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Research Article

Accelerated Fitted Mesh Scheme for Singularly Perturbed Turning Point Boundary Value Problems

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An accelerated fitted mesh scheme is proposed for the numerical solution of the singularly perturbed boundary value problems whose solution exhibits an interior layer near the turning point. To resolve the interior layer, a mesh of the Shishkin type is used with the help of a transition parameter that separates the layer and regular region. A tridiagonal solver is implemented to solve the system of equation. The stability of the described scheme is analyzed, and the truncation error is obtained. The proposed scheme is of almost second-order convergent and accelerated to almost sixth-order convergent by applying the Richardson extrapolation technique. The numerical results obtained by the present scheme have been compared with some existing methods, and it is observed that it gives better accuracy.

1. Introduction

Singularly perturbed differential equations arise in various branches of science and engineering. The well-known examples are the Navier–Stokes equation with large Reynolds number in fluid dynamics, the convective heat transport problems with large Peclet number, and so on. These equations may be divided into singularly perturbed ordinary or partial differential equations and have their physical phenomena in each real-life activities [1–3]. Numerical treatment of the singularly perturbed boundary value problems attracts the attention of researchers because of the presence of boundary and/or interior layers in its solution. In particular, classical finite difference or finite element methods fail to yield satisfactory numerical results on uniform meshes and to obtain stability concerning the perturbation parameter [4, 5].

The singularly perturbed boundary valued ordinary differential equations broadly categorized into reaction-diffusion and convection-diffusion types. The convection-diffusion type has also its different types depending on the kind of layers (boundary and/or interior layers) [3–5]. Hence, the singularly perturbed convection-diffusion boundary valued problems are divided into problems exhibiting right or left boundary layer, interior layer or boundary, and interior layers. The layer resolving numerical methods for singularly perturbed differential equations are usually classified into the fitted operator and fitted mesh methods [6]. In fitted operator methods, exponential fitting factors will be used to control the rapid growth or decay of the numerical solution in layer regions. However, fitted mesh methods use nonuniform and nonlinear meshes, which will be fine in layer regions and coarse outside the layer regions. The well-known layer resolving fitted meshes are Bakhvalov meshes, which are obtained from some nonlinear mesh generating function, and Shishkin meshes which are piecewise-uniform meshes. These two meshes require a priori information about the location and width of the layer [7, 8].

In this study, a layer resolving higher-order fitted mesh method is suggested for the singularly perturbed turning point boundary value problems of the form:

$$\begin{cases} \varepsilon y''(x) + a(x)y'(x) - b(x)y(x) = f(x), & x \in \Omega = (-1,1), \\ y(-1) = l_1, \\ y(1) = l_2, \end{cases} \tag{1}$$

where the coefficient of diffusion term ε satisfy $0 < \varepsilon \ll 1$ and called perturbation parameter. Also, l_1 andl_2 are given constant numbers. Assume that the considered problem has only one turning point at$x = 0 \in \overline{\Omega}$. That is, the coefficient of convection term $a(x)$ vanishes exactly at$x = 0$. For the uniqueness of the solution of the problem in equation (1), we also assume that the functions involved in equation (1) are sufficiently smooth. Moreover, it is assumed that the following conditions are satisfied:

$$\begin{cases} a(0) = 0, \\ a'(0) > 0, \\ a(x) < 0, \quad x \in [-1, 0), \\ a(x) > 0, \quad x \in (0, 1], \\ b(x) \geq \beta \geq 0, \quad x \in [-1, 1], \\ \dfrac{b(0)}{a'(0)} \geq 0. \end{cases} \tag{2}$$

The behavior and location of an interior layer are determined by the conditions provided in equation (2). Numerical treatment of the turning point of the singularly perturbed problems is more problematic than the problems without a turning point. Moreover, problems with turning point problems have attracted more care and have been considered by many researchers both analytically and numerically under various assumptions [9, 10]. These problems have wide applications in the field of physical sciences and engineering such as control theory, electrical networks, lubrication theory, etc. The solution of the proposed problem exhibits an interior layer, to resolve this layer a piecewise-uniform mesh has been generated which is dense in the layer region and coarse otherwise.

In the past few decades, various uniformly convergent numerical schemes based on fitted mesh methods are proposed for solving singularly perturbed convection-diffusion problem types. For more details, one can refer to the book in [6] and literature [1, 9, 10]. These referred books and articles may help us just to get prior knowledge about the nature of the solution of these families of problems and where and why the existing methods fail to work. Further, it is a recent and active research area in engineering and applied science. Though many classical numerical methods such as finite difference methods, finite element methods, and finite volume methods have been developed so far, most of them fail to give a more accurate solution. This difficulty is due to the presence of perturbation parameter that causes the existence of a layer where the solutions vary rapidly and behave smoothly away from the layer [11–14].

Further, interested researcher or reader can find a real time application of the modeled problem of equation (1) and its analytical properties such as existence of the solution, stability and bounds of the considered problem in the literature [1, 7, 9, 15–19]. Moreover, we have been observed that from different presented methods to solve singularly perturbed problems with turning point due to the vanishments of the convection coefficient (a(0)=0), the family of fitted operator methods are not appropriate. This is due to the occurrences of interior layer and behaviors of fitted operator methods. Furthermore the procedures provided in this paper, focused on the numerical technique to produce accurate numerical solution for the families of the problems with interior layers or one can extend to apply on the multiple turning point problems.

2. Description of the Scheme

To define a piecewise-uniform mesh, we consider a positive integer $N \geq 8$. Since the solution of the problem under consideration exhibits an interior layer, we chose the transition parameter τ defined by

$$\tau = \min\left\{\frac{1}{4}, 2\sqrt{\varepsilon}\,\ln(N)\right\}. \tag{3}$$

Now, divide the solution domain $\overline{\Omega} = [-1, 1]$ into three subintervals $[-1, -\tau]$, $[-\tau, \tau]$, and $[\tau, 1]$ with $(N/4)$ points in$[-1, -\tau]$, $[\tau, 1]$, and $(N/2)$ points in $[-\tau, \tau]$. Thus, the mesh spacing in these subintervals is given by

$$h_i = \begin{cases} \dfrac{4(1-\tau)}{N}, & i = 1, 2, \ldots, \dfrac{N}{4}, \dfrac{3N}{4} + 1, \ldots, N, \\ \dfrac{4\tau}{N}, & \dfrac{N}{4} + 1, \ldots, \dfrac{3N}{4}, \end{cases} \tag{4}$$

and the mesh pints are given by

$$x_i = \begin{cases} -1 + ih_i, & i = 0, 1, 2, \ldots, \dfrac{N}{4}, \\ -\tau + \left(i - \dfrac{N}{4}\right)h_i, & \dfrac{N}{4} + 1, \ldots, \dfrac{3N}{4}, \\ \tau + \left(i - \dfrac{3N}{4}\right)h_i, & \dfrac{3N}{4} + 1, \ldots, N. \end{cases} \tag{5}$$

Denoting this discretization of the solution domain by Ω^N, and y_i is the approximation of $y(x_i)$, so that the discretization form of equation (1) on Ω^N, for $i = 1, 2, \ldots, N-1$ is given by

$$\varepsilon\delta^2 y_i + a_i\delta^0 y_i - b_i y_i = f_i, \tag{6}$$

where $\delta^2 y_i = (2/(h_i + h_{i+1}))(\delta^+ y_i - \delta^- y_i)$, $\delta^0 y_i = ((y_{i+1} - y_{i-1})/(h_i + h_{i+1}))$, $\delta^+ y_i = ((y_{i+1} - y_i)/h_{i+1})$, and $\delta^- y_i = ((y_i - y_{i-1})/h_i)$.

Further, equation (6) can be rewritten in the form of three-term recurrence relation of the form:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \tag{7}$$

where $E_i = (2\varepsilon/h_i(h_i + h_{i+1})) - (a_i/(h_i + h_{i+1}))$, $F_i = (2\varepsilon/h_i h_{i+1}) + b_i$, $G_i = (2\varepsilon/h_{i+1}(h_i + h_{i+1})) + (a_i/(h_i + h_{i+1}))$, and $H_i = f_i$.

2.1. Thomas Algorithm. In this section, the stability of solving the tridiagonal system given in equation (7) is provided. Consider the scheme in equation (7), for $i = 1, 2, \ldots, N-1$ and subject to the boundary conditions in equation (1) that can be rewritten as $y(-1) = y_0 = l_1$, and $y(1) = y_N = l_2$. Assume that the solution of equation (7) is given by

$$y_i = W_i y_{i+1} + T_i, \quad i = N-1, N-2, \ldots, 2, 1, \tag{8}$$

where W_i and T_i are to be determined.

Considering equation (7) at the nodal point x_{i-1}, we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1}. \tag{9}$$

Substituting equations (9) into (7) gives $E_i(W_{i-1}y_i + T_{i-1}) - F_i y_i + G_i y_{i+1} = H_i$, which leads to obtaining the equation:

$$y_i = \frac{G_i}{F_i - E_i W_{i-1}} y_{i+1} + \frac{-H_i + E_i T_{i-1}}{F_i - E_i W_{i-1}}. \tag{10}$$

Comparing equations (10) with (8), the two values determined as

$$\begin{aligned} W_i &= \frac{G_i}{F_i - E_i W_{i-1}}, \\ T_i &= \frac{-H_i + E_i T_{i-1}}{F_i - E_i W_{i-1}}. \end{aligned} \tag{11}$$

To solve these recurrence relations for $i = 1, 2, \ldots, N-1$, we need the initial conditions for $W_0 = 0$ and we take $T_0 = y_0 = y(-1) = l_1$. With this starting point of initial values, we compute W_i and T_i for $i = 1, 2, \ldots, N-1$ from equation (11) in the forward process, and then obtain y_i in the backward process from equation (8) and from the boundary condition $y(1) = y_N = l_2$. Further, the conditions for the discrete invariant imbedding algorithm to be stable, if and only if [11]

$$|F_i| \geq |E_i + G_i|. \tag{12}$$

Hence, the Thomas algorithm is stable for the described numerical scheme.

2.2. Truncation Error. In this section, the truncation error for the described method will be investigated. To achieve this investigation, the local truncation error $T(h_i)$ between the exact solution $y(x_i)$, and the approximate solution y_i is given by

$$\begin{aligned} T(h_i) &= \varepsilon y''(x_i) + a(x_i)y'(x_i) - b(x_i)y(x_i) \\ &\quad - \left\{ \frac{2\varepsilon}{h_i + h_{i+1}} (\delta^+ y_i - \delta^- y_i) + a_i \delta^0 y_i - b_i y_i \right\}. \end{aligned} \tag{13}$$

Using Taylor's series expansion to y_i around x_i, we have the approximation for $y_{i\pm1}$ as

$$\begin{aligned} y_{i+1} &= y_i + h_{i+1} y_i' + \frac{h_{i+1}^2}{2} y_i'' + \frac{h_{i+1}^3}{6} y_i''' + \frac{h_{i+1}^4}{24} y_i^{(4)} + O(h_{i+1}^5), \\ y_{i-1} &= y_i - h_i y_i' + \frac{h_i^2}{2} y_i'' - \frac{h_i^3}{6} y_i''' + \frac{h_i^4}{24} y_i^{(4)} + O(h_i^5). \end{aligned} \tag{14}$$

From these two basic equations, we obtain the following:

$$\delta^+ y_i = \frac{y_{i+1} - y_i}{h_{i+1}} = y_i' + \frac{h_{i+1}}{2} y_i'' + \frac{h_{i+1}^2}{6} y_i''' + \frac{h_{i+1}^3}{24} y_i^{(4)} + O(h_{i+1}^4), \tag{15}$$

$$\delta^- y_i = \frac{y_i - y_{i-1}}{h_i} = y_i' - \frac{h_i}{2} y_i'' + \frac{h_i^2}{6} y_i''' - \frac{h_i^3}{24} y_i^{(4)} + (Oh_i^4), \tag{16}$$

$$\delta^0 y_i = \frac{y_{i+1} - y_{i-1}}{h_{i+1} + h_i} = \frac{h_{i+1} - h_i}{2} y_i'' + \frac{h_{i+1}^3 + h_i^3}{6(h_{i+1} + h_i)} y_i''' + \ldots. \tag{17}$$

Substituting equations (15)–(17) into (13) and recall at the nodal point x_i: $y''(x_i) = y_i''$ $a(x_i)y'(x_i) = a_i y_i'$, and $b(x_i)y(x_i) = b_i y_i$, we get

$$T(h_i) = -\frac{h_{i+1} - h_i}{2} y_i'' - \left\{ \frac{\varepsilon}{3}(h_{i+1} - h_i) + \frac{h_{i+1}^3 + h_i^3}{6(h_{i+1} + h_i)} \right\} y_i''' + \ldots. \tag{18}$$

However, $h_{i+1} - h_i = 0$, in the outer and interior layer regions. From the considered piecewise discretization of the solution domain, assume that the value of chosen transition parameter is $\tau = 2\sqrt{\varepsilon}\ln(N)$. Thus, in the neighborhoods between the inner and outer layer region, we have:

$$h_{i+1} - h_i = \frac{4(1-\tau)}{N} - \frac{4\tau}{N} = \frac{4 - 8\tau}{N} = \frac{4 - 8\sqrt{\varepsilon}\ln(N)}{N}. \tag{19}$$

Since the considered problem exhibits an interior layer, the described scheme works on piecewise discretization. Thus, we have to consider the values of the perturbation parameter, $\varepsilon \leq (4/N)$. Substituting this inequality into equation (19) gives

$$h_{i+1} - h_i \leq \frac{4 - 8\sqrt{(4/N)}\ln(N)}{N} = \frac{4\sqrt{N} - 16\ln(N)}{N\sqrt{N}} \leq N^{-2}. \tag{20}$$

Thus, from equations (18) and (20) the norm of truncation error for the formulated scheme is

$$\|T\| \leq \mathrm{C}N^{-2}, \tag{21}$$

where $C = (1/2)\|y_i''\|_\infty$ is arbitrary constant.

Furthermore, within each subinterval $[-1, -\tau]$, $[-\tau, \tau]$ and $[\tau, 1]$, we have the uniform mesh length $h \leq (2/N)$. Thus, $h^2 \leq N^{-2}$. Therefore, the described method is almost second-order convergent. Truncation errors measure how well a finite difference discretization approximates the differential equation. Thus, the described scheme is almost second-order accurate. A finite difference scheme is known as consistent if the limit of truncation error is equal to zero as the mesh size goes to zero, [4]. Hence, this definition of consistency on the described method with the local truncation error in equation (18) is satisfied. Therefore, using this consistency and stability criteria provided in equation (12), the proposed scheme is convergent.

2.3. Richardson Extrapolation. Richardson extrapolation is that whenever the leading term in the error for an approximation scheme is known. By combining two or more approximations obtained from that scheme using different values of mesh lengths: h_i, $(h_i/2)$, $(h_i/4)$, ... to obtain a higher-order approximation and the technique is known as Richardson extrapolation. This procedure is a convergence acceleration technique that consists of considering a linear combination of two computed approximations of a solution.

Particularly in our case, the described numerical scheme is almost second-order convergent as verified in equation (21). With the purpose of this equation, we have

$$\left|y(x_i) - y_i^N\right| \le Ch_i^2, \tag{22}$$

where $y(x_i)$ and y_i^N are exact and approximate solutions, respectively, C is a constant independent of mesh sizes h_i and perturbation parameters. Let us be the mesh obtained by bisecting each mesh interval in Ω^N and denote the approximation of the solution on Ω^{2N} by y_i^{2N}. Then, consider equation (22) works for any $h_i \neq 0$, which implies

$$y(x_i) - y_i^N \approx Ch_i^2 + R^N, \quad x_i \in \overline{\Omega}. \tag{23}$$

So that, it works for any $(h_i/2) \neq 0$ yields

$$y(x_i) - y_i^{2N} \approx C\left(\frac{h_i}{2}\right)^2 + R^{2N}, \quad x_i \in \overline{\Omega}, \tag{24}$$

where the remainders, R^N and R^{2N} are $O(h^4)$.

Eliminating the constant C from equations (23) and (24) leads to $3y(x_i) - (4y_i^{2N} - y_i^N) \cong O(h^4)$, which suggests that

$$\left(y_i^N\right)^{\text{ext}} = \frac{1}{3}\left(4y^{2N} - y^N\right), \tag{25}$$

is also an approximation of $y(x_i)$.

Using this approximation to evaluate the truncation error, we obtain

$$\left|y(x_i) - \left(y_i^N\right)^{\text{ext}}\right| \le Ch_i^4. \tag{26}$$

Now, using these two different solutions which are obtained by the same scheme given by equation (7), we get another third solution in terms of the two by equation (25). This is the Richardson extrapolation technique to accelerate the almost second-order to almost fourth-order convergent. Similar to these procedures eliminating the constant $C_1, C_2, C_3, \ldots$ from the system of equations:

$$\begin{cases} y(x_i) - y_i^N \approx C_1 h_i^2 + C_2 h_i^4 + C_3 h_i^6 + C_4 h_i^8 + \ldots, \\ y(x_i) - y_i^{2N} \approx C_1\left(\frac{h_i}{2}\right)^2 + C_2\left(\frac{h_i}{2}\right)^4 + C_3\left(\frac{h_i}{2}\right)^6 + C_4\left(\frac{h_i}{2}\right)^8 + \ldots, \end{cases}$$
$$\begin{cases} y(x_i) - y_i^{2N} \approx C_1\left(\frac{h_i}{2}\right)^2 + C_2\left(\frac{h_i}{2}\right)^4 + C_3\left(\frac{h_i}{2}\right)^6 + C_4\left(\frac{h_i}{2}\right)^8 + \ldots, \\ y(x_i) - y_i^{4N} \approx C_1\left(\frac{h_i}{4}\right)^2 + C_2\left(\frac{h_i}{4}\right)^4 + C_3\left(\frac{h_i}{4}\right)^6 + C_4\left(\frac{h_i}{4}\right)^8 + \ldots. \end{cases} \tag{27}$$

We obtain the two fourth-orders convergent numerical scheme:

$$\begin{cases} \left(y_i^N\right)^{4\text{ext}} = \frac{1}{3}\left(4y_i^{2N} - y_i^N\right), \\ \left(y_i^{2N}\right)^{4\text{ext}} = \frac{1}{3}\left(4y_i^{4N} - y_i^{2N}\right), \end{cases} \tag{28}$$

which gives the sixth-order convergent:

$$\left(y_i^N\right)^{6\text{ext}} = \frac{1}{15}\left\{16\left(y_i^{2N}\right)^{4\text{ext}} - \left(y_i^N\right)^{4\text{ext}}\right\}. \tag{29}$$

3. Numerical Illustrations

In this section, to endorse the applicability, efficiency, and effectiveness of the formulated scheme, model examples are considered. Furthermore, for the sake of comparison, the test examples have an exact solution [9]. However, it is possible to use the proposed method for modeled examples that have no exact solution and for its comparison double mesh principle will be used. Computational results are provided in the form of tables and figures. The errors presented in the tables are calculated in the maximum norm as

Table 1: Comparison of maximum absolute errors for Example 1.

$\varepsilon \downarrow N \longrightarrow$	16	32	64	128	256	512
Present method						
2^{-8}	$3.2593e-03$	$3.2354e-04$	$1.4027e-05$	$3.5305e-07$	$5.5660e-09$	$7.8408e-11$
2^{-12}	$5.2735e-03$	$1.2840e-03$	$5.1993e-04$	$9.2506e-05$	$1.1128e-05$	$8.4250e-07$
2^{-16}	$6.8477e-03$	$1.5913e-03$	$3.7413e-04$	$1.1250e-04$	$4.4767e-05$	$9.6625e-06$
2^{-20}	$7.1858e-03$	$1.7060e-03$	$4.1918e-04$	$1.0273e-04$	$2.4288e-05$	$8.5800e-06$
2^{-24}	$7.2686e-03$	$1.7292e-03$	$4.2747e-04$	$1.0638e-04$	$2.6433e-05$	$6.4898e-06$
2^{-28}	$7.2868e-03$	$1.7347e-03$	$4.2920e-04$	$1.0700e-04$	$2.6715e-05$	$6.6661e-06$
2^{-32}	$7.2912e-03$	$1.7360e-03$	$4.2962e-04$	$1.0714e-04$	$2.6764e-05$	$6.6884e-06$
Results in [9]						
2^{-8}	$1.77e-01$	$1.02e-01$	$6.00e-02$	$4.06e-02$	$3.10e-02$	$2.64e-02$
2^{-12}	$1.49e-01$	$5.00e-02$	$2.55e-02$	$1.44e-02$	$8.64e-03$	$5.46e-03$
2^{-16}	$1.93e-01$	$4.80e-02$	$1.06e-02$	$4.65e-03$	$2.34e-03$	$1.30e-03$
2^{-20}	$2.05e-01$	$5.44e-02$	$1.32e-02$	$3.02e-03$	$8.48e-04$	$3.72e-04$
2^{-24}	$2.08e-01$	$5.61e-02$	$1.41e-02$	$3.46e-03$	$8.19e-04$	$1.82e-04$
2^{-28}	$2.09e-01$	$5.66e-02$	$1.44e-02$	$3.58e-03$	$8.84e-04$	$2.14e-04$
2^{-32}	$2.09e-01$	$5.67e-02$	$1.44e-02$	$3.61e-03$	$9.00e-04$	$2.23e-04$

Table 2: Comparison of maximum absolute errors for Example 2.

$\varepsilon \downarrow N \longrightarrow$	16	32	64	128	256	512
Present method						
2^{-8}	$7.2136e-05$	$3.0317e-06$	$4.7218e-08$	$7.1839e-10$	$1.1296e-11$	$4.0901e-13$
2^{-12}	$3.8966e-03$	$1.0833e-04$	$4.2209e-06$	$1.5688e-07$	$5.1949e-09$	$1.6333e-10$
2^{-16}	$3.9313e-03$	$1.0915e-04$	$4.2209e-06$	$1.5688e-07$	$5.1949e-09$	$1.6334e-10$
2^{-20}	$3.9336e-03$	$1.0921e-04$	$4.2209e-06$	$1.5688e-07$	$5.1949e-09$	$1.6333e-10$
2^{-24}	$3.9338e-03$	$1.0921e-04$	$4.2209e-06$	$1.5688e-07$	$5.1949e-09$	$1.6335e-10$
2^{-28}	$3.9338e-03$	$1.0921e-04$	$4.2209e-06$	$1.5688e-07$	$5.1949e-09$	$1.6333e-10$
2^{-32}	$3.9338e-03$	$1.0921e-04$	$4.2209e-06$	$1.5688e-07$	$5.1949e-09$	$1.6334e-10$
Results in [9]						
2^{-8}	$5.43e-02$	$1.48e-02$	$3.24e-03$	$6.61e-04$	$1.88e-04$	$5.10e-05$
2^{-12}	$6.31e-02$	$1.57e-02$	$3.47e-03$	$9.45e-04$	$2.75e-04$	$7.71e-05$
2^{-16}	$6.63e-02$	$1.62e-02$	$3.66e-03$	$1.02e-03$	$2.98e-04$	$8.38e-05$
2^{-20}	$6.67e-02$	$1.61e-02$	$3.73e-03$	$1.04e-03$	$3.03e-04$	$8.54e-05$
2^{-24}	$6.69e-02$	$1.61e-02$	$3.75e-03$	$1.04e-03$	$3.05e-04$	$8.58e-05$
2^{-28}	$6.69e-02$	$1.61e-02$	$3.75e-03$	$1.04e-03$	$3.05e-04$	$8.59e-05$
2^{-32}	$6.69e-02$	$1.61e-02$	$3.75e-03$	$1.04e-03$	$3.05e-04$	$8.60e-05$

$$\|E\|^N = \max_i \left| y(x_i) - y_i \right|, \quad \text{for } i = 0, 1, 2, \ldots, N, \tag{30}$$

where $y(x_i)$ and y_i are the exact and approximate solutions of the given problem at x_i. The computed rate of convergence is defined by the formula [4]:

$$R = \frac{\log\left(\|E\|^N\right) - \log\left(\|E\|^{2N}\right)}{\log 2}. \tag{31}$$

Example 1. Consider the singularly perturbed problem given by

$$\begin{cases} \varepsilon y''(x) + x y'(x) = -\pi x \sin(\pi x) - \varepsilon \pi^2 (\cos \pi x), & -1 < x < 1, \\ y(-1) = -2, \\ y(1) = 0. \end{cases} \tag{32}$$

The solution exhibits an interior layer of width $O(\sqrt{\varepsilon}\,|\ln \varepsilon|)$ at $x = 0$.

The exact solution to the problem is $y(x) = \cos(\pi x) + (\operatorname{erf}(x/\sqrt{2\varepsilon})/\operatorname{erf}(1/\sqrt{2\varepsilon}))$.

Example 2. Consider the singular perturbation problem,

$$\begin{cases} \varepsilon y''(x) + 2x y'(x) = 0, & -1 < x < 1, \\ y(-1) = -1, \\ y(1) = 1. \end{cases} \tag{33}$$

The solution exhibits an interior layer of width $O(\sqrt{\varepsilon}\,|\ln \varepsilon|)$ at $x = 0$.

The exact solution to the problem is $y(x) = (\operatorname{erf}(x/\sqrt{\varepsilon})/\operatorname{erf}(1/\sqrt{\varepsilon}))$.

The computed maximum absolute errors and rate of convergences are presented in Tables 1–4 The more recently

Table 3: Comparison of maximum absolute errors for Example 2.

$\varepsilon\downarrow$	$N = 128$		$N = 256$		$N = 512$	
	Results in [9]	Present method	Results in [9]	Present method	Results in [9]	Present method
2^{-4}	$1.58e-02$	$4.5980e-10$	$9.80e-04$	$1.6310e-10$	$8.03e-05$	$6.7635e-13$
2^{-8}	$1.61e-02$	$7.1839e-10$	$1.04e-03$	$1.6310e-10$	$8.54e-05$	$4.0901e-13$
2^{-10}	$1.61e-02$	$4.7218e-08$	$1.04e-03$	$7.1838e-10$	$8.59e-05$	$1.1305e-11$
2^{-12}	$1.61e-02$	$1.5688e-07$	$1.05e-03$	$5.1949e-09$	$8.60e-05$	$1.6333e-10$

Table 4: Computed rate of convergence for Example 2.

$\varepsilon\downarrow N\longrightarrow$	16	32	64	128	256
2^{-8}	4.5725	6.0046	6.0384	5.9909	4.7875
2^{-12}	5.1687	4.6817	4.7498	4.9164	4.9912
2^{-16}	5.1706	4.6926	4.7498	4.9164	4.9911
2^{-20}	5.1707	4.6934	4.7498	4.9164	4.9912
2^{-24}	5.1707	4.6934	4.7498	4.9164	4.9912
2^{-28}	5.1707	4.6934	4.7498	4.9164	4.9912
2^{-32}	5.1707	4.6934	4.7498	4.9164	4.9912

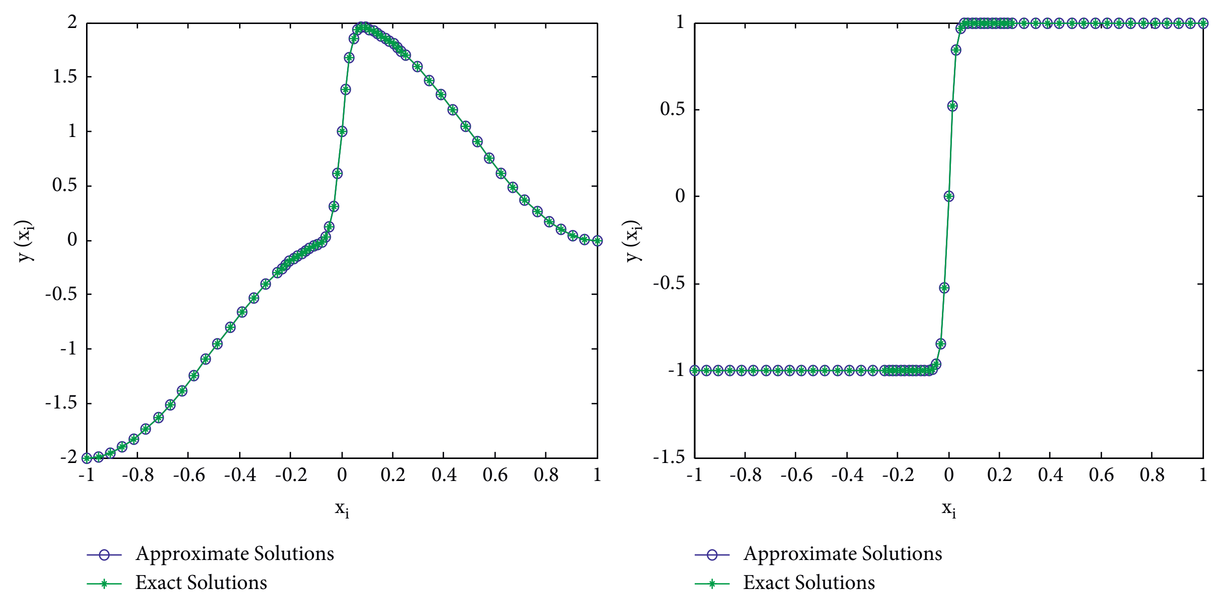

Figure 1: Graphs of approximate and exact solutions for Examples 1 and 2 respectively, when $\varepsilon = 2^{-10}$ and $N = 64$.

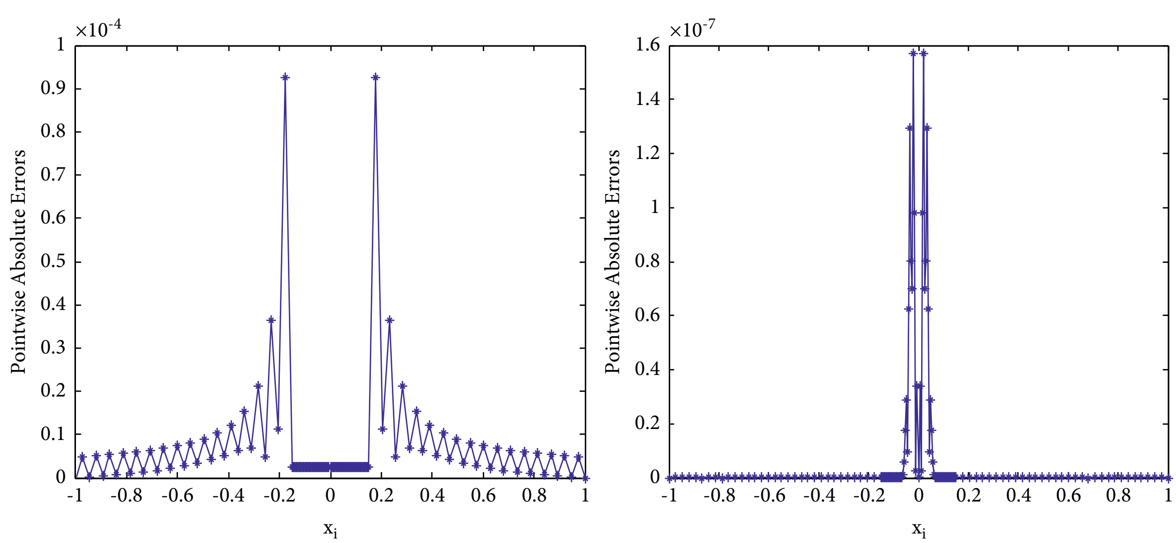

Figure 2: Point wise absolute errors for Examples 1 and 2 respectively, when $\varepsilon = 2^{-12}$ and $N = 128$.

existing results and the computational results for examples under consideration are compared in Tables 1–3. These results clearly show that the described scheme is more accurate than other existing methods. Also, Table 4 shows that the conformation of the accelerated rate of with the theoretical descriptions. Moreover, the described scheme is shown to be accelerated to almost uniformly convergent of sixth order. From the results presented in the Tables, it can be se en that the errors decrease as the perturbation parameter ε, decreases and after a certain, ε it stabilized which confirms the uniform convergence. The effectiveness and accuracy of the described scheme are confirmed by the comparison with existing results. Figure 1, verify that the physical behavior of the solution exhibits an interior layer. Further, Figure 2 indicates the effects of perturbation parameter on the solution profile that causes the maximum errors occurred for the two considered examples respectively.

4. Conclusion

In this paper, singularly perturbed turning point boundary value problems are solved numerically by the accelerated Shishkin mesh type. This is sufficiently used to resolve the problems under consideration and gives accurate results even for small perturbation parameters closer and closer to zero. The stability of the described scheme is analyzed and the truncation error is obtained that guarantees its convergence. The proposed scheme is of almost second-order convergent and accelerated to almost sixth-order convergent by applying the Richardson extrapolation technique. The numerical results obtained by the present scheme have been compared with some existing methods and it is observed that the proposed scheme gives better accuracy.

Data Availability

No data were used to support this study.

Consent

Not applicable.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Authors' Contributions

The author of this manuscript contributes to proposing the research title, describing, making MATLAB coding to provide the numerical results, analysis, and approving the final manuscript.

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