xrightarrow{\alpha^{-2}} & P^{-1}\oplus N\otimes_B Q^{-1} & \longrightarrow & X & \longrightarrow & 0 \end{array}, \quad (25)$$

When we combine (24) and (25), we get the exact sequence shown below.

$$\cdots \longrightarrow P^{-2}\oplus N\otimes_B Q^{-2} \xrightarrow{\alpha^{-2}} P^{-1}\oplus N\otimes_B Q^{-1} - \xrightarrow{\alpha^{-1}} P^0\oplus N\otimes_B Q^0 \xrightarrow{\alpha^0} P^1\oplus N\otimes_B Q^1 \xrightarrow{\alpha^1}\cdots. \quad (26)$$

with $\mathrm{Ker}\alpha^0 = X$.

We now construct an exact sequence similar to (26) for a left B-module Y. Since each $M\otimes_A P^i$ has finite projective dimension as B-module by assumption on M, and $\mathrm{Ker} d'^i$ is a Gorenstein-projective B-module, it follows that $Ext_B^1(\mathrm{Ker} d'^i, M\otimes_A P^i) = 0$. Thus, by ([26], Lemma 1.6 (ii)) again, we obtain the exact sequence as follows:

$$0 \longrightarrow Y \longrightarrow \left(M\otimes_A P^0\right)\oplus Q^0 \xrightarrow{\beta^0} \left(M\otimes_A P^1\right)\oplus Q^1 \xrightarrow{\beta^1} \cdots. \quad (27)$$

with $\beta^i = \begin{pmatrix} \mathrm{Id}_M\otimes d^i & \sigma^i \\ 0 & d'^i \end{pmatrix}$, and $\sigma^i\colon Q^i \longrightarrow M\otimes_A P^{i+1}$, $\forall\, i\in\mathbb{Z}$, such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M\otimes_A \mathrm{Coker} g & \longrightarrow & M\otimes_A P^0 & \xrightarrow{\mathrm{Id}\otimes d^0} & M\otimes_A P^1 & \xrightarrow{\mathrm{Id}\otimes d^1} & \cdots \\ & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle \binom{\mathrm{Id}}{0}} & & \downarrow{\scriptstyle \binom{\mathrm{Id}}{0}} & & \\ 0 & \longrightarrow & Y & \longrightarrow & M\otimes_A P^0\oplus Q^0 & \xrightarrow{\beta^0} & M\otimes_A P^1\oplus Q^1 & \xrightarrow{\beta^1} & \cdots \end{array}, \quad (28)$$

is commutative. The dual argument gives the commutative diagram with the exact rows

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & M\otimes_A P^{-2} & \xrightarrow{\mathrm{Id}\otimes d^{-2}} & M\otimes_A P^{-1} & \xrightarrow{\mathrm{Id}\otimes d^{-1}} & M\otimes_A \mathrm{Coker} g & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \binom{\mathrm{Id}}{0}} & & \downarrow{\scriptstyle \binom{\mathrm{Id}}{0}} & & \downarrow{\scriptstyle i_2} & & \\ \cdots & \longrightarrow & M\otimes_A P^{-2}\oplus Q^{-2} & \xrightarrow{\beta^{-2}} & M\otimes_A P^{-1}\oplus Q^{-1} & \longrightarrow & Y & \longrightarrow & 0 \end{array}, \quad (29)$$

As a result, combining (28) and (29) yields the following exact sequence, which is similar to the following equation:

$$\cdots \longrightarrow M\otimes_A P^{-2}\oplus Q^{-2} \xrightarrow{\beta^{-2}} M\otimes_A P^{-1}\oplus Q^{-1} \xrightarrow{\beta^{-1}} M\otimes_A P^{0}\oplus Q^{0} \xrightarrow{\beta^{0}} M\otimes_A P^{1}\oplus Q^{1} \xrightarrow{\beta^{1}} \cdots \tag{30}$$

with $\mathrm{Ker}\beta^0 = Y$.

Glue together the exact sequences (26) and (30) to obtain the following sequence:

$$\mathcal{T}^{\bullet}\colon \cdots \longrightarrow \mathrm{T}_A\left(P^{-1}\right)\oplus \mathrm{T}_B\left(Q^{-1}\right) \xrightarrow{\left(\alpha^{-1}\,\beta^{-1}\right)} \mathrm{T}_A\left(P^{0}\right)\oplus \mathrm{T}_B\left(Q^{0}\right) \xrightarrow{\left(\alpha^{0}\,\beta^{0}\right)} \cdots \tag{31}$$

with $\mathrm{Ker}\left(\alpha^0\,\beta^0\right) = (X, Y, f, g)$.

The morphism $\left(\alpha^i\,\beta^i\right)\ \forall i\in\mathbb{Z}$, is a $\Delta_{(0,0)}$-map because

$$\begin{array}{ccc}
M\otimes_A\left(P^i\oplus N\otimes_B Q^i\right) & \xrightarrow{\begin{pmatrix}\mathrm{Id}_{M\otimes P^i} & 0\\ 0 & 0\end{pmatrix}} & M\otimes_A P^i\oplus Q^i\\
\Big\downarrow{\scriptstyle \mathrm{Id}_M\otimes\begin{pmatrix} d^i & 0\\ \gamma^i & \mathrm{Id}_N\otimes d'^i\end{pmatrix}} & & \Big\downarrow{\scriptstyle \begin{pmatrix}\mathrm{Id}_M\otimes d^i & \sigma^i\\ 0 & d'^i\end{pmatrix}}\\
M\otimes_A\left(P^{i+1}\oplus N\otimes_B Q^{i+1}\right) & \xrightarrow{\begin{pmatrix}\mathrm{Id}_{M\otimes P^{i+1}} & 0\\ 0 & 0\end{pmatrix}} & M\otimes_A P^{i+1}\oplus Q^{i+1}
\end{array}, \tag{32}$$

and

$$\begin{array}{ccc}
N\otimes_B\left(M\otimes_A P^i\oplus Q^i\right) & \xrightarrow{\begin{pmatrix}0 & 0\\ 0 & \mathrm{Id}_{N\otimes Q^i}\end{pmatrix}} & P^i\oplus N\otimes_B Q^i\\
\Big\downarrow{\scriptstyle \mathrm{Id}_N\otimes\begin{pmatrix}\mathrm{Id}_M\otimes d^i & \sigma^i\\ 0 & d'^i\end{pmatrix}} & & \Big\downarrow{\scriptstyle \begin{pmatrix} d^i & 0\\ \gamma^i & \mathrm{Id}_N\otimes d'^i\end{pmatrix}}\\
N\otimes_B\left(M\otimes_A P^{i+1}\oplus Q^{i+1}\right) & \xrightarrow{\begin{pmatrix}0 & 0\\ 0 & \mathrm{Id}_{N\otimes Q^{i+1}}\end{pmatrix}} & P^{i+1}\oplus N\otimes_B Q^{i+1}
\end{array}, \tag{33}$$

are commutative diagrams.

Since the complexes (26) and (30) are exact, it follows from Lemma 1 (1) that the sequence $\mathcal{T}^{\bullet}$ is exact. The object (X, Y, f, g) arises as the kernel of the morphism $\left(\alpha^0\,\beta^0\right)$, and we see from Lemma 1 (2) that $f = i_2\circ(\mathrm{Id}_M\otimes\pi_1)$ and $g = i_1\circ(\mathrm{Id}_N\otimes\pi_2)$. However, based on the commutative diagram of A-modules shown below,

$$\begin{array}{ccccccc}
N\otimes_B Y & \xrightarrow{g} & X & \xrightarrow{\pi_1} & \mathrm{Coker}g & \longrightarrow & 0\\
\Big\downarrow{\scriptstyle \mathrm{Id}_N\otimes\pi_2} & \nearrow{\scriptstyle i_1} & & & & & \\
N\otimes_B \mathrm{Coker}f & & & & & &
\end{array} \tag{34}$$

We know that g is uniquely determined by $i_1\circ(\mathrm{Id}_N\otimes\pi_2)$. Similarly, f is uniquely determined by $i_1\circ(\mathrm{Id}_M\otimes\pi_1)$.

We are now left with showing that $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathcal{T}^{\bullet}, (X', Y', f', g'))$ is exact for each projective $\Delta_{(0,0)}$-module (X', Y', f', g'). We can deduce from Lemma 2 that it is enough to show that $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathcal{T}^{\bullet}, \mathrm{T}_A(P))$ and $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathcal{T}^{\bullet}, \mathrm{T}_B(Q))$ are exact for each projective A-module P and for each projective B-module Q. By Lemma 5 (1) the sequence $0\longrightarrow \mathrm{Z}_B(M\otimes_A P)\longrightarrow \mathrm{T}_A(P)\longrightarrow \mathrm{Z}_A(P)\longrightarrow 0$ is exact. Since each term in the complex $\mathcal{T}^{\bullet}$ is a projective $\Delta_{(0,0)}$-module, the sequence

$$0\longrightarrow \mathrm{Hom}_{\Delta_{(0,0)}}\left(\mathcal{T}^{\bullet}, \mathrm{Z}_B(M\otimes_A P)\right)\longrightarrow \mathrm{Hom}_{\Delta_{(0,0)}}\left(\mathcal{T}^{\bullet}, \mathrm{T}_A(P)\right)\longrightarrow \mathrm{Hom}_{\Delta_{(0,0)}}\left(\mathcal{T}^{\bullet}, \mathrm{Z}_A(P)\right)\longrightarrow 0, \tag{35}$$

is exact. By Lemma 5 (2) we have the following equations,

$$\mathrm{Hom}_{\Delta_{(0,0)}}(\mathcal{T}^{\bullet}, \mathrm{Z}_A(P)) \cong \mathrm{Hom}_A(\mathcal{P}^{\bullet}, P). \quad (36)$$

The complex $\mathrm{Hom}_A(\mathcal{P}^{\bullet}, P)$ is exact because $\mathcal{P}^{\bullet}$ is a complete projective resolution. Thus, the complex $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathcal{T}^{\bullet}, \mathrm{Z}_A(P))$ is exact. Also, by Lemma 5 (2), we have

$$\mathrm{Hom}_{\Delta_{(0,0)}}(\mathcal{T}^{\bullet}, \mathrm{Z}_B(M \otimes_A P)) \cong \mathrm{Hom}_B(\mathcal{Q}^{\bullet}, M \otimes_A P). \quad (37)$$

To show the exactness of $\mathrm{Hom}_B(\mathcal{Q}^{\bullet}, M \otimes_A P)$, we know that a B-module $M \otimes_A P$ has finite projective dimension, since $M \otimes_A P$ is isomorphic to direct summand of a direct sum of copies of M. Thus, $\mathrm{Hom}_B(\mathcal{Q}^{\bullet}, M \otimes_A P)$ is exact by [18], [Proposition 2.3], which implies $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathcal{T}^{\bullet}, \mathrm{Z}_B(M \otimes_A P))$ is exact. Hence from the exact sequence of complexes in (35) it follows that the complex $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathcal{T}^{\bullet}, \mathrm{T}_A(P))$ is exact. Similarly, the complex

$$\mathrm{Hom}_{\Delta_{(0,0)}}(\mathcal{T}^{\bullet}, \mathrm{T}_B(Q)), \quad (38)$$

is exact. Thus, $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathcal{T}^{\bullet}, (X', Y', f', g'))$ is exact for each projective $\Delta_{(0,0)}$-module (X', Y', f', g'). Therefore, a $\Delta_{(0,0)}$-module (X, Y, f, g) is a Gorenstein-projective. □

If the converse of Theorem 2 holds, then Gorenstein-projective modules over $\Delta_{(0,0)}$ will be fully determined. However, whether the converse is true or not is an open problem.

Corollary 1. *Let* $\Delta_{(0,0)} = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$*be Morita ring, and* (X, Y, f, g) $\Delta_{(0,0)}$*-module. If Coker$g$, Coker$f$ are Gorenstein-projective A-modules such that Coker$g \cong$ Imf and Coker$f \cong$ Img, then $(X, Y, f, g)$is a Gorenstein-projective $\Delta_{(0,0)}$-module.*

4. Application

In this section, we study when the class of all Gorenstein-projective A-modules and B-modules coincides with the class of projective A-modules and B-modules, respectively.

If each finitely generated projective left R-module is projective, then a ring R is said to be left CM-free. And R is said to be strongly left CM-free if each Gorenstein-projective left module is projective(see [12]).

The results that follow provide sufficient conditions for the algebras A and B to inherit the stronglyCM-freeness of $\Delta_{(0,0)}$.

Proposition 3. *Let* $\Delta_{(0,0)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$*be Morita ring.*

(1) Assume that $M_A$has finite flat dimension, ${}_AN$is projective, and $\Delta_{(0,0)}$is a strongly CM-free, then$A$is a strongly CM-free.

(2) Assume that $N_B$has finite flat dimension, ${}_BM$is projective, and $\Delta_{(0,0)}$is a strongly CM-free, then$B$is a strongly CM-free.

Proof

(1) Assume ${}_AN$ is projective and $\Delta_{(0,0)}$ is a strongly CM-free. Let X be a Gorenstein-projective A-module. Because M_A has a finite projective dimension, Proposition 2 (1) asserts that $\mathrm{T}_A(X) = (X, M \otimes_A X, \mathrm{Id}_{M \otimes_A X}, 0)$ is a Gorenstein-projective $\Delta_{(0,0)}$-module. The assumption that $\Delta_{(0,0)}$ is a strongly CM-free implies that $\mathrm{T}_A(X)$ is a projective $\Delta_{(0,0)}$-module. By Lemma 2 (1), $\mathrm{T}_A(X) = \mathrm{T}_A(P)$ for some projective A-module Por$\mathrm{T}_A(X) = \mathrm{T}_B(Q) = (N \otimes_B Q, Q, 0, 1)$ for some projective B-module Q. Hence $X = P$, or $X = N \otimes_B Q$. An A-module $N \otimes_B Q$ is projective because it is isomorphic to a direct summand of a direct sum of copies of ${}_AN$ and ${}_AN$ is projective. Thus, X is a projective A-module. Therefore, A is a strongly CM-free.

(2) Assume ${}_BM$ is projective and $\Delta_{(0,0)}$ is a strongly CM-free. Let Y be a Gorenstein-projective B-module. By similar argument as in(1), Y is a projective B-module. Therefore, B is a strongly CM-free. □

As a consequence we have the following corollary.

Corollary 2. *Let* $\Delta_{(0,0)} = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$*be Morita ring. If $\Delta_{(0,0)}$is a strongly CM-free, then$A$is a strongly CM-free.*

Data Availability

No datasets were generated or analyzed during the study.

Conflicts of Interest

The author declares no conflicts of interest.

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