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Research Article

Study of Graphene Networks and Line Graph of Graphene Networks via *NM*-Polynomial and Topological Indices

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The topological invariants are related to the molecular graph of the chemical structure and are numerical numbers that help us to understand the topology of the concerned chemical structure. With the help of these numbers, many properties of graphene can be guessed without preforming any experiment. Huge amount of calculations are required to obtain topological invariants for graphene, but by applying basic calculus roles, neighborhood M -polynomial of graphene gives its indices. The aim of this work is to compute neighborhood degree-dependent indices for the graph of graphene and the line graph of subdivision graph of graphene. Firstly, we establish neighborhood M-polynomial of these families of graphs, and then, by applying basic calculus, we obtain several neighborhood degree-dependent indices. Our results play an important role to understand graphene and enhance its abilities.

1. Introduction

Graph is the union of edges (lines) and vertices (nodes), and the study of graphs is known as the graph theory [1, 2]. In this study, we study the properties of graphs and invent applications of different families of graphs [3, 4]. Today, the graph theory got applications in almost all areas of sciences and engineering, and applying the graph theory to solve problems of chemistry is known as the chemical graph theory [5, 6].

In the chemical graph theory, topological indices are graph invariants that remain the same up to graph isomorphism and help us to attain properties of molecular graphs without any lab work. The first topological index was introduced by Wiener in 1947 when he was working on the boiling point of alkane [7]. Today, this index is named the Wiener index, which is a distance-based topological index. After Wiener, Randić introduced the first degree-depend index, which was firstly named the path index, but today, it is known as the Randić index [8]. This index got huge attention of researchers due to its amazing applications. After the promising success of the Randić index, Gutman coined the idea of Zagreb indices and many research papers are written on this index (see [9] and references therein).

Since now, a single index can give all the information about any molecular graph, so till now, more than 150 indices are defined and studied [10–12]. Since huge amount of work is required to obtain topological indices, so the research found a shortcut to obtain them and introduced M-polynomial [13] and this polynomial gives easy way to compute almost all degree-based indices [14, 15]. Many studies have been performed so far on M-polynomial [16, 17], and still, there are many known interesting macular graphs whose M-polynomial can be established. Following M-polynomial, neighborhood M -polynomial was introduced in Ref. [18], which is helpful in determining neighborhood degree sum-based graphical indices.

The aim of this work is to compute neighborhood degree-dependent indices for the graph of graphene and the line graph of subdivision graph of graphene. Firstly, we establish neighborhood M-polynomial of these families of graphs, and then, by applying basic calculus, we obtain several neighborhood degree-dependent indices.

2. Primaries

Throughout this work, we consider only finite, simple, undirected, and connected graphs with a vertex set $V(G)$ and an edge set $E(G)$ [19, 20]. The degree $d_G(v)$ of a vertex v is the number of vertices adjacent to v [21, 22]. The neighborhood degree sum-based graphical indices depend upon the neighborhood degree, which is defined by $d_G(e) = d_G(u) + d_G(v) - 2$ for an edge $e = uv$, where u and v are vertices of the edge e. Neighborhood degree sum δ_u [23, 24] of a vertex u is defined as the sum of degrees of all vertices that are adjacent to the vertex. The degree of a vertex is the total number of edges incident to the vertex. The line graph $L(G)$ of a graph G is the graph whose vertex set corresponds to the edges of G such that two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent (see for details [25, 26] and references therein). The subdivision graph $S(G)$ of a graph G is the graph obtained from G by replacing each of its edges by a path of length two (see [27, 28]).

The neighborhood M -polynomial of a graph G is defined as [18]

$$NM(G;x,y) = \sum_{i\le j} m_{i,j} x^i y^j, \tag{1}$$

where $m_{i,j}$ is the total number of edges $uv \in E(G)$ such that $\{\delta_u, \delta_v\} = \{i, j\}$. We use $NM(G)$ for $NM(G;x,y)$ in this work.

The neighborhood degree-based graphical indices defined on the edge set of a graph G can be expressed as

$$I(G) = \sum_{uv\in E(G)} f(\delta_u, \delta_v), \tag{2}$$

where $f(\delta_u, \delta_v)$ is the function of δ_u, δ_v used in the definition of neighborhood degree-based indices. The above result can also be written as

$$I(G) = \sum_{i\le j} m_{i,j} f(i,j). \tag{3}$$

By taking $f(\delta_u, \delta_v) = \delta_u + \delta_v, \delta_u\delta_v, \delta_u^2 + \delta_v^2$, $(1/\delta_u\delta_v)$, $(\delta_u\delta_v)^\alpha$, $\delta_u\delta_v(\delta_u + \delta_v)$, $(\delta_u/\delta_v) + (\delta_v/\delta_u)$, $(2/\delta_u + \delta_v)$, $(\delta_u\delta_v/\delta_u + \delta_v)$, $(\delta_u\delta_v/\delta_u + \delta_v - 2)^3$, $[\delta_u + \delta_v]^2$, $[\delta_u\delta_v]^2$, $(\delta_u + \delta_v/\delta_u\delta_v)$, $(2\sqrt{\delta_u\delta_v}/\delta_u + \delta_v)$ in equation (2), we get the third version of the Zagreb index, the neighborhood second Zagreb index, the neighborhood forgotten graphical index, the neighborhood second modified Zagreb index, the neighborhood general Randić index, the third NDe index, the fifth NDe index, the neighborhood Harmonic index, the neighborhood inverse sum index, the Sanskruti index, the fifth hyper M_1 Zagreb index, the fifth hyper M_2 Zagreb index, the fifth arithmetic-geometric index, and the fifth geometric-arithmetic index, respectively [29–32].

All the abovementioned indices can be computed directly from NM-polynomial, and the relations of some neighborhood degree-based graphical indices with the NM -polynomial are shown in Table 1, where $D_x(f(x,y)) = x((\partial(f(x,y)))/\partial x)$, $D_y(f(x,y)) = y((\partial(f(x,y)))/\partial y)$, $S_x(f(x,y)) = \int_0^x ((f(t,y))/t)dt$, $S_y(f(x,y)) = \int_0^y ((f(x,t))/t)dt$, and $J(f(x,y)) = f(x,x)$, $Q_\alpha(f(x,y)) = x^\alpha f(x,y)$ are the operators.

3. Neighborhood M -Polynomial of Graphene Networks

In this section, we find neighborhood M-polynomial of graphene and calculate neighborhood degree-based graphical indices of graphene by using its neighborhood M -polynomial.

3.1. Graphene Networks. Graphene is an atomic scale honeycomb lattice made of the carbon atoms. Graphene is denoted by $G_{t,s}$, where t is the number of rows of benzene rings and s is the number of benzene rings in each row. Graphene has $2st + 2s + 2t$ vertices and $3st + 2s + 2t - 1$ edges [33]. Figure 1 shows the graphene $G_{t,s}$.

Theorem 1. *Let $G_{t,s}$ be a graphene network, with $t > 1$ and $s > 1$; then, its neighborhood M -polynomial is given by*

$$\begin{aligned} NM(G_{t,s}) = {} & 4x^4y^5 + tx^5y^5 + 8x^5y^7 + (2t-4)x^5y^8 \\ & + (4s-8)x^6y^7 + 2sx^7y^9 + (t-2)x^8y^8 \\ & + (2t-4)x^8y^9 + (3st-4s-4t+5)x^9y^9. \end{aligned} \tag{4}$$

Proof. Graphene has $2st + 2s + 2t$ vertices and $3st + 2s + 2t - 1$ edges. The edge set of $G_{t,s}$, for $t > 1$ and $s > 1$, can be partitioned as follows:

$$\begin{aligned} |E_{4,5}| &= |\{uv \in E(G_{t,s}): \delta_u = 4, \delta_v = 5\}| = 4 = m_{4,5}, \\ |E_{5,5}| &= |\{uv \in E(G_{t,s}): \delta_u = 5, \delta_v = 5\}| = t = m_{5,5}, \\ |E_{5,7}| &= |\{uv \in E(G_{t,s}): \delta_u = 5, \delta_v = 7\}| = 8 = m_{5,7}, \\ |E_{5,8}| &= |\{uv \in E(G_{t,s}): \delta_u = 5, \delta_v = 8\}| = 2t-4 = m_{5,8}, \\ |E_{6,7}| &= |\{uv \in E(G_{t,s}): \delta_u = 6, \delta_v = 7\}| = 4s-8 = m_{6,7}, \\ |E_{7,9}| &= |\{uv \in E(G_{t,s}): \delta_u = 7, \delta_v = 9\}| = 2s = m_{7,9}, \\ |E_{8,8}| &= |\{uv \in E(G_{t,s}): \delta_u = 8, \delta_v = 8\}| = t-2 = m_{8,8}, \\ |E_{8,9}| &= |\{uv \in E(G_{t,s}): \delta_u = 8, \delta_v = 9\}| = 2t-4 = m_{8,9}, \\ |E_{9,9}| &= |\{uv \in E(G_{t,s}): \delta_u = 9, \delta_v = 9\}| \\ &= 3st-4s-4t+5 = m_{9,9}. \end{aligned} \tag{5}$$

Thus, the neighborhood M -polynomial of $G_{t,s}$, for $t > 1$ and $s > 1$, is

Table 1: Derivation of some neighborhood degree-based graphical indices [18].

Graphical index	$f(x, y)$	Derivation from $NM(G)$
M_1'	$x + y$	$(D_x + D_y)(NM(G))\|_{x=y=1}$
M_2^*	xy	$(D_x D_y)(NM(G))\|_{x=y=1}$
F_N^*	$x^2 + y^2$	$(D_x^2 + D_y^2)(NM(G))\|_{x=y=1}$
M_2^{nm}	$1/xy$	$(S_x S_y)(NM(G))\|_{x=y=1}$
NR_α	$(xy)^\alpha$	$(D_x^\alpha D_y^\alpha)(NM(G))\|_{x=y=1}$
ND_3	$xy(x + y)$	$D_x D_y (D_x + D_y)(NM(G))\|_{x=y=1}$
ND_5	$(x^2 + y^2)/xy$	$(D_x S_y + S_x D_y)(NM(G))\|_{x=y=1}$
NH	$2/(x + y)$	$(2S_x J)(NM(G))\|_{x=y=1}$
NI	$xy/(x + y)$	$(S_x J D_x D_y)(NM(G))\|_{x=y=1}$
S	$xy/(x + y - 2)^3$	$(S_x^3 Q_{-2} J D_x^3 D_y^3)(NM(G))\|_{x=y=1}$
$HM_1G_5(G)$	$(x + y)^2$	$(D_x^2 + D_y^2 + 2D_x D_y)(NM(G))\|_{x=y=1}$
$HM_2G_5(G)$	$(xy)^2$	$D_x D_y (D_x D_y)(NM(G))\|_{x=y=1}$
$AG_5(G)$	$(x + y)/(2\sqrt{xy})$	$(1/2)S_x^{(1/2)} S_y^{(1/2)} (D_x + D_y)(NM(G))\|_{x=y=1}$
$GA_5(G)$	$(2\sqrt{xy})/(x + y)$	$(2S_x J D_x^{1/2} D_y^{1/2} (NM(G))\|_{x=1}$

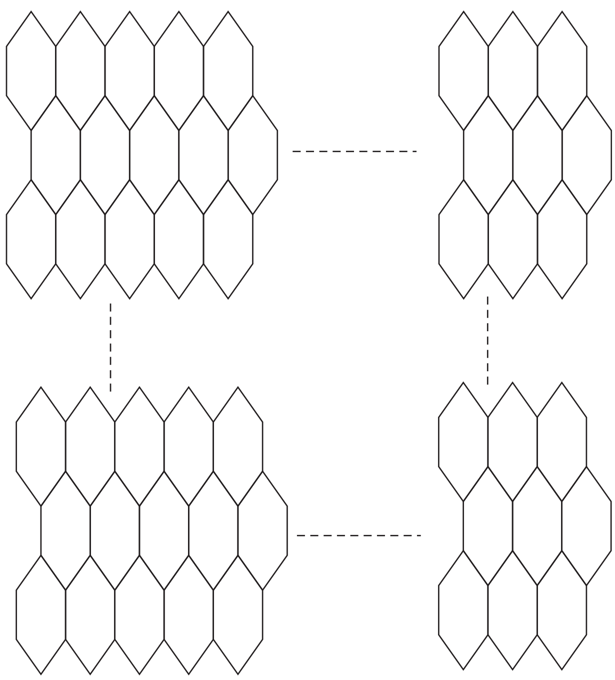

Figure 1: Graphene $G_{t,s}$.

$$NM(G_{t,s}) = \sum_{i\le j} m_{i,j} x^i y^j,$$

$$\begin{aligned} NM(G_{t,s}) = {} & 4x^4y^5 + tx^5y^5 + 8x^5y^7 + (2t-4)x^5y^8 \\ & + (4s-8)x^6y^7 + 2sx^7y^9 + (t-2)x^8y^8 \\ & + (2t-4)x^8y^9 + (3st-4s-4t+5)x^9y^9, \end{aligned} \tag{6}$$

This is the required neighborhood M -polynomial of $G_{t,s}$, for $t > 1$ and $s > 1$. □

Theorem 2. *Let $G_{t,s}$ be a graphene network, with $t = 1$ and $s > 1$; then, its neighborhood M -polynomial is given by*

$$\begin{aligned} NM(G_{t,s}) = {} & 2x^4y^4 + 4x^4y^5 + 4x^5y^7 \\ & + (4s-8)x^6y^7 + (s-1)x^7y^7. \end{aligned} \tag{7}$$

Proof. Graphene has $2st + 2s + 2t$ vertices and $3st + 2s + 2t - 1$ edges. The edge set of $G_{t,s}$, for $t = 1$ and $s > 1$, can be partitioned as follows:

$$\begin{aligned} |E_{4,4}| &= \left|\{uv \in E(G_{t,s}) \colon \delta_u = 4, \delta_v = 4\}\right| = 2 = m_{4,4}, \\ |E_{4,5}| &= \left|\{uv \in E(G_{t,s}) \colon \delta_u = 4, \delta_v = 5\}\right| = 4 = m_{4,5}, \\ |E_{5,7}| &= \left|\{uv \in E(G_{t,s}) \colon \delta_u = 5, \delta_v = 7\}\right| = 4 = m_{5,7}, \\ |E_{6,7}| &= \left|\{uv \in E(G_{t,s}) \colon \delta_u = 6, \delta_v = 7\}\right| = 4s - 8 = m_{6,7}, \\ |E_{7,7}| &= \left|\{uv \in E(G_{t,s}) \colon \delta_u = 7, \delta_v = 7\}\right| = s - 1 = m_{7,7}. \end{aligned} \tag{8}$$

Thus, the neighborhood M -polynomial of $G_{t,s}$, for $t = 1$ and $s > 1$, is

$$\begin{aligned} NM(G_{t,s}) &= \sum_{i\le j} m_{i,j} x^i y^j, \\ NM(G_{t,s}) &= 2x^4y^4 + 4x^4y^5 + 4x^5y^7 \\ & \quad + (4s-8)x^6y^7 + (s-1)x^7y^7. \end{aligned} \tag{9}$$

This is the required neighborhood M -polynomial of $G_{t,s}$, for $t = 1$ and $s > 1$. □

Theorem 3. *Let $G_{t,s}$ be a graphene network, with $t > 2$ and $s = 1$; then, its neighborhood M -polynomial is given by*

$$\begin{aligned} NM(G_{t,s}) = {} & 2x^4y^4 + 4x^4y^5 + (t-2)x^5y^5 + 4x^5y^7 \\ & + (2t-4)x^5y^8 + 2x^7y^8 + (2t-5)x^8y^8. \end{aligned} \tag{10}$$

Proof. Graphene has $2st + 2s + 2t$ vertices and $3st + 2s + 2t - 1$ edges. The edge set of $G_{t,s}$, for $t > 2$ and $s = 1$, can be partitioned as follows:

$$
\begin{aligned}
|E_{4,4}| &= |\{uv \in E(G_{t,s}) \colon \delta_u = 4, \delta_v = 4\}| = 2 = m_{4,4},\\
|E_{4,5}| &= |\{uv \in E(G_{t,s}) \colon \delta_u = 4, \delta_v = 5\}| = 4 = m_{4,5},\\
|E_{5,5}| &= |\{uv \in E(G_{t,s}) \colon \delta_u = 5, \delta_v = 5\}| = t-2 = m_{5,5},\\
|E_{5,7}| &= |\{uv \in E(G_{t,s}) \colon \delta_u = 5, \delta_v = 7\}| = 4 = m_{5,7},\\
|E_{5,8}| &= |\{uv \in E(G_{t,s}) \colon \delta_u = 5, \delta_v = 8\}| = 2t-4 = m_{5,8},\\
|E_{7,8}| &= |\{uv \in E(G_{t,s}) \colon \delta_u = 7, \delta_v = 8\}| = 2 = m_{7,8},\\
|E_{8,8}| &= |\{uv \in E(G_{t,s}) \colon \delta_u = 8, \delta_v = 8\}| = 2t-5 = m_{8,8}.
\end{aligned}
\tag{11}
$$

Thus, the neighborhood M -polynomial of $G_{t,s}$, for $t > 2$ and $s = 1$, is

$$
\begin{aligned}
NM(G_{t,s}) &= \sum_{i \le j} m_{i,j} x^i y^j,\\
NM(G_{t,s}) &= 2x^4y^4 + 4x^4y^5 + (t-2)x^5y^5\\
&\quad + 4x^5y^7 + (2t-4)x^5y^8 + 2x^7y^8 + (2t-5)x^8y^8,
\end{aligned}
\tag{12}
$$

this is the required neighborhood M -polynomial of $G_{t,s}$, for $t > 2$ and $s = 1$.

Now, we calculate neighborhood degree-based graphical indices of graphene by using its neighborhood M -polynomial. □

Corollary 1. *Let $G_{t,s}$ be a graphene network, with $t > 1$ and $s > 1$; then, its neighborhood degree-based graphical indices are given by*

(1) $M_1' = 54st + 12s + 14t - 34$

(2) $M_2^* = 243st - 30s - 11t - 147$

(3) $F_N^* = 486st - 48s - 2t - 306$

(4) $M_2^{nm} = (1/81)(3st - 4s - 4t + 5) + (2/21)(s-2) + (269/2880)(t-2) + (2/63)s + (1/25)t + (3/7)$

(5) $NR_\alpha = (20)^\alpha 4 + (25)^\alpha (t) + (35)^\alpha 8 + (40)^\alpha 2(t-2) + (42)^\alpha 4(s-2) + (63)^\alpha 2s + (64)^\alpha (t-2) + (72)^\alpha 2 (t-2) + (81)^\alpha (3st - 4s - 4t + 5)$

(6) $ND_3 = 4374st - 1632s - 1070t - 2022$

(7) $ND_5 = 2(3st - 4s - 4t + 5) + (170/21)(s-2) + (943/90)(t-2) + (260/63)s + 2t + (879/35)$

(8) $NH = (1/9)(3st - 4s - 4t + 5) + (8/13)(s-2) + (1181/1768)(t-2) + (1/4)s + (1/5)t + (20/9)$

(9) $NI = (9/2)(3st - 4s - 4t + 5) + (168/13)(s-2) + (4116/221)(t-2) + (63/8)s + (5/2)t + (290/9)$

(10) $S = 129.7463379(3st - 4s - 4t + 5) + 222.6536439(s-2) + 412.8858222(t-2) + (729/4)s + (15625/512)t + 436.2944606$

(11) $HM_1G_5(G_{t,s}) = 972st - 108s - 24t - 600$

(12) $HM_2G_5(G_{t,s}) = 19683st - 11250s - 7955t - 5235$

(13) $AG_5(G_{t,s}) = (3st - 4s - 4t + 5) + 4.011887099(s-2) + 5.058949692(t-2) + 2.015810523s + t + 12.1384032$

(14) $GA_5(G_{t,s}) = (3st - 4s - 4t + 5) + ((8\sqrt{42})/13)(s-2) + 4.942553816(t-2) + ((3\sqrt{7})/4)s + t + 11.86333834$

Proof. NM -Polynomial of $G_{t,s}$, for $t > 1$ and $s > 1$, is given by

$$
\begin{aligned}
NM(G_{t,s}) &= 4x^4y^5 + tx^5y^5 + 8x^5y^7 + 2(t-2)x^5y^8\\
&\quad + 4(s-2)x^6y^7 + 2sx^7y^9 + (t-2)x^8y^8\\
&\quad + 2(t-2)x^8y^9 + (3st - 4s - 4t + 5)x^9y^9.
\end{aligned}
\tag{13}
$$

Let

$$NM(G_{t,s}) = f(x,y). \tag{14}$$

Then, we have

$$
\begin{aligned}
f(x,y) = NM(G_{t,s}) &= 4x^4y^5 + tx^5y^5 + 8x^5y^7\\
&\quad + 2(t-2)x^5y^8 + 4(s-2)x^6y^7 + 2sx^7y^9\\
&\quad + (t-2)x^8y^8 + 2(t-2)x^8y^9\\
&\quad + (3st - 4s - 4t + 5)x^9y^9.
\end{aligned}
\tag{15}
$$

(1) M_1' is defined as

$$M_1' = (D_x + D_y)(NM(G))|_{x=y=1}. \tag{16}$$

Now, by using equation (15), we have

$$
\begin{aligned}
M_1' &= (D_x + D_y)(f(x,y))|_{x=y=1},\\
M_1' &= (D_x + D_y)\big(4x^4y^5 + tx^5y^5 + 8x^5y^7 + 2(t-2)x^5y^8 + 4(s-2)x^6y^7 + 2sx^7y^9 + (t-2)x^8y^8 + 2(t-2)x^8y^9 + (3st - 4s - 4t + 5)x^9y^9\big)|_{x=y=1},\\
M_1' &= \big[16x^4y^5 + 20x^4y^5 + 5tx^5y^5 + 5tx^5y^5 + 40x^5y^7 + 56x^5y^7 + 10(t-2)x^5y^8 + 16(t-2)x^5y^8 + 24(s-2)x^6y^7 + 28(s-2)x^6y^7 + 14sx^7y^9\\
&\quad + 18sx^7y^9 + 8(t-2)x^8y^8 + 8(t-2)x^8y^8 + 16(t-2)x^8y^9 + 18(t-2)x^8y^9 + 9(3st - 4s - 4t + 5)x^9y^9 + 9(3st - 4s - 4t + 5)x^9y^9\big]|_{x=y=1},\\
M_1' &= 54st + 12s + 14t - 34.
\end{aligned}
\tag{17}
$$

(2) M_2^* is defined as

$$M_2^* = (D_xD_y)(NM(G))|_{x=y=1}. \tag{18}$$

Now, by using equation (15), we have

$$\begin{aligned}
M_2^* &= \left(D_xD_y\right)(f(x,y))|_{x=y=1},\\
M_2^* &= \left(D_xD_y\right)\left(4x^4y^5 + tx^5y^5 + 8x^5y^7 + 2(t-2)x^5y^8 + 4(s-2)x^6y^7 + 2sx^7y^9\right.\\
&\quad \left.+ (t-2)x^8y^8 + 2(t-2)x^8y^9 + (3st-4s-4t+5)x^9y^9\right)|_{x=y=1},\\
M_2^* &= \left[80x^4y^5 + 25tx^5y^5 + 280x^5y^7 + 80(t-2)x^5y^8 + 168(s-2)x^6y^7 + 126sx^7y^9\right.\\
&\quad \left.+ 64(t-2)x^8y^8 + 144(t-2)x^8y^9 + 81(3st-4s-4t+5)x^9y^9\right]|_{x=y=1},\\
M_2^* &= 243st - 30s - 11t - 147.
\end{aligned} \tag{19}$$

(3) F_N^* is defined as

$$F_N^* = \left(D_x^2 + D_y^2\right)(NM(G))|_{x=y=1}. \tag{20}$$

Now, by using equation (15), we have

$$\begin{aligned}
F_N^* &= \left(D_x^2 + D_y^2\right)(f(x,y))|_{x=y=1},\\
F_N^* &= \left(D_x^2 + D_y^2\right)\left(4x^4y^5 + tx^5y^5 + 8x^5y^7 + 2(t-2)x^5y^8 + 4(s-2)x^6y^7 + 2sx^7y^9 + (t-2)x^8y^8 + 2(t-2)x^8y^9\right.\\
&\quad \left.+ (3st-4s-4t+5)x^9y^9\right)|_{x=y=1},\\
F_N^* &= \left[64x^4y^5 + 100x^4y^5 + 25tx^5y^5 + 25tx^5y^5 + 200x^5y^7 + 392x^5y^7 + 50(t-2)x^5y^8 + 128(t-2)x^5y^8\right.\\
&\quad + 144(s-2)x^6y^7 + 196(s-2)x^6y^7 + 98sx^7y^9 + 162sx^7y^9 + 64(t-2)x^8y^8 + 64(t-2)x^8y^8\\
&\quad \left.+ 128(t-2)x^8y^9 + 162(t-2)x^8y^9 + 81(3st-4s-4t+5)x^9y^9 + 81(3st-4s-4t+5)x^9y^9\right]|_{x=y=1},\\
F_N^* &= 486st - 48s - 2t - 306.
\end{aligned} \tag{21}$$

(4) M_2^{nm} is defined as

$$M_2^{nm} = \left(S_xS_y\right)(NM(G))|_{x=y=1}. \tag{22}$$

Now, by using equation (15), we have

$$\begin{aligned}
M_2^{nm} &= \left(S_xS_y\right)(f(x,y))|_{x=y=1},\\
M_2^{nm} &= \left(S_xS_y\right)\left(4x^4y^5 + tx^5y^5 + 8x^5y^7 + 2(t-2)x^5y^8 + 4(s-2)x^6y^7 + 2sx^7y^9 + (t-2)x^8y^8 + 2(t-2)x^8y^9\right.\\
&\quad \left.+ (3st-4s-4t+5)x^9y^9\right)|_{x=y=1},\\
M_2^{nm} &= \left[\frac{4}{20}x^4y^5 + \frac{1}{25}tx^5y^5 + \frac{8}{35}x^5y^7 + \frac{2}{40}(t-2)x^5y^8 + \frac{4}{42}(s-2)x^6y^7 + \frac{2}{63}sx^7y^9 + \frac{1}{64}(t-2)x^8y^8 + \frac{2}{72}(t-2)x^8y^9\right.\\
&\quad \left.+ \frac{1}{81}(3st-4s-4t+5)x^9y^9\right]|_{x=y=1},\\
M_2^{nm} &= \frac{1}{81}(3st-4s-4t+5) + \frac{2}{21}(s-2) + \frac{269}{2880}(t-2) + \frac{2}{63}s + \frac{1}{25}t + \frac{3}{7}.
\end{aligned} \tag{23}$$

(5) NR_α is defined as

$$NR_\alpha = \left(D_x^\alpha D_y^\alpha\right)(NM(G))|_{x=y=1}. \tag{24}$$

Now, by using equation (15), we have

$$\begin{aligned}
NR_\alpha &= \left(D_x^\alpha D_y^\alpha\right)(f(x,y))|_{x=y=1},\\
NR_\alpha &= \left(D_x^\alpha D_y^\alpha\right)\left(4x^4y^5 + tx^5y^5 + 8x^5y^7 + 2(t-2)x^5y^8 + 4(s-2)x^6y^7 + 2sx^7y^9 + (t-2)x^8y^8 + 2(t-2)x^8y^9 + (3st-4s-4t+5)x^9y^9\right)|_{x=y=1},\\
NR_\alpha &= \Big[(20)^\alpha 4x^4y^5 + (25)^\alpha (t)x^5y^5 + (35)^\alpha 8x^5y^7 + (40)^\alpha 2(t-2)x^5y^8 + (42)^\alpha 4(s-2)x^6y^7 + (63)^\alpha 2sx^7y^9\\
&\quad + (64)^\alpha (t-2)x^8y^8 + (72)^\alpha 2(t-2)x^8y^9 + (81)^\alpha (3st-4s-4t+5)x^9y^9\Big]|_{x=y=1},\\
NR_\alpha &= (20)^\alpha 4 + (25)^\alpha (t) + (35)^\alpha 8 + (40)^\alpha 2(t-2) + (42)^\alpha 4(s-2) + (63)^\alpha 2s + (64)^\alpha (t-2) + (72)^\alpha 2(t-2) + (81)^\alpha (3st-4s-4t+5).
\end{aligned} \tag{25}$$

(6) ND_3 is defined as

$$ND_3 = \left(D_xD_y\right)\left(D_x + D_y\right)(NM(G))|_{x=y=1}. \tag{26}$$

Now, by using equation (15), we have

$$\begin{aligned}
ND_3 &= \left(D_xD_y\right)\left(D_x + D_y\right)(f(x,y))|_{x=y=1},\\
ND_3 &= \left(D_xD_y\right)\left(D_x + D_y\right)\Big(4x^4y^5 + tx^5y^5 + 8x^5y^7 + 2(t-2)x^5y^8 + 4(s-2)x^6y^7 + 2sx^7y^9 + (t-2)x^8y^8\\
&\quad + 2(t-2)x^8y^9 + (3st-4s-4t+5)x^9y^9\Big)|_{x=y=1},\\
ND_3 &= \left(D_xD_y\right)\Big[16x^4y^5 + 20x^4y^5 + 5tx^5y^5 + 5tx^5y^5 + 40x^5y^7 + 56x^5y^7 + 10(t-2)x^5y^8 + 16(t-2)x^5y^8\\
&\quad + 24(s-2)x^6y^7 + 28(s-2)x^6y^7 + 14sx^7y^9 + 18sx^7y^9 + 8(t-2)x^8y^8 + 8(t-2)x^8y^8 + 16(t-2)x^8y^9\\
&\quad + 18(t-2)x^8y^9 + 9(3st-4s-4t+5)x^9y^9 + 9(3st-4s-4t+5)x^9y^9\Big]|_{x=y=1},\\
ND_3 &= \Big[720x^4y^5 + 250tx^5y^5 + 3360x^5y^7 + 1040(t-2)x^5y^8 + 2184(s-2)x^6y^7 + 2016sx^7y^9\\
&\quad + 1024(t-2)x^8y^8 + 2448(t-2)x^8y^9 + 1458(3st-4s-4t+5)x^9y^9\Big]|_{x=y=1},\\
ND_3 &= 4374st - 1632s - 1070t - 2022.
\end{aligned} \tag{27}$$

(7) ND_5 is defined as

$$ND_5 = \left(D_xS_y + S_xD_y\right)(NM(G))|_{x=y=1}. \tag{28}$$

Now, by using equation (15), we have

$$\begin{aligned}
ND_5 &= \left(D_xS_y + S_xD_y\right)(f(x,y))|_{x=y=1}.\\
ND_5 &= \left(D_xS_y + S_xD_y\right)\Big(4x^4y^5 + tx^5y^5 + 8x^5y^7 + 2(t-2)x^5y^8 + 4(s-2)x^6y^7 + 2sx^7y^9 + (t-2)x^8y^8 + 2(t-2)x^8y^9\\
&\quad + (3st-4s-4t+5)x^9y^9\Big)|_{x=y=1},\\
ND_5 &= \Big[\frac{16}{5}x^4y^5 + \frac{20}{4}x^4y^5 + tx^5y^5 + tx^5y^5 + \frac{40}{7}x^5y^7 + \frac{56}{5}x^5y^7 + \frac{10}{8}(t-2)x^5y^8 + \frac{16}{5}(t-2)x^5y^8\\
&\quad + \frac{24}{7}(s-2)x^6y^7 + \frac{28}{6}(s-2)x^6y^7 + \frac{14}{9}sx^7y^9 + \frac{18}{7}sx^7y^9 + (t-2)x^8y^8 + (t-2)x^8y^8 + \frac{16}{9}(t-2)x^8y^9\\
&\quad + \frac{18}{8}(t-2)x^8y^9 + (3st-4s-4t+5)x^9y^9 + (3st-4s-4t+5)x^9y^9\Big]|_{x=y=1},\\
ND_5 &= 2(3st-4s-4t+5) + \frac{170}{21}(s-2) + \frac{943}{90}(t-2) + \frac{260}{63}s + 2t + \frac{879}{35}.
\end{aligned} \tag{29}$$

(8) NH is defined as

$$NH = (2S_x J)(NM(G))|_{x=y=1}. \quad (30)$$

Now, by using equation (15), we have

$$\begin{aligned}
NH &= (2S_x J)(f(x,y))|_{x=y=1},\\
NH &= (2S_x J)\big(4x^4y^5 + tx^5y^5 + 8x^5y^7 + 2(t-2)x^5y^8 + 4(s-2)x^6y^7 + 2sx^7y^9 + (t-2)x^8y^8 + 2(t-2)x^8y^9\\
&\quad + (3st-4s-4t+5)x^9y^9\big)|_{x=y=1},\\
NH &= \Big[\frac{8}{9}x^9 + \frac{2}{10}tx^{10} + \frac{16}{12}x^{12} + \frac{4}{13}(t-2)x^{13} + \frac{8}{13}(s-2)x^{13} + \frac{4}{16}sx^{16} + \frac{2}{16}(t-2)x^{16}\\
&\quad + \frac{4}{17}(t-2)x^{17} + \frac{2}{18}(3st-4s-4t+5)x^{18}\Big]|_{x=1},\\
NH &= \frac{1}{9}(3st-4s-4t+5) + \frac{8}{13}(s-2) + \frac{1181}{1768}(t-2) + \frac{1}{4}s + \frac{1}{5}t + \frac{20}{9}.
\end{aligned} \quad (31)$$

(9) NI is defined as

$$NI = (S_x J D_x D_y)(NM(G))|_{x=y=1}. \quad (32)$$

Now, by using equation (15), we have

$$\begin{aligned}
NI &= (S_x J D_x D_y)(f(x,y))|_{x=y=1},\\
NI &= (S_x J D_x D_y)\big(4x^4y^5 + tx^5y^5 + 8x^5y^7 + 2(t-2)x^5y^8 + 4(s-2)x^6y^7 + 2sx^7y^9 + (t-2)x^8y^8 + 2(t-2)x^8y^9\\
&\quad + (3st-4s-4t+5)x^9y^9\big)|_{x=y=1},\\
NI &= (S_x J)\big[80x^4y^5 + 25tx^5y^5 + 280x^5y^7 + 80(t-2)x^5y^8 + 168(s-2)x^6y^7 + 126sx^7y^9 + 64(t-2)x^8y^8\\
&\quad + 144(t-2)x^8y^9 + 81(3st-4s-4t+5)x^9y^9\big]|_{x=y=1},\\
NI &= \Big[\frac{80}{9}x^9 + \frac{25}{10}tx^{10} + \frac{280}{12}x^{12} + \frac{80}{13}(t-2)x^{13} + \frac{168}{13}(s-2)x^{13} + \frac{126}{16}sx^{16} + \frac{64}{16}(t-2)x^{16} + \frac{144}{17}(t-2)x^{17}\\
&\quad + \frac{81}{18}(3st-4s-4t+5)x^{18}\Big]|_{x=1},\\
NI &= \frac{9}{2}(3st-4s-4t+5) + \frac{168}{13}(s-2) + \frac{4116}{221}(t-2) + \frac{63}{8}s + \frac{5}{2}t + \frac{290}{9}.
\end{aligned} \quad (33)$$

(10) S is defined as

$$S = (S_x^3 Q_{-2} J D_x^3 D_y^3)(NM(G))|_{x=y=1}. \quad (34)$$

Now, by using equation (15), we have

$$
\begin{aligned}
S &= \left(S_x^3 Q_{-2} J D_x^3 D_y^3\right)(f(x, y))\big|_{x=y=1},\\
S &= \left(S_x^3 Q_{-2} J D_x^3 D_y^3\right)\Big(4x^4y^5 + tx^5y^5 + 8x^5y^7 + 2(t-2)x^5y^8 + 4(s-2)x^6y^7 + 2sx^7y^9 + (t-2)x^8y^8\\
&\quad + 2(t-2)x^8y^9 + (3st-4s-4t+5)x^9y^9\Big)\Big|_{x=y=1},\\
S &= \Big[\frac{32000}{343}x^7 + \frac{15625}{512}tx^8 + \frac{343000}{1000}x^{10} + \frac{128000}{1331}(t-2)x^{11} + \frac{296352}{1331}(s-2)x^{11} + \frac{500094}{2744}sx^{14}\\
&\quad + \frac{262144}{2744}(t-2)x^{14} + \frac{746496}{3375}(t-2)x^{15} + \frac{531441}{4096}(3st-4s-4t+5)x^{16}\Big]\Big|_{x=1},\\
S &= 129.7463379(3st-4s-4t+5) + 222.6536439(s-2) + 412.8858222(t-2) + \frac{729}{4}s + \frac{15625}{512}t + 436.2944606.
\end{aligned}
\tag{35}
$$

(11) Fifth hyper M_1 Zagreb index is defined as

$$HM_1G_5(G) = \left(D_x^2 + D_y^2 + 2D_xD_y\right)(NM(G))\big|_{x=y=1}. \tag{36}$$

Now, by using equation (15), we have

$$
\begin{aligned}
HM_1G_5\left(G_{t,s}\right) &= \left(D_x^2 + D_y^2 + 2D_xD_y\right)(f(x, y))\big|_{x=y=1},\\
HM_1G_5\left(G_{t,s}\right) &= \left(D_x^2 + D_y^2 + 2D_xD_y\right)\Big(4x^4y^5 + tx^5y^5 + 8x^5y^7 + 2(t-2)x^5y^8 + 4(s-2)x^6y^7 + 2sx^7y^9 + (t-2)x^8y^8\\
&\quad + 2(t-2)x^8y^9 + (3st-4s-4t+5)x^9y^9\Big)\big|_{x=y=1},\\
HM_1G_5\left(G_{t,s}\right) &= \Big[\Big\{64x^4y^5 + 100x^4y^5 + 25tx^5y^5 + 25tx^5y^5 + 200x^5y^7 + 392x^5y^7 + 50(t-2)x^5y^8\\
&\quad + 128(t-2)x^5y^8 + 144(s-2)x^6y^7 + 196(s-2)x^6y^7 + 98sx^7y^9 + 162sx^7y^9\\
&\quad + 64(t-2)x^8y^8 + 64(t-2)x^8y^8 + 128(t-2)x^8y^9 + 162(t-2)x^8y^9 + 81(3st-4s-4t+5)x^9y^9\\
&\quad + 81(3st-4s-4t+5)x^9y^9\Big\} + 2\Big\{80x^4y^5 + 25tx^5y^5 + 280x^5y^7 + 80(t-2)x^5y^8\\
&\quad + 168(s-2)x^6y^7 + 126sx^7y^9 + 64(t-2)x^8y^8 + 144(t-2)x^8y^9 + 81(3st-4s-4t+5)x^9y^9\Big\}\Big]\big|_{x=y=1},\\
HM_1G_5\left(G_{t,s}\right) &= 972st - 108s - 24t - 600.
\end{aligned}
\tag{37}
$$

(12) Fifth hyper M_2 Zagreb index is defined as

$$HM_2G_5(G) = D_xD_y\left(D_xD_y\right)(NM(G))\big|_{x=y=1}. \tag{38}$$

Now, by using equation (15), we have

$$
\begin{aligned}
HM_2G_5\left(G_{t,s}\right) &= D_xD_y\left(D_xD_y\right)(f(x, y))\big|_{x=y=1},\\
HM_2G_5\left(G_{t,s}\right) &= D_xD_y\left(D_xD_y\right)\Big(4x^4y^5 + tx^5y^5 + 8x^5y^7 + 2(t-2)x^5y^8 + 4(s-2)x^6y^7 + 2sx^7y^9\\
&\quad + (t-2)x^8y^8 + 2(t-2)x^8y^9 + (3st-4s-4t+5)x^9y^9\Big)\Big|_{x=y=1},\\
HM_2G_5\left(G_{t,s}\right) &= \left(D_xD_y\right)\Big[80x^4y^5 + 25tx^5y^5 + 280x^5y^7 + 80(t-2)x^5y^8 + 168(s-2)x^6y^7 + 126sx^7y^9\\
&\quad + 64(t-2)x^8y^8 + 144(t-2)x^8y^9 + 81(3st-4s-4t+5)x^9y^9\Big]\Big|_{x=y=1},\\
HM_2G_5\left(G_{t,s}\right) &= \Big[1600x^4y^5 + 625tx^5y^5 + 9800x^5y^7 + 3200(t-2)x^5y^8 + 7056(s-2)x^6y^7 + 7938sx^7y^9\\
&\quad + 4096(t-2)x^8y^8 + 10368(t-2)x^8y^9 + 6561(3st-4s-4t+5)x^9y^9\Big]\Big|_{x=y=1},\\
HM_2G_5\left(G_{t,s}\right) &= 19683st - 11250s - 7955t - 5235.
\end{aligned}
\tag{39}
$$

(13) Fifth arithmetic-geometric index is defined as

$$AG_5(G) = \frac{1}{2} S_x^{1/2} S_y^{1/2} \left(D_x + D_y\right)(NM(G))|_{x=y=1}. \quad (40)$$

Now, by using equation (15), we have

$$\begin{aligned}
AG_5\left(G_{t,s}\right) &= \frac{1}{2} S_x^{1/2} S_y^{1/2} \left(D_x + D_y\right)(f(x,y))\big|_{x=y=1},\\
AG_5\left(G_{t,s}\right) &= \frac{1}{2} S_x^{1/2} S_y^{1/2} \left(D_x + D_y\right)\big(4x^4y^5 + tx^5y^5 + 8x^5y^7 + 2(t-2)x^5y^8 + 4(s-2)x^6y^7 + 2sx^7y^9 + (t-2)x^8y^8\\
&\quad +2(t-2)x^8y^9 + (3st-4s-4t+5)x^9y^9\big)\big|_{x=y=1},\\
AG_5\left(G_{t,s}\right) &= \frac{1}{2} S_x^{1/2} S_y^{1/2}\big[16x^4y^5 + 20x^4y^5 + 5tx^5y^5 + 5tx^5y^5 + 40x^5y^7 + 56x^5y^7 + 10(t-2)x^5y^8 + 16(t-2)x^5y^8\\
&\quad +24(s-2)x^6y^7 + 28(s-2)x^6y^7 + 14sx^7y^9 + 18sx^7y^9 + 8(t-2)x^8y^8 + 8(t-2)x^8y^8 + 16(t-2)x^8y^9\\
&\quad +18(t-2)x^8y^9 + 9(3st-4s-4t+5)x^9y^9 + 9(3st-4s-4t+5)x^9y^9\big]\big|_{x=y=1},\\
AG_5\left(G_{t,s}\right) &= \left[\frac{36}{2\sqrt{20}}x^4y^5 + \frac{10}{2\sqrt{25}}tx^5y^5 + \frac{96}{2\sqrt{35}}x^5y^7 + \frac{26}{2\sqrt{40}}(t-2)x^5y^8 + \frac{52}{2\sqrt{42}}(s-2)x^6y^7 + \frac{32}{2\sqrt{63}}sx^7y^9\right.\\
&\quad \left.+\frac{16}{2\sqrt{64}}(t-2)x^8y^8 + \frac{34}{2\sqrt{72}}(t-2)x^8y^9 + \frac{18}{2\sqrt{81}}(3st-4s-4t+5)x^9y^9\right]\Bigg|_{x=y=1},\\
AG_5\left(G_{t,s}\right) &= (3st-4s-4t+5) + 4.011887099(s-2) + 5.058949692(t-2) + 2.015810523s + t + 12.1384032.
\end{aligned} \quad (41)$$

(14) Fifth geometric-arithmetic index is defined as

$$GA_5(G) = 2S_xJ\left(D_x^{1/2}D_y^{1/2}\right)(NM(G))|_{x=1}, \quad (42)$$

Now, by using equation (15), we have

$$\begin{aligned}
GA_5\left(G_{t,s}\right) &= 2S_xJ\left(D_x^{1/2}D_y^{1/2}\right)(f(x,y))\big|_{x=1},\\
GA_5\left(G_{t,s}\right) &= 2S_xJ\left(D_x^{1/2}D_y^{1/2}\right)\big(4x^4y^5 + tx^5y^5 + 8x^5y^7 + 2(t-2)x^5y^8 + 4(s-2)x^6y^7 + 2sx^7y^9 + (t-2)x^8y^8\\
&\quad +2(t-2)x^8y^9 + (3st-4s-4t+5)x^9y^9\big)\big|_{x=1},\\
GA_5\left(G_{t,s}\right) &= (2S_xJ)\big[4\sqrt{20}\,x^4y^5 + \sqrt{25}\,tx^5y^5 + 8\sqrt{35}\,x^5y^7 + 2\sqrt{40}\,(t-2)x^5y^8 + 4\sqrt{42}\,(s-2)x^6y^7\\
&\quad +2\sqrt{63}\,sx^7y^9 + \sqrt{64}\,(t-2)x^8y^8 + 2\sqrt{72}\,(t-2)x^8y^9 + \sqrt{81}\,(3st-4s-4t+5)x^9y^9\big]\big|_{x=1},\\
GA_5\left(G_{t,s}\right) &= \left[\frac{8\sqrt{20}}{9}x^9 + \frac{2\sqrt{25}}{10}tx^{10} + \frac{16\sqrt{35}}{12}x^{12} + \frac{4\sqrt{40}}{13}(t-2)x^{13}c + \frac{8\sqrt{42}}{13}(s-2)x^{13} + \frac{4\sqrt{63}}{16}sx^{16}\right.\\
&\quad \left.+\frac{2\sqrt{64}}{16}(t-2)x^{16} + \frac{4\sqrt{72}}{17}(t-2)x^{17} + \frac{2\sqrt{81}}{18}(3st-4s-4t+5)x^{18}\right]\Bigg|_{x=1},\\
GA_5\left(G_{t,s}\right) &= (3st-4s-4t+5) + \frac{8\sqrt{42}}{13}(s-2) + 4.942553816(t-2) + \frac{3\sqrt{7}}{4}s + t + 11.86333834.
\end{aligned} \quad (43)$$

□

Corollary 2. *Let $G_{t,s}$ be a graphene network, with $t = 1$ and $s > 1$; then, its neighborhood degree-based graphical indices are given by*

(1) $M_1' = 66s - 18$
(2) $M_2^* = 217s - 133$
(3) $F_N^* = 438s - 254$
(4) $M_2^{nm} = (2/21)(s-2) + (1/49)(s-1) + (123/280)$
(5) $NR_\alpha = (16)^\alpha 2 + (20)^\alpha 4 + (35)^\alpha 4 + (42)^\alpha 4(s-2) + (49)^\alpha (s-1)$
(6) $ND_3 = 2870s - 2398$
(7) $ND_5 = (170/21)(s-2) + 2(s-1) + (723/35)$
(8) $NH = (8/13)(s-2) + (1/7)(s-1) + (37/18)$
(9) $NI = (168/13)(s-2) + (7/2)(s-1) + (221/9)$
(10) $S = 222.6536439(s-2) + 68.08391204(s-1) + 302.7203866$
(11) $HM_1G_5(G_{t,s}) = 872s - 520$
(12) $HM_2G_5(G_{t,s}) = 9457s - 9501$
(13) $AG_5(G_{t,s}) = (13\sqrt{42}/21)(s-2) + (s-1) + 10.08166278$
(14) $GA_5(G_{t,s}) = (8\sqrt{42}/13)(s-2) + (s-1) + 9.919285149$

Proof. NM -Polynomial of $G_{t,s}$, for $t = 1$ and $s > 1$, is given by

$$NM(G_{t,s}) = 2x^4y^4 + 4x^4y^5 + 4x^5y^7 + 4(s-2)x^6y^7 + (s-1)x^7y^7. \quad (44)$$

Let,

$$NM(G_{t,s}) = f(x,y), \quad (45)$$

then, we have

$$f(x,y) = NM(G_{t,s}) = 2x^4y^4 + 4x^4y^5 + 4x^5y^7 + 4(s-2)x^6y^7 + (s-1)x^7y^7. \quad (46)$$

(1) M_1' is defined as

$$M_1' = (D_x + D_y)(NM(G))|_{x=y=1}. \quad (47)$$

Now, by using equation (46), we have

$$\begin{aligned} M_1' &= (D_x + D_y)(f(x,y))|_{x=y=1}, \\ M_1' &= (D_x + D_y)(2x^4y^4 + 4x^4y^5 + 4x^5y^7 + 4(s-2)x^6y^7 + (s-1)x^7y^7)|_{x=y=1}, \\ M_1' &= [8x^4y^4 + 8x^4y^4 + 16x^4y^5 + 20x^4y^5 + 20x^5y^7 + 28x^5y^7 + 24(s-2)x^6y^7 + 28(s-2)x^6y^7 + 7(s-1)x^7y^7 + 7(s-1)x^7y^7]|_{x=y=1}, \\ M_1' &= 66s - 18. \end{aligned} \quad (48)$$

(2) M_2^* is defined as

$$M_2^* = (D_xD_y)(NM(G))|_{x=y=1}. \quad (49)$$

Now, by using equation (46), we have

$$\begin{aligned} M_2^* &= (D_xD_y)(f(x,y))|_{x=y=1}, \\ M_2^* &= (D_xD_y)(2x^4y^4 + 4x^4y^5 + 4x^5y^7 + 4(s-2)x^6y^7 + (s-1)x^7y^7)|_{x=y=1}, \\ M_2^* &= [32x^4y^4 + 80x^4y^5 + 140x^5y^7 + 168(s-2)x^6y^7 + 49(s-1)x^7y^7]|_{x=y=1}, \\ M_2^* &= 217s - 133. \end{aligned} \quad (50)$$

(3) F_N^* is defined as

$$F_N^* = (D_x^2 + D_y^2)(NM(G))|_{x=y=1}. \quad (51)$$

Now, by using equation (46), we have

$$\begin{aligned}
F_N^* &= \left(D_x^2 + D_y^2\right)(f(x,y))|_{x=y=1},\\
F_N^* &= \left(D_x^2 + D_y^2\right)\left(2x^4y^4 + 4x^4y^5 + 4x^5y^7 + 4(s-2)x^6y^7 + (s-1)x^7y^7\right)|_{x=y=1},\\
F_N^* &= \left[32x^4y^4 + 32x^4y^4 + 64x^4y^5 + 100x^4y^5 + 100x^5y^7 + 196x^5y^7 + 144(s-2)x^6y^7 + 196(s-2)x^6y^7\right.\\
&\quad \left.+490(s-1)x^7y^7 + 49(s-1)x^7y^7\right]|_{x=y=1},\\
F_N^* &= 438s - 254.
\end{aligned} \tag{52}$$

(4) M_2^{nm} is defined as

$$M_2^{nm} = \left(S_xS_y\right)(NM(G))|_{x=y=1}. \tag{53}$$

Now, by using equation (46), we have

$$\begin{aligned}
M_2^{nm} &= \left(S_xS_y\right)(f(x,y))|_{x=y=1},\\
M_2^{nm} &= \left(S_xS_y\right)\left(2x^4y^4 + 4x^4y^5 + 4x^5y^7 + 4(s-2)x^6y^7 + (s-1)x^7y^7\right)|_{x=y=1},\\
M_2^{nm} &= \left[\frac{2}{16}x^4y^4 + \frac{4}{20}x^4y^5 + \frac{4}{35}x^5y^7 + \frac{4}{42}(s-2)x^6y^7 + \frac{1}{49}(s-1)x^7y^7\right]|_{x=y=1},\\
M_2^{nm} &= \frac{2}{21}(s-2) + \frac{1}{49}(s-1) + \frac{123}{280}.
\end{aligned} \tag{54}$$

(5) NR_α is defined as

$$NR_\alpha = \left(D_x^\alpha D_y^\alpha\right)(NM(G))|_{x=y=1}. \tag{55}$$

Now, by using equation (46), we have

$$\begin{aligned}
NR_\alpha &= \left(D_x^\alpha D_y^\alpha\right)(f(x,y))|_{x=y=1},\\
NR_\alpha &= \left(D_x^\alpha D_y^\alpha\right)\left(2x^4y^4 + 4x^4y^5 + 4x^5y^7 + 4(s-2)x^6y^7 + (s-1)x^7y^7\right)|_{x=y=1},\\
NR_\alpha &= \left[(16)^\alpha 2x^4y^4 + (20)^\alpha 4x^4y^5 + (35)^\alpha 4x^5y^7 + (42)^\alpha 4(s-2)x^6y^7 + (49)^\alpha (s-1)x^7y^7\right]|_{x=y=1},\\
NR_\alpha &= (16)^\alpha 2 + (20)^\alpha 4 + (35)^\alpha 4 + (42)^\alpha 4(s-2) + (49)^\alpha (s-1).
\end{aligned} \tag{56}$$

(6) ND_3 is defined as

$$ND_3 = \left(D_xD_y\right)\left(D_x + D_y\right)(NM(G))|_{x=y=1}. \tag{57}$$

Now, by using equation (46), we have

$$\begin{aligned}
ND_3 &= \left(D_xD_y\right)\left(D_x + D_y\right)(f(x,y))|_{x=y=1},\\
ND_3 &= \left(D_xD_y\right)\left(D_x + D_y\right)\left(2x^4y^4 + 4x^4y^5 + 4x^5y^7 + 4(s-2)x^6y^7 + (s-1)x^7y^7\right)|_{x=y=1},\\
ND_3 &= \left(D_xD_y\right)\left[8x^4y^4 + 8x^4y^4 + 16x^4y^5 + 20x^4y^5 + 20x^5y^7 + 28x^5y^7 + 24(s-2)x^6y^7\right.\\
&\quad \left. + 28(s-2)x^6y^7 + 7(s-1)x^7y^7 + 7(s-1)x^7y^7\right]|_{x=y=1},\\
ND_3 &= \left[256x^4y^4 + 720x^4y^5 + 1680x^5y^7 + 2184(s-2)x^6y^7 + 686(s-1)x^7y^7\right]|_{x=y=1},\\
ND_3 &= 2870s - 2398.
\end{aligned} \tag{58}$$

(7) ND_5 is defined as

$$ND_5 = \left(D_xS_y + S_xD_y\right)(NM(G))|_{x=y=1}. \tag{59}$$

Now, by using equation (46), we have

$$\begin{aligned}
ND_5 &= \left(D_xS_y + S_xD_y\right)(f(x,y))|_{x=y=1},\\
ND_5 &= \left(D_xS_y + S_xD_y\right)\left(2x^4y^4 + 4x^4y^5 + 4x^5y^7 + 4(s-2)x^6y^7 + (s-1)x^7y^7\right)|_{x=y=1},\\
ND_5 &= \left[2x^4y^4 + 2x^4y^4 + \frac{16}{5}x^4y^5 + \frac{20}{4}x^4y^5 + \frac{20}{7}x^5y^7 + \frac{28}{5}x^5y^7 + \frac{24}{7}(s-2)x^6y^7 + \frac{28}{6}(s-2)x^6y^7 + (s-1)x^7y^7 + (s-1)x^7y^7\right]|_{x=y=1},\\
ND_5 &= \frac{170}{21}(s-2) + 2(s-1) + \frac{723}{35}.
\end{aligned} \tag{60}$$

(8) NH is defined as

$$NH = \left(2S_xJ\right)(NM(G))|_{x=y=1}. \tag{61}$$

Now, by using equation (46), we have

$$\begin{aligned}
NH &= \left(2S_xJ\right)(f(x,y))|_{x=y=1},\\
NH &= \left(2S_xJ\right)\left(2x^4y^4 + 4x^4y^5 + 4x^5y^7 + 4(s-2)x^6y^7 + (s-1)x^7y^7\right)|_{x=y=1},\\
NH &= \left[\frac{4}{8}x^8 + \frac{8}{9}x^9 + \frac{8}{12}x^{12} + \frac{8}{13}(s-2)x^{13} + \frac{2}{14}(s-1)x^{14}\right]|_{x=1},\\
NH &= \frac{8}{13}(s-2) + \frac{1}{7}(s-1) + \frac{37}{18}.
\end{aligned} \tag{62}$$

(9) NI is defined as

$$NI = \left(S_xJD_xD_y\right)(NM(G))|_{x=y=1}. \tag{63}$$

Now, by using equation (46), we have

$$\begin{aligned} NI &= \left(S_x J D_x D_y\right)(f(x,y))\big|_{x=y=1}, \\ NI &= \left(S_x J D_x D_y\right)\left(2x^4y^4 + 4x^4y^5 + 4x^5y^7 + 4(s-2)x^6y^7 + (s-1)x^7y^7\right)\big|_{x=y=1}, \\ NI &= \left(S_x J\right)\left[32x^4y^4 + 80x^4y^5 + 140x^5y^7 + 168(s-2)x^6y^7 + 49(s-1)x^7y^7\right]\big|_{x=y=1}, \\ NI &= \left[\frac{32}{8}x^8 + \frac{80}{9}x^9 + \frac{140}{12}x^{12} + \frac{168}{13}(s-2)x^{13} + \frac{49}{14}(s-1)x^{14}\right]\big|_{x=1}, \\ NI &= \frac{168}{13}(s-2) + \frac{7}{2}(s-1) + \frac{221}{9}. \end{aligned} \tag{64}$$

(10) S is defined as

$$S = \left(S_x^3 Q_{-2} J D_x^3 D_y^3\right)(NM(G))\big|_{x=y=1}. \tag{65}$$

Now, by using equation (46), we have

$$\begin{aligned} S &= \left(S_x^3 Q_{-2} J D_x^3 D_y^3\right)(f(x,y))\big|_{x=y=1}, \\ S &= \left(S_x^3 Q_{-2} J D_x^3 D_y^3\right)\left(2x^4y^4 + 4x^4y^5 + 4x^5y^7 + 4(s-2)x^6y^7 + (s-1)x^7y^7\right)\big|_{x=y=1}, \\ S &= \left[\frac{8192}{216}x^6 + \frac{32000}{343}x^7 + \frac{171500}{1000}x^{10} + \frac{296352}{1331}(s-2)x^{11} + \frac{117649}{1728}(s-1)x^{12}\right]\big|_{x=1}, \\ S &= 222.6536439(s-2) + 68.08391204(s-1) + 302.7203866. \end{aligned} \tag{66}$$

(11) Fifth hyper M_1 Zagreb index is defined as

$$HM_1G_5(G) = \left(D_x^2 + D_y^2 + 2D_xD_y\right)(NM(G))\big|_{x=y=1}. \tag{67}$$

Now, by using equation (46), we have

$$\begin{aligned} HM_1G_5\left(G_{t,s}\right) &= \left(D_x^2 + D_y^2 + 2D_xD_y\right)(f(x,y))\big|_{x=y=1}, \\ HM_1G_5\left(G_{t,s}\right) &= \left(D_x^2 + D_y^2 + 2D_xD_y\right)\left(2x^4y^4 + 4x^4y^5 + 4x^5y^7 + 4(s-2)x^6y^7 + (s-1)x^7y^7\right)\big|_{x=y=1}, \\ HM_1G_5\left(G_{t,s}\right) &= \left[\left\{32x^4y^4 + 32x^4y^4 + 64x^4y^5 + 100x^4y^5 + 100x^5y^7 + 196x^5y^7 + 144(s-2)x^6y^7 + 196(s-2)x^6y^7\right.\right. \\ &\quad \left.\left. + 49(s-1)x^7y^7 + 49(s-1)x^7y^7\right\} + 2\left\{32x^4y^4 + 80x^4y^5 + 140x^5y^7 + 168(s-2)x^6y^7 + 49(s-1)x^7y^7\right\}\right]\big|_{x=y=1}, \\ HM_1G_5\left(G_{t,s}\right) &= 872s - 520. \end{aligned} \tag{68}$$

(12) Fifth hyper M_2 Zagreb index is defined as

$$HM_2G_5(G) = D_xD_y\left(D_xD_y\right)(NM(G))\big|_{x=y=1}. \tag{69}$$

Now, by using equation (46), we have

$$\begin{aligned}
HM_2G_5(G_{t,s}) &= D_xD_y(D_xD_y)(f(x,y))|_{x=y=1},\\
HM_2G_5(G_{t,s}) &= D_xD_y(D_xD_y)\left(2x^4y^4+4x^4y^5+4x^5y^7+4(s-2)x^6y^7+(s-1)x^7y^7\right)|_{x=y=1},\\
HM_2G_5(G_{t,s}) &= (D_xD_y)\left[32x^4y^4+80x^4y^5+140x^5y^7+168(s-2)x^6y^7+49(s-1)x^7y^7\right]|_{x=y=1},\\
HM_2G_5(G_{t,s}) &= \left[512x^4y^4+1600x^4y^5+4900x^5y^7+7056(s-2)x^6y^7+2401(s-1)x^7y^7\right]|_{x=y=1},\\
HM_2G_5(G_{t,s}) &= 9457s-9501.
\end{aligned} \tag{70}$$

(13) Fifth arithmetic-geometric index is defined as

$$AG_5(G) = \frac{1}{2}S_x^{1/2}S_y^{1/2}(D_x+D_y)(NM(G))|_{x=y=1}. \tag{71}$$

Now, by using equation (46), we have

$$\begin{aligned}
AG_5(G_{t,s}) &= \frac{1}{2}S_x^{1/2}S_y^{1/2}(D_x+D_y)(f(x,y))|_{x=y=1},\\
AG_5(G_{t,s}) &= \frac{1}{2}S_x^{1/2}S_y^{1/2}(D_x+D_y)\left(2x^4y^4+4x^4y^5+4x^5y^7+4(s-2)x^6y^7+(s-1)x^7y^7\right)|_{x=y=1},\\
AG_5(G_{t,s}) &= \frac{1}{2}S_x^{1/2}S_y^{1/2}\left[8x^4y^4+8x^4y^4+16x^4y^5+20x^4y^5+20x^5y^7+28x^5y^7+24(s-2)x^6y^7+28(s-2)x^6y^7\right.\\
&\quad \left.+7(s-1)x^7y^7+7(s-1)x^7y^7\right]|_{x=y=1},\\
AG_5(G_{t,s}) &= \left[\frac{16}{2\sqrt{16}}x^4y^4+\frac{36}{2\sqrt{20}}x^4y^5+\frac{48}{2\sqrt{35}}x^5y^7+\frac{52}{2\sqrt{42}}(s-2)x^6y^7+\frac{14}{2\sqrt{49}}(s-1)x^7y^7\right]|_{x=y=1},\\
AG_5(G_{t,s}) &= \frac{13\sqrt{42}}{21}(s-2)+(s-1)+10.08166278.
\end{aligned} \tag{72}$$

(14) Fifth geometric-arithmetic index is defined as

$$GA_5(G) = 2S_xJ\left(D_x^{1/2}D_y^{1/2}\right)(NM(G))|_{x=1}. \tag{73}$$

Now, by using equation (46), we have

$$\begin{aligned}
GA_5(G_{t,s}) &= 2S_xJ\left(D_x^{1/2}D_y^{1/2}\right)(f(x,y))|_{x=1},\\
GA_5(G_{t,s}) &= 2S_xJ\left(D_x^{1/2}D_y^{1/2}\right)\left(2x^4y^4+4x^4y^5+4x^5y^7+4(s-2)x^6y^7+(s-1)x^7y^7\right)|_{x=1},\\
GA_5(G_{t,s}) &= \left[\frac{4\sqrt{16}}{8}x^8+\frac{8\sqrt{20}}{9}x^9+\frac{8\sqrt{35}}{12}x^{12}+\frac{8\sqrt{42}}{13}(s-2)x^{13}+\frac{2\sqrt{49}}{14}(s-1)x^{14}\right]|_{x=1},\\
GA_5(G_{t,s}) &= \frac{8\sqrt{42}}{13}(s-2)+(s-1)+9.919285149.
\end{aligned} \tag{74}$$

□

Corollary 3. *Let $G_{t,s}$ be a graphene network, with $t > 2$ and $s = 1$; then, its neighborhood degree-based graphical indices are given by*

(1) $M_1' = 68t - 22$

(2) $M_2^* = 233t - 166$

(3) $F_N^* = 484t - 346$

(4) $M_2^{nm} = (1/64)(2t-5) + (9/100)(t-2) + (19/40)$
(5) $NR_\alpha = (16)^\alpha 2 + (20)^\alpha 4 + (25)^\alpha (t-2) + (35)^\alpha 4 + (40)^\alpha 2(t-2) + (56)^\alpha 2 + (64)^\alpha (2t-5)$
(6) $ND_3 = 3338t - 3364$
(7) $ND_5 = 2(2t-5) + (129/20)(t-2) + (3457/140)$
(8) $NH = (1/8)(2t-5) + (33/65)(t-2) + (209/90)$
(9) $NI = 4(2t-5) + (225/26)(t-2) + (1441/45)$
(10) $S = 95.5335277(2t-5) + 126.6858726(t-2) + 462.5892987$
(11) $HM_1G_5(G_{t,s}) = 950t - 678$
(12) $HM_2G_5(G_{t,s}) = 12017t - 14846$
(13) $AG_5(G_{t,s}) = (2t-5) + 3.055480479(t-2) + 12.0861221$
(14) $GA_5(G_{t,s}) = (2t-5) + 2.946017022(t-2) + 11.91483576$

Proof. NM -Polynomial of $G_{t,s}$, for $t>2$ and $s=1$, is given by

$$NM(G_{t,s}) = 2x^4y^4 + 4x^4y^5 + (t-2)x^5y^5 + 4x^5y^7 + 2(t-2)x^5y^8 + 2x^7y^8 + (2t-5)x^8y^8. \quad (75)$$

Let,

$$NM(G_{t,s}) = f(x,y). \quad (76)$$

Then, we have

$$f(x,y) = NM(G_{t,s}) = 2x^4y^4 + 4x^4y^5 + (t-2)x^5y^5 + 4x^5y^7 + 2(t-2)x^5y^8 + 2x^7y^8 + (2t-5)x^8y^8. \quad (77)$$

(1) M_1' is defined as

$$M_1' = (D_x + D_y)(NM(G))|_{x=y=1}. \quad (78)$$

Now, by using equation (77), we have

$$\begin{aligned}
M_1' &= (D_x + D_y)(f(x,y))|_{x=y=1},\\
M_1' &= (D_x + D_y)\left(2x^4y^4 + 4x^4y^5 + (t-2)x^5y^5 + 4x^5y^7 + 2(t-2)x^5y^8 + 2x^7y^8 + (2t-5)x^8y^8\right)|_{x=y=1},\\
M_1' &= \left[8x^4y^4 + 8x^4y^4 + 16x^4y^5 + 20x^4y^5 + 5(t-2)x^5y^5 + 5(t-2)x^5y^5 + 20x^5y^7 + 28x^5y^7\right.\\
&\left. +10(t-2)x^5y^8 + 16(t-2)x^5y^8 + 14x^7y^8 + 16x^7y^8 + 8(2t-5)x^8y^8 + 8(2t-5)x^8y^8\right]|_{x=y=1},\\
M_1' &= 68t - 22.
\end{aligned} \quad (79)$$

(2) M_2^* is defined as

$$M_2^* = (D_xD_y)(NM(G))|_{x=y=1}. \quad (80)$$

Now, by using equation (77), we have

$$\begin{aligned}
M_2^* &= (D_xD_y)(f(x,y))|_{x=y=1},\\
M_2^* &= (D_xD_y)\left(2x^4y^4 + 4x^4y^5 + (t-2)x^5y^5 + 4x^5y^7 + 2(t-2)x^5y^8 + 2x^7y^8 + (2t-5)x^8y^8\right)|_{x=y=1},\\
M_2^* &= \left[32x^4y^4 + 80x^4y^5 + 25(t-2)x^5y^5 + 140x^5y^7 + 80(t-2)x^5y^8 + 112x^7y^8 + 64(2t-5)x^8y^8\right]|_{x=y=1},\\
M_2^* &= 233t - 166.
\end{aligned} \quad (81)$$

(3) F_N^* is defined as

$$F_N^* = (D_x^2 + D_y^2)(NM(G))|_{x=y=1}. \quad (82)$$

Now, by using equation (77), we have

$$\begin{aligned}
F_N^* &= \left(D_x^2 + D_y^2\right)(f(x,y))\big|_{x=y=1},\\
F_N^* &= \left(D_x^2 + D_y^2\right)\left(2x^4y^4 + 4x^4y^5 + (t-2)x^5y^5 + 4x^5y^7 + 2(t-2)x^5y^8 + 2x^7y^8 + (2t-5)x^8y^8\right)\big|_{x=y=1},\\
F_N^* &= \big[32x^4y^4 + 32x^4y^4 + 64x^4y^5 + 100x^4y^5 + 25(t-2)x^5y^5 + 25(t-2)x^5y^5 + 100x^5y^7 + 196x^5y^7 + 50(t-2)x^5y^8\\
&\quad +128(t-2)x^5y^8 + 98x^7y^8 + 128x^7y^8 + 64(2t-5)x^8y^8 + 64(2t-5)x^8y^8\big]\big|_{x=y=1},\\
F_N^* &= 484t - 346.
\end{aligned} \tag{83}$$

(4) M_2^{nm} is defined as

$$M_2^{nm} = \left(S_xS_y\right)(NM(G))|_{x=y=1}. \tag{84}$$

Now, by using equation (77), we have

$$\begin{aligned}
M_2^{nm} &= \left(S_xS_y\right)(f(x,y))|_{x=y=1},\\
M_2^{nm} &= \left(S_xS_y\right)\left(2x^4y^4 + 4x^4y^5 + (t-2)x^5y^5 + 4x^5y^7 + 2(t-2)x^5y^8 + 2x^7y^8 + (2t-5)x^8y^8\right)|_{x=y=1},\\
M_2^{nm} &= \left[\frac{2}{16}x^4y^4 + \frac{4}{20}x^4y^5 + \frac{1}{25}(t-2)x^5y^5 + \frac{4}{35}x^5y^7 + \frac{2}{40}(t-2)x^5y^8 + \frac{2}{56}x^7y^8 + \frac{1}{64}(2t-5)x^8y^8\right]\Big|_{x=y=1},\\
M_2^{nm} &= \frac{1}{64}(2t-5) + \frac{9}{100}(t-2) + \frac{19}{40}.
\end{aligned} \tag{85}$$

(5) NR_α is defined as

$$NR_\alpha = \left(D_x^\alpha D_y^\alpha\right)(NM(G))|_{x=y=1}. \tag{86}$$

Now, by using equation (77), we have

$$\begin{aligned}
NR_\alpha &= \left(D_x^\alpha D_y^\alpha\right)(f(x,y))|_{x=y=1},\\
NR_\alpha &= \left(D_x^\alpha D_y^\alpha\right)\left(2x^4y^4 + 4x^4y^5 + (t-2)x^5y^5 + 4x^5y^7 + 2(t-2)x^5y^8 + 2x^7y^8 + (2t-5)x^8y^8\right)|_{x=y=1},\\
NR_\alpha &= \left[(16)^\alpha 2x^4y^4 + (20)^\alpha 4x^4y^5 + (25)^\alpha (t-2)x^5y^5 + (35)^\alpha 4x^5y^7 + (40)^\alpha 2(t-2)x^5y^8 + (56)^\alpha 2x^7y^8 + (64)^\alpha (2t-5)x^8y^8\right]\big|_{x=y=1},\\
NR_\alpha &= (16)^\alpha 2 + (20)^\alpha 4 + (25)^\alpha (t-2) + (35)^\alpha 4 + (40)^\alpha 2(t-2), +(56)^\alpha 2 + (64)^\alpha (2t-5).
\end{aligned} \tag{87}$$

(6) ND_3 is defined as

$$ND_3 = \left(D_xD_y\right)\left(D_x + D_y\right)(NM(G))|_{x=y=1}. \tag{88}$$

Now, by using equation (77), we have

$$\begin{aligned}
ND_3 &= \left(D_xD_y\right)\left(D_x+D_y\right)(f(x,y))|_{x=y=1},\\
ND_3 &= \left(D_xD_y\right)\left(D_x+D_y\right)\left(2x^4y^4+4x^4y^5+(t-2)x^5y^5+4x^5y^7+2(t-2)x^5y^8+2x^7y^8+(2t-5)x^8y^8\right)|_{x=y=1},\\
ND_3 &= \left(D_xD_y\right)\left[8x^4y^4+8x^4y^4+16x^4y^5+20x^4y^5+5(t-2)x^5y^5+5(t-2)x^5y^5+20x^5y^7+28x^5y^7\right.\\
&\quad \left.+10(t-2)x^5y^8+16(t-2)x^5y^8+14x^7y^8+16x^7y^8+8(2t-5)x^8y^8+8(2t-5)x^8y^8\right]|_{x=y=1}.\\
ND_3 &= \left[256x^4y^4+720x^4y^5+250(t-2)x^5y^5+1680x^5y^7+1040(t-2)x^5y^8+1680x^7y^8+1024(2t-5)x^8y^8\right]|_{x=y=1},\\
ND_3 &= 3338t-3364.
\end{aligned} \tag{89}$$

(7) ND_5 is defined as

$$ND_5 = \left(D_xS_y+S_xD_y\right)(NM(G))|_{x=y=1}. \tag{90}$$

Now, by using equation (77), we have

$$\begin{aligned}
ND_5 &= \left(D_xS_y+S_xD_y\right)(f(x,y))|_{x=y=1},\\
ND_5 &= \left(D_xS_y+S_xD_y\right)\left(2x^4y^4+4x^4y^5+(t-2)x^5y^5+4x^5y^7+2(t-2)x^5y^8+2x^7y^8+(2t-5)x^8y^8\right)|_{x=y=1},\\
ND_5 &= \left[2x^4y^4+2x^4y^4+\frac{16}{5}x^4y^5+\frac{20}{4}x^4y^5+(t-2)x^5y^5+(t-2)x^5y^5+\frac{20}{7}x^5y^7+\frac{28}{5}x^5y^7+\frac{10}{8}(t-2)x^5y^8\right.\\
&\quad \left.+\frac{16}{5}(t-2)x^5y^8+\frac{14}{8}x^7y^8+\frac{16}{7}x^7y^8+(2t-5)x^8y^8+(2t-5)x^8y^8\right]|_{x=y=1},\\
ND_5 &= 2(2t-5)+\frac{129}{20}(t-2)+\frac{3457}{140}.
\end{aligned} \tag{91}$$

(8) NH is defined as

$$NH = (2S_xJ)(NM(G))|_{x=y=1}. \tag{92}$$

Now, by using equation (77), we have

$$\begin{aligned}
NH &= (2S_xJ)(f(x,y))|_{x=y=1},\\
NH &= (2S_xJ)\left(2x^4y^4+4x^4y^5+(t-2)x^5y^5+4x^5y^7+2(t-2)x^5y^8+2x^7y^8+(2t-5)x^8y^8\right)|_{x=y=1},\\
NH &= \left[\frac{4}{8}x^8+\frac{8}{9}x^9+\frac{2}{10}(t-2)x^{10}+\frac{8}{12}x^{12}+\frac{4}{13}(t-2)x^{13}+\frac{4}{15}x^{15}+\frac{2}{16}(2t-5)x^{16}\right]|_{x=1},\\
NH &= \frac{1}{8}(2t-5)+\frac{33}{65}(t-2)+\frac{209}{90}.
\end{aligned} \tag{93}$$

(9) NI is defined as

$$NI = \left(S_xJD_xD_y\right)(NM(G))|_{x=y=1}. \tag{94}$$

Now, by using equation (77), we have

$$
\begin{aligned}
NI &= \left(S_x J D_x D_y\right)(f(x,y))|_{x=y=1},\\
NI &= \left(S_x J D_x D_y\right)\left(2x^4y^4 + 4x^4y^5 + (t-2)x^5y^5 + 4x^5y^7 + 2(t-2)x^5y^8 + 2x^7y^8 + (2t-5)x^8y^8\right)|_{x=y=1},\\
NI &= (S_x J)\left[32x^4y^4 + 80x^4y^5 + 25(t-2)x^5y^5 + 140x^5y^7 + 80(t-2)x^5y^8 + 112x^7y^8 + 64(2t-5)x^8y^8\right]|_{x=y=1},\\
NI &= \left[\frac{32}{8}x^8 + \frac{80}{9}x^9 + \frac{25}{10}(t-2)x^{10} + \frac{140}{12}x^{12} + \frac{80}{13}(t-2)x^{13} + \frac{112}{15}x^{15} + \frac{64}{16}(2t-5)x^{16}\right]|_{x=1},\\
NI &= 4(2t-5) + \frac{225}{26}(t-2) + \frac{1441}{45}.
\end{aligned}
\tag{95}
$$

(10) S is defined as

$$S = \left(S_x^3 Q_{-2} J D_x^3 D_y^3\right)(NM(G))|_{x=y=1}. \tag{96}$$

Now, by using equation (77), we have

$$
\begin{aligned}
S &= \left(S_x^3 Q_{-2} J D_x^3 D_y^3\right)(f(x,y))|_{x=y=1},\\
S &= \left(S_x^3 Q_{-2} J D_x^3 D_y^3\right)\left(2x^4y^4 + 4x^4y^5 + (t-2)x^5y^5 + 4x^5y^7 + 2(t-2)x^5y^8 + 2x^7y^8 + (2t-5)x^8y^8\right)|_{x=y=1},\\
S &= \left[\frac{8192}{216}x^6 + \frac{32000}{343}x^7 + \frac{15625}{512}(t-2)x^8 + \frac{171500}{1000}x^{10} + \frac{128000}{1331}(t-2)x^{11} + \frac{351232}{2197}x^{13} + \frac{262144}{2744}(2t-5)x^{14}\right]|_{x=1},\\
S &= 95.5335277(2t-5) + 126.6858726(t-2) + 462.5892987.
\end{aligned}
\tag{97}
$$

(11) Fifth hyper M_1 Zagreb index is defined as

$$HM_1G_5(G) = \left(D_x^2 + D_y^2 + 2D_xD_y\right)(NM(G))|_{x=y=1}. \tag{98}$$

Now, by using equation (77), we have

$$
\begin{aligned}
HM_1G_5(G_{t,s}) &= \left(D_x^2 + D_y^2 + 2D_xD_y\right)(f(x,y))|_{x=y=1},\\
HM_1G_5(G_{t,s}) &= \left(D_x^2 + D_y^2 + 2D_xD_y\right)\left(2x^4y^4 + 4x^4y^5 + (t-2)x^5y^5 + 4x^5y^7 + 2(t-2)x^5y^8 + 2x^7y^8 + (2t-5)x^8y^8\right)|_{x=y=1},\\
HM_1G_5(G_{t,s}) &= \Big[\big\{32x^4y^4 + 32x^4y^4 + 64x^4y^5 + 100x^4y^5 + 25(t-2)x^5y^5 + 25(t-2)x^5y^5 + 100x^5y^7 + 196x^5y^7\\
&\quad + 50(t-2)x^5y^8 + 128(t-2)x^5y^8 + 98x^7y^8 + 128x^7y^8 + 64(2t-5)x^8y^8 + 64(2t-5)x^8y^8\big\}\\
&\quad + 2\big\{32x^4y^4 + 80x^4y^5 + 25(t-2)x^5y^5 + 140x^5y^7 + 80(t-2)x^5y^8 + 112x^7y^8 + 64(2t-5)x^8y^8\big\}\Big]|_{x=y=1},\\
HM_1G_5(G_{t,s}) &= 950t - 678.
\end{aligned}
\tag{99}
$$

(12) Fifth hyper M_2 Zagreb index is defined as

$$HM_2G_5(G) = D_xD_y\left(D_xD_y\right)(NM(G))|_{x=y=1}. \tag{100}$$

Now, by using equation (77), we have

$$
\begin{aligned}
HM_2G_5(G_{t,s}) &= D_xD_y(D_xD_y)(f(x,y))|_{x=y=1},\\
HM_2G_5(G_{t,s}) &= D_xD_y(D_xD_y)\left(2x^4y^4 + 4x^4y^5 + (t-2)x^5y^5 + 4x^5y^7 + 2(t-2)x^5y^8 + 2x^7y^8 + (2t-5)x^8y^8\right)|_{x=y=1},\\
HM_2G_5(G_{t,s}) &= (D_xD_y)\left[32x^4y^4 + 80x^4y^5 + 25(t-2)x^5y^5 + 140x^5y^7 + 80(t-2)x^5y^8 + 112x^7y^8 + 64(2t-5)x^8y^8\right]|_{x=y=1},\\
HM_2G_5(G_{t,s}) &= \left[512x^4y^4 + 1600x^4y^5 + 625(t-2)x^5y^5 + 4900x^5y^7 + 3200(t-2)x^5y^8 + 6272x^7y^8 + 4096(2t-5)x^8y^8\right]|_{x=y=1},\\
HM_2G_5(G_{t,s}) &= 12017t - 14846.
\end{aligned}
\tag{101}
$$

(13) Fifth arithmetic-geometric index is defined as

$$AG_5(G) = \frac{1}{2}S_x^{1/2}S_y^{1/2}(D_x + D_y)(NM(G))|_{x=y=1}. \tag{102}$$

Now, by using equation (77), we have

$$
\begin{aligned}
AG_5(G_{t,s}) &= \frac{1}{2}S_x^{1/2}S_y^{1/2}(D_x + D_y)(f(x,y))|_{x=y=1},\\
AG_5(G_{t,s}) &= \frac{1}{2}S_x^{1/2}S_y^{1/2}(D_x + D_y)\left(2x^4y^4 + 4x^4y^5 + (t-2)x^5y^5 + 4x^5y^7 + 2(t-2)x^5y^8 + 2x^7y^8 + (2t-5)x^8y^8\right)|_{x=y=1},\\
AG_5(G_{t,s}) &= \frac{1}{2}S_x^{1/2}S_y^{1/2}\left[8x^4y^4 + 8x^4y^4 + 16x^4y^5 + 20x^4y^5 + 5(t-2)x^5y^5 + 5(t-2)x^5y^5 + 20x^5y^7 + 28x^5y^7\right.\\
&\quad \left. +10(t-2)x^5y^8 + 16(t-2)x^5y^8 + 14x^7y^8 + 16x^7y^8 + 8(2t-5)x^8y^8 + 8(2t-5)x^8y^8\right]|_{x=y=1},\\
AG_5(G_{t,s}) &= \left[\frac{16}{2\sqrt{16}}x^4y^4 + \frac{36}{2\sqrt{20}}x^4y^5 + \frac{10}{2\sqrt{25}}(t-2)x^5y^5 + \frac{48}{2\sqrt{35}}x^5y^7 + \frac{26}{2\sqrt{40}}(t-2)x^5y^8\right.\\
&\quad \left. + \frac{30}{2\sqrt{56}}x^7y^8 + \frac{16}{2\sqrt{64}}(2t-5)x^8y^8\right]|_{x=y=1},\\
AG_5(G_{t,s}) &= (2t-5) + 3.055480479(t-2) + 12.0861221.
\end{aligned}
\tag{103}
$$

(14) Fifth geometric-arithmetic index is defined as

$$GA_5(G) = 2S_xJ\left(D_x^{1/2}D_Y^{1/2}\right)(NM(G))|_{x=1}. \tag{104}$$

Now, by using equation (77), we have

$$
\begin{aligned}
GA_5(G_{t,s}) &= 2S_xJ\left(D_x^{1/2}D_y^{1/2}\right)(f(x,y))|_{x=1},\\
GA_5(G_{t,s}) &= 2S_xJ\left(D_x^{1/2}D_y^{1/2}\right)\left(2x^4y^4 + 4x^4y^5 + (t-2)x^5y^5 + 4x^5y^7 + 2(t-2)x^5y^8 + 2x^7y^8 + (2t-5)x^8y^8\right)|_{x=1},\\
GA_5(G_{t,s}) &= \left[\frac{4\sqrt{16}}{8}x^8 + \frac{8\sqrt{20}}{9}x^9 + \frac{2\sqrt{25}}{10}(t-2)x^{10} + \frac{8\sqrt{35}}{12}x^{12} + \frac{4\sqrt{40}}{13}(t-2)x^{13} + \frac{4\sqrt{56}}{15}x^{15} + \frac{2\sqrt{64}}{16}(2t-5)x^{16}\right]|_{x=1},\\
GA_5(G_{t,s}) &= (2t-5) + 2.946017022(t-2) + 11.91483576.
\end{aligned}
\tag{105}
$$

□

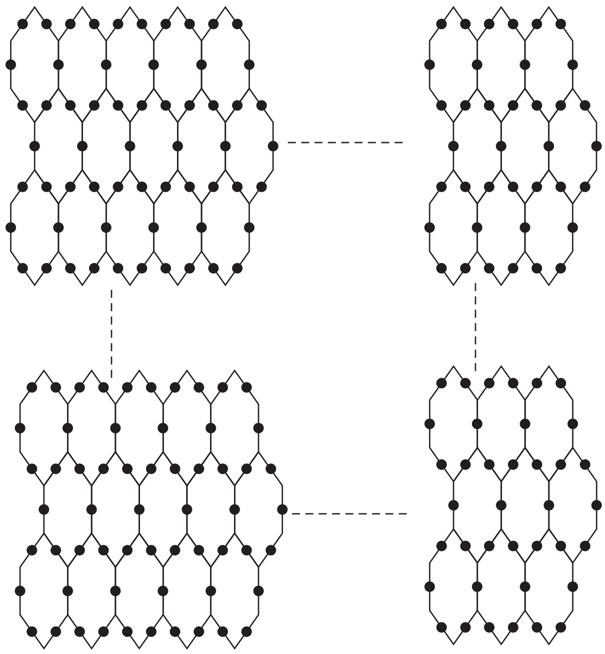

Figure 2: Subdivision graph of $G_{t,s}$.

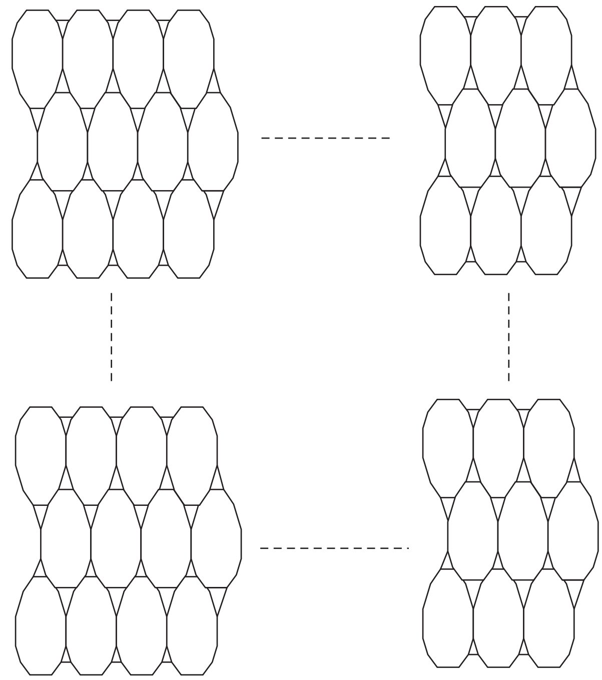

Figure 3: Line graph of the subdivision graph of $G_{t,s}$.

4. Neighborhood M -Polynomial of the Line Graph of the Subdivision Graph of Graphene Networks

In this section, we find neighborhood M -polynomial of the line graph of the subdivision graph of graphene and calculate neighborhood degree-based graphical indices of the line graph of the subdivision graph of graphene by using its neighborhood M -polynomial.

4.1. Line Graph of the Subdivision Graph of Graphene Networks. The subdivision graph of graphene $G_{t,s}$ is denoted by $S(G_{t,s})$, and the line graph of the subdivision graph of graphene is denoted by $L(S(G_{t,s}))$. The line graph of the subdivision graph of graphene has $2(3st + 2s + 2t - 1)$ vertices and $9st + 4s + 4t - 5$ edges [34]. The subdivision graph of $G_{t,s}$ and the line graph of the subdivision graph of $G_{t,s}$ are shown in Figures 2 and 3 respectively.

Theorem 4. *Let $L(S(G_{t,s}))$ be the line graph of the subdivision graph of graphene $G_{t,s}$, with $t \neq 1$ and $s > 1$; then, its neighborhood M -polynomial is given by*

$$NM\left(L\left(S\left(G_{t,s}\right)\right)\right) = (t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8 + 4(s+t-2)x^8y^9 + (9st - 8s - 5t + 1)x^9y^9. \quad (106)$$

Proof. The line graph of the subdivision graph of graphene has $2(3st + 2s + 2t - 1)$ vertices and $9st + 4s + 4t - 5$ edges. The edge set of $L(S(G_{t,s}))$, for $t \neq 1$ and $s > 1$, can be partitioned as follows:

$$
\begin{aligned}
|E_{4,4}| &= \left|\{uv \in E(L(S(G_{t,s}))) \colon \delta_u = 4, \delta_v = 4\}\right| = t + 6 = m_{4,4},\\
|E_{4,5}| &= \left|\{uv \in E(L(S(G_{t,s}))) \colon \delta_u = 4, \delta_v = 5\}\right| = 2(t + 2) = m_{4,5},\\
|E_{5,5}| &= \left|\{uv \in E(L(S(G_{t,s}))) \colon \delta_u = 5, \delta_v = 5\}\right| = 2(s - 2) = m_{5,5},\\
|E_{5,8}| &= \left|\{uv \in E(L(S(G_{t,s}))) \colon \delta_u = 5, \delta_v = 8\}\right| = 2(2s + t - 2) = m_{5,8},\\
|E_{8,8}| &= \left|\{uv \in E(L(S(G_{t,s}))) \colon \delta_u = 8, \delta_v = 8\}\right| = 2s = m_{8,8}.\\
|E_{8,9}| &= \left|\{uv \in E(L(S(G_{t,s}))) \colon \delta_u = 8, \delta_v = 9\}\right| = 4(s + t - 2) = m_{8,9},\\
|E_{9,9}| &= \left|\{uv \in E(L(S(G_{t,s}))) \colon \delta_u = 9, \delta_v = 9\}\right| = 9st - 8s - 5t + 1 = m_{9,9},
\end{aligned}
\tag{107}
$$

Thus, the neighborhood M -polynomial of $L(S(G_{t,s}))$, for $t \neq 1$ and $s > 1$, is

$$
\begin{aligned}
NM(L(S(G_{t,s}))) &= \sum_{i\leq j} m_{i,j} x^i y^j m,\\
NM(L(S(G_{t,s}))) &= (t + 6)x^4 y^4 + 2(t + 2)x^4 y^5 + 2(s - 2)x^5 y^5 + 2(2s + t - 2)x^5 y^8 + 2sx^8 y^8 + 4(s + t - 2)x^8 y^9 + (9st - 8s - 5t + 1)x^9 y^9.
\end{aligned}
\tag{108}
$$

This is the required neighborhood M -polynomial of $L(S(G_{t,s}))$, for $t \neq 1$ and $s > 1$. □

Theorem 5. *Let $L(S(G_{t,s}))$ be the line graph of the subdivision graph of graphene $G_{t,s}$, with $t \neq 1$ and $s = 1$; then, its neighborhood M -polynomial is given by*

$$
NM(L(S(G_{t,s}))) = (t + 8)x^4 y^4 + 2tx^4 y^5 + 2tx^5 y^8 + 2x^8 y^8 + 4(t - 1)x^8 y^9 + (4t - 7)x^9 y^9. \tag{109}
$$

Proof. The line graph of the subdivision graph of graphene has $2(3st + 2s + 2t - 1)$ vertices and $9st + 4s + 4t - 5$ edges. The edge set of $L(S(G_{t,s}))$, for $t \neq 1$ and $s = 1$, can be partitioned as follows:'

$$
\begin{aligned}
|E_{4,4}| &= \left|\{uv \in E(L(S(G_{t,s}))) \colon \delta_u = 4, \delta_v = 4\}\right| = t + 8 = m_{4,4},\\
|E_{4,5}| &= \left|\{uv \in E(L(S(G_{t,s}))) \colon \delta_u = 4, \delta_v = 5\}\right| = 2t = m_{4,5},\\
|E_{5,8}| &= \left|\{uv \in E(L(S(G_{t,s}))) \colon \delta_u = 5, \delta_v = 8\}\right| = 2t = m_{5,8},\\
|E_{8,8}| &= \left|\{uv \in E(L(S(G_{t,s}))) \colon \delta_u = 8, \delta_v = 8\}\right| = 2 = m_{8,8},\\
|E_{8,9}| &= \left|\{uv \in E(L(S(G_{t,s}))) \colon \delta_u = 8, \delta_v = 9\}\right| = 4(t - 1) = m_{8,9},\\
|E_{9,9}| &= \left|\{uv \in E(L(S(G_{t,s}))) \colon \delta_u = 9, \delta_v = 9\}\right| = 4t - 7 = m_{9,9}.
\end{aligned}
\tag{110}
$$

Thus, the neighborhood M -polynomial of $L(S(G_{t,s}))$, for $t \neq 1$ and $s = 1$, is

$$
\begin{aligned}
NM(L(S(G_{t,s}))) &= \sum_{i\leq j} m_{i,j} x^i y^j,\\
NM(L(S(G_{t,s}))) &= (t + 8)x^4 y^4 + 2tx^4 y^5 + 2tx^5 y^8\\
&\quad + 2x^8 y^8 + 4(t - 1)x^8 y^9 + (4t - 7)x^9 y^9.
\end{aligned}
\tag{111}
$$

This is the required neighborhood M -polynomial of $L(S(G_{t,s}))$, for $t \neq 1$ and $s = 1$. □

Theorem 6. *Let $L(S(G_{t,s}))$ be the line graph of the subdivision graph of graphene $G_{t,s}$, with $t = 1$ and $s > 1$; then, its neighborhood M -polynomial is given by*

$$
\begin{aligned}
NM(L(S(G_{t,s}))) &= 10x^4 y^4 + 4x^4 y^5 + 2(s - 2)x^5 y^5\\
&\quad + 4(s - 1)x^5 y^8 + 2(s - 1)x^8 y^8\\
&\quad + 4(s - 1)x^8 y^9 + (s - 1)x^9 y^9.
\end{aligned}
\tag{112}
$$

Proof. The line graph of the subdivision graph of graphene has $2(3st+2s+2t-1)$ vertices and $9st+4s+4t-5$ edges. The edge set of $L(S(G_{t,s}))$, for $t=1$ and $s>1$, can be partitioned as follows:

$$\begin{aligned}
|E_{4,4}| &= |\{uv\in E(L(S(G_{t,s}))): \delta_u=4, \delta_v=4\}| = 10 = m_{4,4},\\
|E_{4,5}| &= |\{uv\in E(L(S(G_{t,s}))): \delta_u=4, \delta_v=5\}| = 4 = m_{4,5},\\
|E_{5,5}| &= |\{uv\in E(L(S(G_{t,s}))): \delta_u=5, \delta_v=5\}| = 2(s-2) = m_{5,5},\\
|E_{5,8}| &= |\{uv\in E(L(S(G_{t,s}))): \delta_u=5, \delta_v=8\}| = 4(s-1) = m_{5,8},\\
|E_{8,8}| &= |\{uv\in E(L(S(G_{t,s}))): \delta_u=8, \delta_v=8\}| = 2(s-1) = m_{8,8},\\
|E_{8,9}| &= |\{uv\in E(L(S(G_{t,s}))): \delta_u=8, \delta_v=9\}| = 4(s-1) = m_{8,9},\\
|E_{9,9}| &= |\{uv\in E(L(S(G_{t,s}))): \delta_u=9, \delta_v=9\}| = s-1 = m_{9,9}.
\end{aligned} \tag{113}$$

Thus, the neighborhood M -polynomial of $L(S(G_{t,s}))$, for $t=1$ and $s>1$, is

$$\begin{aligned}
NM(L(S(G_{t,s}))) &= \sum_{i\le j} m_{i,j}x^iy^j,\\
NM(L(S(G_{t,s}))) &= 10x^4y^4+4x^4y^5+2(s-2)x^5y^5+4(s-1)x^5y^8 2(s-1)x^8y^8+4(s-1)x^8y^9+(s-1)x^9y^9.
\end{aligned} \tag{114}$$

This is the required neighborhood M -polynomial of $L(S(G_{t,s}))$, for $t=1$ and $s>1$.

Now, we calculate neighborhood degree-based graphical indices of the line graph of the subdivision graph of graphene by using its neighborhood M -polynomial. □

Corollary 4. *Let $L(S(G_{t,s}))$ be the line graph of the subdivision graph of graphene network, with $t\neq 1$ and $s>1$; then, its neighborhood degree-based graphical indices are given by*

(1) $M_1' = 162st+28s+30t-126$

(2) $M_2^* = 729st-22s+19t-579$

(3) $F_N^* = 1458st-4s+62t-1198$

(4) $M_2^{nm} = (1/81)(9st-8s-5t+1)+(1/20)(2s+t-2)(1/18)+(s+t-2)+(2/25)(s-2)+(1/16)(t+6)+(1/10)(t+2)+(1/32)s$

(5) $NR_\alpha = (16)^\alpha(t+6)+(20)^\alpha 2(t+2)+(25)^\alpha 2(s-2)+(40)^\alpha 2(2s+t-2)+(64)^\alpha 2s+(72)^\alpha 4(s+t-2)+(81)^\alpha(9st-8s-5t+1)$

(6) $ND_3 = 13122st-2140s-866t-9926$

(7) $ND_5 = 2(9st-8s-5t+1)+(89/20)(2s+t-2)+(145/18)(s+t-2)+4(s-2)+2(t+6)+(41/10)(t+2)+4s$

(8) $NH = (1/9)(9st-8s-5t+1)+(4/13)(2s+t-2)+(8/17)(s+t-2)+(2/5)(s-2)+(1/4)(t+6)+(4/9)(t+2)+(1/4)s$

(9) $NI = (9/2)(9st-8s-5t+1)+(80/13)(2s+t-2)+(288/17)(s+t-2)+5(s-2)+2(t+6)+(40/9)(t+2)+8s$

(10) $S = 129.7463379(9st-8s-5t+1)+96.16829452(2s+t-2)+(55296/125)(s+t-2)+(15625/256)(s-2)+(512/27)(t+6)+46.64723032(t+2)+(65536/343)s.$

(11) $HM_1G_5(L(S(G_{t,s}))) = 2916st-48s+100t-2356$

(12) $HM_2G_5(L(S(G_{t,s}))) = 59049st-15910s-7813t-40675$

(13) $AG_5(L(S(G_{t,s}))) = (9st-8s-5t+1)+((13\sqrt{10})/20)(2s+t-2)+((17\sqrt{2})/6)(s+t-2)+2(s-2)+(t+6)+((9\sqrt{5})/10)(t+2)+2s.$

(14) $GA_5(L(S(G_{t,s}))) = (9st-8s-5t+1)+((8\sqrt{10})/13)(2s+t-2)+((48\sqrt{2})/17)(s+t-2)+2(s-2)+(t+6)+((8\sqrt{5})/9)(t+2)+2s$

Proof. NM -Polynomial of $L(S(G_{t,s}))$, for $t \neq 1$ and $s > 1$, is given by

$$NM(L(S(G_{t,s}))) = (t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8 + 4(s+t-2)x^8y^9 + (9st-8s-5t+1)x^9y^9. \quad (115)$$

Let

$$NM(L(S(G_{t,s}))) = f(x,y), \quad (116)$$

and then, we have

$$f(x,y) = NM(L(S(G_{t,s}))) = (t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8 + 4(s+t-2)x^8y^9 + (9st-8s-5t+1)x^9y^9. \quad (117)$$

(1) M_1' is defined as

$$M_1' = (D_x + D_y)(NM(G))|_{x=y=1}. \quad (118)$$

Now, by using equation (117), we have

$$\begin{aligned}
M_1' &= (D_x + D_y)(f(x,y))|_{x=y=1},\\
M_1' &= (D_x + D_y)\big((t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8 + 4(s+t-2)x^8y^9\\
&\quad +(9st-8s-5t+1)x^9y^9\big)|_{x=y=1},\\
M_1' &= \big[4(t+6)x^4y^4 + 4(t+6)x^4y^4 + 8(t+2)x^4y^5 + 10(t+2)x^4y^5 + 10(s-2)x^5y^5 + 10(s-2)x^5y^5\\
&\quad +10(2s+t-2)x^5y^8 + 16(2s+t-2)x^5y^8 + 16sx^8y^8 + 16sx^8y^8\\
&\quad +32(s+t-2)x^8y^9 + 36(s+t-2)x^8y^9 + 9(9st-8s-5t+1)x^9y^9 + 9(9st-8s-5t+1)x^9y^9\big]|_{x=y=1},\\
M_1' &= 162st + 28s + 30t - 126.
\end{aligned} \quad (119)$$

(2) M_2^* is defined as

$$M_2^* = (D_xD_y)(NM(G))|_{x=y=1}. \quad (120)$$

Now, by using equation (117), we have

$$\begin{aligned}
M_2^* &= (D_xD_y)(f(x,y))|_{x=y=1},\\
M_2^* &= (D_xD_y)\big((t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8 + 4(s+t-2)x^8y^9\\
&\quad + (9st-8s-5t+1)x^9y^9\big)|_{x=y=1},\\
M_2^* &= \big[16(t+6)x^4y^4 + 40(t+2)x^4y^5 + 50(s-2)x^5y^5 + 80(2s+t-2)x^5y^8 + 128sx^8y^8 + 288(s+t-2)x^8y^9\\
&\quad + 81(9st-8s-5t+1)x^9y^9\big]|_{x=y=1},\\
M_2^* &= 729st - 22s + 19t - 579.
\end{aligned} \quad (121)$$

(3) F_N^* is defined as

$$F_N^* = \left(D_x^2 + D_y^2\right)(NM(G))|_{x=y=1}. \tag{122}$$

Now, by using equation (117), we have

$$\begin{aligned}
F_N^* &= \left(D_x^2 + D_y^2\right)(f(x,y))|_{x=y=1},\\
F_N^* &= \left(D_x^2 + D_y^2\right)\Big((t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8 + 4(s+t-2)x^8y^9\\
&\quad +(9st-8s-5t+1)x^9y^9\Big)|_{x=y=1},\\
F_N^* &= \big[16(t+6)x^4y^4 + 16(t+6)x^4y^4 + 32(t+2)x^4y^5 + 50(t+2)x^4y^5 + 50(s-2)x^5y^5 + 50(s-2)x^5y^5\\
&\quad +50(2s+t-2)x^5y^8 + 128(2s+t-2)x^5y^8 + 128sx^8y^8 + 128sx^8y^8 + 256(s+t-2)x^8y^9\\
&\quad +324(s+t-2)x^8y^9 + 81(9st-8s-5t+1)x^9y^9 + 81(9st-8s-5t+1)x^9y^9\big]|_{x=y=1},\\
F_N^* &= 1458st - 4s + 62t - 1198.
\end{aligned} \tag{123}$$

(4) M_2^{nm} is defined as

$$M_2^{nm} = \left(S_xS_y\right)(NM(G))|_{x=y=1}. \tag{124}$$

Now, by using equation (117), we have

$$\begin{aligned}
M_2^{nm} &= \left(S_xS_y\right)(f(x,y))|_{x=y=1},\\
M_2^{nm} &= \left(S_xS_y\right)\Big((t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8 + 4(s+t-2)x^8y^9\\
&\quad +(9st-8s-5t+1)x^9y^9\Big)|_{x=y=1},\\
M_2^{nm} &= \Big[\frac{1}{16}(t+6)x^4y^4 + \frac{2}{20}(t+2)x^4y^5 + \frac{2}{25}(s-2)x^5y^5 + \frac{2}{40}(2s+t-2)x^5y^8 + \frac{2}{64}sx^8y^8 + \frac{4}{72}(s+t-2)x^8y^9\\
&\quad +\frac{1}{81}(9st-8s-5t+1)x^9y^9\Big]|_{x=y=1},\\
M_2^{nm} &= \frac{1}{81}(9st-8s-5t+1) + \frac{1}{20}(2s+t-2) + \frac{1}{18}(s+t-2) + \frac{2}{25}(s-2) + \frac{1}{16}(t+6) + \frac{1}{10}(t+2) + \frac{1}{32}s.
\end{aligned} \tag{125}$$

(5) NR_α is defined as

$$NR_\alpha = \left(D_x^\alpha D_y^\alpha\right)(NM(G))|_{x=y=1}. \tag{126}$$

Now, by using equation (117), we have

$$\begin{aligned}
NR_\alpha &= \left(D_x^\alpha D_y^\alpha\right)(f(x,y))|_{x=y=1},\\
NR_\alpha &= \left(D_x^\alpha D_y^\alpha\right)\Big((t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8 + 4(s+t-2)x^8y^9\\
&\quad +(9st-8s-5t+1)x^9y^9\Big)|_{x=y=1},\\
NR_\alpha &= \big[(16)^\alpha(t+6)x^4y^4 + (20)^\alpha 2(t+2)x^4y^5 + (25)^\alpha 2(s-2)x^5y^5\\
&\quad +(40)^\alpha 2(2s+t-2)x^5y^8 + (64)^\alpha 2sx^8y^8\\
&\quad +(72)^\alpha 4(s+t-2)x^8y^9 + (81)^\alpha(9st-8s-5t+1)x^9y^9\big]|_{x=y=1},\\
NR_\alpha &= (16)^\alpha(t+6) + (20)^\alpha 2(t+2) + (25)^\alpha 2(s-2) + (40)^\alpha 2(2s+t-2) + (64)^\alpha 2s\\
&\quad +(72)^\alpha 4(s+t-2) + (81)^\alpha(9st-8s-5t+1).
\end{aligned} \tag{127}$$

(6) ND_3 is defined as

$$ND_3 = \left(D_xD_y\right)\left(D_x+D_y\right)(NM(G))|_{x=y=1}. \tag{128}$$

Now, by using equation (117), we have

$$\begin{aligned}
ND_3 &= \left(D_xD_y\right)\left(D_x+D_y\right)(f(x,y))|_{x=y=1},\\
ND_3 &= \left(D_xD_y\right)\left(D_x+D_y\right)\Big((t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8\\
&\quad +4(s+t-2)x^8y^9 + (9st-8s-5t+1)x^9y^9\Big)|_{x=y=1},\\
ND_3 &= \left(D_xD_y\right)\big[4(t+6)x^4y^4 + 4(t+6)x^4y^4 + 8(t+2)x^4y^5 + 10(t+2)x^4y^5 + 10(s-2)x^5y^5 + 10(s-2)x^5y^5\\
&\quad +10(2s+t-2)x^5y^8 + 16(2s+t-2)x^5y^8 + 16sx^8y^8 + 16sx^8y^8 + 32(s+t-2)x^8y^9\\
&\quad +36(s+t-2)x^8y^9 + 9(9st-8s-5t+1)x^9y^9 + 9(9st-8s-5t+1)x^9y^9\big]|_{x=y=1},\\
ND_3 &= \big[128(t+6)x^4y^4 + 360(t+2)x^4y^5 + 500(s-2)x^5y^5 + 1040(2s+t-2)x^5y^8 + 2048sx^8y^8\\
&\quad +4896(s+t-2)x^8y^9 + 1458(9st-8s-5t+1)x^9y^9\big]|_{x=y=1},\\
ND_3 &= 13122st - 2140s - 866t - 9926.
\end{aligned} \tag{129}$$

(7) ND_5 is defined as

$$ND_5 = \left(D_xS_y + S_xD_y\right)(NM(G))|_{x=y=1}. \tag{130}$$

Now, by using equation (117), we have

$$ND_5 = \left(D_xS_y + S_xD_y\right)(f(x,y))|_{x=y=1},$$

$$ND_5 = \left(D_xS_y + S_xD_y\right)\Big((t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8$$
$$+4(s+t-2)x^8y^9 + (9st-8s-5t+1)x^9y^9\Big)|_{x=y=1},$$

$$ND_5 = \Big[(t+6)x^4y^4 + (t+6)x^4y^4 + \frac{8}{5}(t+2)x^4y^5 + \frac{10}{4}(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(s-2)x^5y^5$$
$$+\frac{10}{8}(2s+t-2)x^5y^8 + \frac{16}{5}(2s+t-2)x^5y^8 + 2sx^8y^8 + 2sx^8y^8 + \frac{32}{9}(s+t-2)x^8y^9$$
$$+\frac{36}{8}(s+t-2)x^8y^9 + (9st-8s-5t+1)x^9y^9 + (9st-8s-5t+1)x^9y^9\Big]|_{x=y=1},$$

$$ND_5 = 2(9st-8s-5t+1) + \frac{89}{20}(2s+t-2) + \frac{145}{18}(s+t-2) + 4(s-2) + 2(t+6) + \frac{41}{10}(t+2) + 4s. \tag{131}$$

(8) NH is defined as

$$NH = (2S_xJ)(NM(G))|_{x=y=1}. \tag{132}$$

Now, by using equation (117), we have

$$NH = (2S_xJ)(f(x,y))|_{x=y=1},$$

$$NH = (2S_xJ)\Big((t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8 + 4(s+t-2)x^8y^9$$
$$+(9st-8s-5t+1)x^9y^9\Big)|_{x=y=1},$$

$$NH = \Big[\frac{2}{8}(t+6)x^8 + \frac{4}{9}(t+2)x^9 + \frac{4}{10}(s-2)x^{10} + \frac{4}{13}(2s+t-2)x^{13} + \frac{4}{16}sx^{16} + \frac{8}{17}(s+t-2)x^{17}$$
$$+\frac{2}{18}(9st-8s-5t+1)x^{18}\Big]|_{x=1},$$

$$NH = \frac{1}{9}(9st-8s-5t+1) + \frac{4}{13}(2s+t-2) + \frac{8}{17}(s+t-2) + \frac{2}{5}(s-2) + \frac{1}{4}(t+6) + \frac{4}{9}(t+2) + \frac{1}{4}s. \tag{133}$$

(9) NI is defined as

$$NI = \left(S_xJD_xD_y\right)(NM(G))|_{x=y=1}. \tag{134}$$

Now, by using equation (117), we have

$$
\begin{aligned}
NI &= \left(S_x J D_x D_y\right)(f(x,y))|_{x=y=1},\\
NI &= \left(S_x J D_x D_y\right)\Big((t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8\\
&\quad +4(s+t-2)x^8y^9 + (9st-8s-5t+1)x^9y^9\Big)|_{x=y=1},\\
NI &= (S_x J)\big[16(t+6)x^4y^4 + 40(t+2)x^4y^5 + 50(s-2)x^5y^5 + 80(2s+t-2)x^5y^8 + 128sx^8y^8\\
&\quad +288(s+t-2)x^8y^9 + 81(9st-8s-5t+1)x^9y^9\big]|_{x=y=1},\\
NI &= \left[\frac{16}{8}(t+6)x^8 + \frac{40}{9}(t+2)x^9 + \frac{50}{10}(s-2)x^{10} + \frac{80}{13}(2s+t-2)x^{13} + \frac{128}{16}sx^{16}\right.\\
&\quad \left.+\frac{288}{17}(s+t-2)x^{17} + \frac{81}{18}(9st-8s-5t+1)x^{18}\right]|_{x=1},\\
NI &= \frac{9}{2}(9st-8s-5t+1) + \frac{80}{13}(2s+t-2) + \frac{288}{17}(s+t-2) + 5(s-2) + 2(t+6) + \frac{40}{9}(t+2) + 8s.
\end{aligned}
\tag{135}
$$

(10) S is defined as

$$S = \left(S_x^3 Q_{-2} J D_x^3 D_y^3\right)(NM(G))|_{x=y=1}. \tag{136}$$

Now, by using equation (117), we have

$$
\begin{aligned}
S &= \left(S_x^3 Q_{-2} J D_x^3 D_y^3\right)(f(x,y))|_{x=y=1},\\
S &= \left(S_x^3 Q_{-2} J D_x^3 D_y^3\right)\Big((t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8\\
&\quad +4(s+t-2)x^8y^9 + (9st-8s-5t+1)x^9y^9\Big)|_{x=y=1},\\
S &= \left[\frac{4096}{216}(t+6)x^6 + \frac{16000}{343}(t+2)x^7 + \frac{31250}{512}(s-2)x^8 + \frac{128000}{1331}(2s+t-2)x^{11} + \frac{524288}{2744}sx^{14}\right.\\
&\quad \left.+\frac{1492992}{3375}(s+t-2)x^{15} + \frac{531441}{4096}(9st-8s-5t+1)x^{16}\right]|_{x=1},\\
S &= 129.7463379(9st-8s-5t+1) + 96.16829452(2s+t-2) + \frac{55296}{125}(s+t-2) + \frac{15625}{256}(s-2)\\
&\quad +\frac{512}{27}(t+6) + 46.64723032(t+2) + \frac{65536}{343}s.
\end{aligned}
\tag{137}
$$

(11) Fifth hyper M_1 Zagreb index is defined as

$$HM_1G_5(G) = \left(D_x^2 + D_y^2 + 2D_xD_y\right)(NM(G))|_{x=y=1}. \tag{138}$$

Now, by using equation (117), we have

$$HM_1G_5(L(S(G_{t,s}))) = (D_x^2 + D_y^2 + 2D_xD_y)(f(x,y))|_{x=y=1},$$

$$HM_1G_5(L(S(G_{t,s}))) = (D_x^2 + D_y^2 + 2D_xD_y)((t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8 + 4(s+t-2)x^8y^9 + (9st-8s-5t+1)x^9y^9)|_{x=y=1},$$

$$\begin{aligned} HM_1G_5(L(S(G_{t,s}))) = {} & [\{16(t+6)x^4y^4 + 16(t+6)x^4y^4 + 32(t+2)x^4y^5 + 50(t+2)x^4y^5 + 50(s-2)x^5y^5 \\ & + 50(s-2)x^5y^5 + 50(2s+t-2)x^5y^8 + 128(2s+t-2)x^5y^8 + 128sx^8y^8 + 128sx^8y^8 \\ & + 256(s+t-2)x^8y^9 + 324(s+t-2)x^8y^9 + 81(9st-8s-5t+1)x^9y^9 \\ & + 81(9st-8s-5t+1)x^9y^9\} + 2\{16(t+6)x^4y^4 + 40(t+2)x^4y^5 + 50(s-2)x^5y^5 \\ & + 80(2s+t-2)x^5y^8 + 128sx^8y^8 + 288(s+t-2)x^8y^9 + 81(9st-8s-5t+1)x^9y^9\}]|_{x=y=1}, \end{aligned} \tag{139}$$

$$HM_1G_5(L(S(G_{t,s}))) = 2916st - 48s + 100t - 2356.$$

(12) Fifth hyper M_2 Zagreb index is defined as

$$HM_2G_5(G) = D_xD_y(D_xD_y)(NM(G))|_{x=y=1}. \tag{140}$$

Now, by using equation (117), we have

$$HM_2G_5(L(S(G_{t,s}))) = D_xD_y(D_xD_y)(f(x,y))|_{x=y=1},$$

$$HM_2G_5(L(S(G_{t,s}))) = D_xD_y(D_xD_y)((t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8 + 4(s+t-2)x^8y^9 + (9st-8s-5t+1)x^9y^9)|_{x=y=1},$$

$$\begin{aligned} HM_2G_5(L(S(G_{t,s}))) = {} & (D_xD_y)[16(t+6)x^4y^4 + 40(t+2)x^4y^5 + 50(s-2)x^5y^5 + 80(2s+t-2)x^5y^8 \\ & + 128sx^8y^8 + 288(s+t-2)x^8y^9 + 81(9st-8s-5t+1)x^9y^9]|_{x=y=1}, \end{aligned} \tag{141}$$

$$HM_2G_5(L(S(G_{t,s}))) = [256(t+6)x^4y^4 + 800(t+2)x^4y^5 + 1250(s-2)x^5y^5 + 3200(2s+t-2)x^5y^8 + 8192sx^8y^8 + 20736(s+t-2)x^8y^9 + 6561(9st-8s-5t+1)x^9y^9]|_{x=y=1},$$

$$HM_2G_5(L(S(G_{t,s}))) = 59049st - 15910s - 7813t - 40675.$$

(13) Fifth arithmetic-geometric index is defined as

$$AG_5(G) = \frac{1}{2}S_x^{1/2}S_y^{1/2}(D_x + D_y)(NM(G))|_{x=y=1}. \tag{142}$$

Now, by using equation (117), we have

$$AG_5\left(L\left(S\left(G_{t,s}\right)\right)\right) = \frac{1}{2}S_x^{1/2}S_{\backslash y}^{1/2}\left(D_x + D_y\right)(f(x, y))|_{x=y=1},$$

$$AG_5\left(L\left(S\left(G_{t,s}\right)\right)\right) = \frac{1}{2}S_x^{1/2}S_y^{1/2}\left(D_x + D_y\right)\Big((t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8$$
$$+4(s+t-2)x^8y^9 + (9st-8s-5t+1)x^9y^9\Big)|_{x=y=1},$$

$$AG_5\left(L\left(S\left(G_{t,s}\right)\right)\right) = \frac{1}{2}S_x^{1/2}S_y^{1/2}\big[4(t+6)x^4y^4 + 4(t+6)x^4y^4 + 8(t+2)x^4y^5 + 10(t+2)x^4y^5 + 10(s-2)x^5y^5$$
$$+10(s-2)x^5y^5 + 10(2s+t-2)x^5y^8 + 16(2s+t-2)x^5y^8 + 16sx^8y^8 + 16sx^8y^8$$
$$+32(s+t-2)x^8y^9 + 36(s+t-2)x^8y^9 + 9(9st-8s-5t+1)x^9y^9 + 9(9st-8s-5t+1)x^9y^9\big]|_{x=y=1},$$

$$AG_5\left(L\left(S\left(G_{t,s}\right)\right)\right) = \Bigg[\frac{8}{2\sqrt{16}}(t+6)x^4y^4 + \frac{18}{2\sqrt{20}}(t+2)x^4y^5 + \frac{20}{2\sqrt{25}}(s-2)x^5y^5 + \frac{26}{2\sqrt{40}}(2s+t-2)x^5y^8$$
$$+\frac{32}{2\sqrt{64}}sx^8y^8 + \frac{68}{2\sqrt{72}}(s+t-2)x^8y^9 + \frac{18}{2\sqrt{81}}(9st-8s-5t+1)x^9y^9\Bigg]|_{x=y=1},$$

$$AG_5\left(L\left(S\left(G_{t,s}\right)\right)\right) = (9st-8s-5t+1) + \frac{13\sqrt{10}}{20}(2s+t-2) + \frac{17\sqrt{2}}{6}(s+t-2) + 2(s-2) + (t+6) + \frac{9\sqrt{5}}{10}(t+2) + 2s. \tag{143}$$

(14) Fifth geometric-arithmetic index is defined as

$$GA_5(G) = 2S_xJ\left(D_x^{1/2}D_x^{1/2}\right)(NM(G))|_{x=1}. \tag{144}$$

Now, by using equation (117), we have

$$GA_5\left(L\left(S\left(G_{t,s}\right)\right)\right) = 2S_xJ\left(D_x^{1/2}D_y^{1/2}\right)(f(x, y))|_{x=1},$$

$$GA_5\left(L\left(S\left(G_{t,s}\right)\right)\right) = 2S_xJ\left(D_x^{1/2}D_y^{1/2}\right)\Big((t+6)x^4y^4 + 2(t+2)x^4y^5 + 2(s-2)x^5y^5 + 2(2s+t-2)x^5y^8 + 2sx^8y^8$$
$$+4(s+t-2)x^8y^9 + (9st-8s-5t+1)x^9y^9\Big)|_{x=1,}$$

$$GA_5\left(L\left(S\left(G_{t,s}\right)\right)\right) = \Bigg[\frac{2\sqrt{16}}{8}(t+6)x^8 + \frac{4\sqrt{20}}{9}(t+2)x^9 + \frac{4\sqrt{25}}{10}(s-2)x^{10} + \frac{4\sqrt{40}}{13}(2s+t-2)x^{13} + \frac{4\sqrt{64}}{16}sx^{16} \tag{145}$$
$$+\frac{8\sqrt{72}}{17}(s+t-2)x^{17} + \frac{2\sqrt{81}}{18}(9st-8s-5t+1)x^{18}\Bigg]|_{x=1},$$

$$GA_5\left(L\left(S\left(G_{t,s}\right)\right)\right) = (9st-8s-5t+1) + \frac{8\sqrt{10}}{13}(2s+t-2)\frac{48\sqrt{2}}{17}(s+t-2) + 2(s-2) + (t+6)\frac{8\sqrt{5}}{9}(t+2) + 2s.$$

□

Corollary 5. *Let $L(S(G_{t,s}))$ be the line graph of the subdivision graph of graphene network, with $t \neq 1$ and $s = 1$; then, its neighborhood degree-based graphical indices are given by*

(1) $M_1' = 192t - 98$
(2) $M_2^* = 748t - 599$
(3) $F_N^* = 1520t - 1202$
(4) $M_2^{nm} = (1/81)(4t - 7) + (1/16)(t + 8) + (1/18)(t - 1) + (3/20)t + (1/32)$
(5) $NR_\alpha = (16)^\alpha (t + 8) + (20)^\alpha 2t + (40)^\alpha 2t + (64)^\alpha 2 + (72)^\alpha 4(t - 1) + (81)^\alpha (4t - 7)$
(6) $ND_3 = 12256t - 12030$
(7) $ND_5 = 2(4t - 7) + 2(t + 8) + (145/18)(t - 1) + (171/20)t + 4$
(8) $NH = (1/9)(4t - 7) + (1/4)(t + 8) + (8/17)(t - 1) + (88/117)t + (1/4)$
(9) $NI = (9/2)(4t - 7) + 2(t + 8) + (288/17)(t - 1) + (1240/117)t + 8$
(10) $S = 129.7463379(4t - 7) + (512/27)(t + 8) + 442.368(t - 1) + 142.8155248t + 191.0670554$
(11) $HM_1G_5(L(S(G_{t,s}))) = 3016t - 2400$
(12) $HM_2G_5(L(S(G_{t,s}))) = 51236t - 56423$
(13) $AG_5(L(S(G_{t,s}))) = (4t - 7) + (t + 8) + ((17\sqrt{2})/6)(t - 1) + 4.067941659t + 2$
(14) $GA_5(L(S(G_{t,s}))) = (4t - 7) + (t + 8) + (48\sqrt{2}/17)(t - 1) + 3.93363300t + 2$

Proof. NM -Polynomial of $L(S(G_{t,s}))$, for $t \neq 1$ and $s = 1$, is given by

$$NM\left(L\left(S\left(G_{t,s}\right)\right)\right) = (t + 8)x^4y^4 + 2tx^4y^5 + 2tx^5y^8 + 2x^8y^8 + 4(t - 1)x^8y^9 + (4t - 7)x^9y^9. \tag{146}$$

Let

$$NM\left(L\left(S\left(G_{t,s}\right)\right)\right) = f(x, y), \tag{147}$$

and then, we have

$$f(x, y) = NM\left(L\left(S\left(G_{t,s}\right)\right)\right) = (t + 8)x^4y^4 + 2tx^4y^5 + 2tx^5y^8 + 2x^8y^8 + 4(t - 1)x^8y^9 + (4t - 7)x^9y^9. \tag{148}$$

(1) M_1' is defined as

$$M_1' = \left(D_x + D_y\right)(NM(G))|_{x=y=1}. \tag{149}$$

Now, by using equation (148), we have

$$\begin{aligned}
M_1' &= \left(D_x + D_y\right)(f(x, y))|_{x=y=1},\\
M_1' &= \left(D_x + D_y\right)\left((t + 8)x^4y^4 + 2tx^4y^5 + 2tx^5y^8 + 2x^8y^8 + 4(t - 1)x^8y^9 + (4t - 7)x^9y^9\right)|_{x=y=1},\\
M_1' &= \left[4(t + 8)x^4y^4 + 4(t + 8)x^4y^4 + 8tx^4y^5 + 10tx^4y^5 + 10tx^5y^8 + 16tx^5y^8 + 16x^8y^8 + 16x^8y^8\right.\\
&\quad \left.+32(t - 1)x^8y^9 + 36(t - 1)x^8y^9 + 9(4t - 7)x^9y^9 + 9(4t - 7)x^9y^9\right]|_{x=y=1},\\
M_1' &= 192t - 98.
\end{aligned} \tag{150}$$

(2) M_2^* is defined as

$$M_2^* = \left(D_xD_y\right)(NM(G))|_{x=y=1}. \tag{151}$$

Now, by using equation (148), we have

$$\begin{aligned}
M_2^* &= \left(D_x D_y\right)(f(x,y))\big|_{x=y=1},\\
M_2^* &= \left(D_x D_y\right)\left((t+8)x^4y^4 + 2tx^4y^5 + 2tx^5y^8 + 2x^8y^8 + 4(t-1)x^8y^9 + (4t-7)x^9y^9\right)\big|_{x=y=1},\\
M_2^* &= \left[16(t+8)x^4y^4 + 40tx^4y^5 + 80tx^5y^8 + 128x^8y^8 + 288(t-1)x^8y^9 + 81(4t-7)x^9y^9\right]\big|_{x=y=1},\\
M_2^* &= 748t - 599.
\end{aligned} \tag{152}$$

(3) F_N^* is defined as

$$F_N^* = \left(D_x^2 + D_y^2\right)(NM(G))\big|_{x=y=1}. \tag{153}$$

Now, by using equation (148), we have

$$\begin{aligned}
F_N^* &= \left(D_x^2 + D_y^2\right)(f(x,y))\big|_{x=y=1},\\
F_N^* &= \left(D_x^2 + D_y^2\right)\left((t+8)x^4y^4 + 2tx^4y^5 + 2tx^5y^8 + 2x^8y^8 + 4(t-1)x^8y^9 + (4t-7)x^9y^9\right)\big|_{x=y=1},\\
F_N^* &= \left[16(t+8)x^4y^4 + 16(t+8)x^4y^4 + 32tx^4y^5 + 50tx^4y^5 + 50tx^5y^8 + 128tx^5y^8 + 128x^8y^8 + 128x^8y^8\right.\\
&\quad \left.+256(t-1)x^8y^9 + 324(t-1)x^8y^9 + 81(4t-7)x^9y^9 + 81(4t-7)x^9y^9\right]\big|_{x=y=1},\\
F_N^* &= 1520t - 1202.
\end{aligned} \tag{154}$$

(4) M_2^{nm} is defined as

$$M_2^{nm} = \left(S_x S_y\right)(NM(G))\big|_{x=y=1}. \tag{155}$$

Now, by using equation (148), we have

$$\begin{aligned}
M_2^{nm} &= \left(S_x S_y\right)(f(x,y))\big|_{x=y=1},\\
M_2^{nm} &= \left(S_x S_y\right)\left((t+8)x^4y^4 + 2tx^4y^5 + 2tx^5y^8 + 2x^8y^8 + 4(t-1)x^8y^9 + (4t-7)x^9y^9\right)\big|_{x=y=1},\\
M_2^{nm} &= \left[\frac{1}{16}(t+8)x^4y^4 + \frac{2}{20}tx^4y^5 + \frac{2}{40}tx^5y^8 + \frac{2}{64}x^8y^8 + \frac{4}{72}(t-1)x^8y^9 + \frac{1}{81}(4t-7)x^9y^9\right]\big|_{x=y=1},\\
M_2^{nm} &= \frac{1}{81}(4t-7) + \frac{1}{16}(t+8) + \frac{1}{18}(t-1) + \frac{3}{20}t + \frac{1}{32}.
\end{aligned} \tag{156}$$

(5) NR_α is defined as

$$NR_\alpha = \left(D_x^\alpha D_y^\alpha\right)(NM(G))\big|_{x=y=1}. \tag{157}$$

Now, by using equation (148), we have

$$\begin{aligned}
NR_\alpha &= \left(D_x^\alpha D_y^\alpha\right)(f(x,y))\big|_{x=y=1},\\
NR_\alpha &= \left(D_x^\alpha D_y^\alpha\right)\left((t+8)x^4y^4 + 2tx^4y^5 + 2tx^5y^8 + 2x^8y^8 + 4(t-1)x^8y^9 + (4t-7)x^9y^9\right)\big|_{x=y=1},\\
NR_\alpha &= \left[(16)^\alpha(t+8)x^4y^4 + (20)^\alpha 2tx^4y^5 + (40)^\alpha 2tx^5y^8 + (64)^\alpha 2x^8y^8 + (72)^\alpha 4(t-1)x^8y^9 + (81)^\alpha(4t-7)x^9y^9\right]\big|_{x=y=1},\\
NR_\alpha &= (16)^\alpha(t+8) + (20)^\alpha 2t + (40)^\alpha 2t + (64)^\alpha 2 + (72)^\alpha 4(t-1) + (81)^\alpha(4t-7).
\end{aligned} \tag{158}$$

(6) ND_3 is defined as

$$ND_3 = \left(D_x D_y\right)\left(D_x + D_y\right)(NM(G))\big|_{x=y=1}. \tag{159}$$

Now, by using equation (148), we have

$$\begin{aligned} ND_3 &= (D_xD_y)(D_x+D_y)(f(x,y))|_{x=y=1},\\ ND_3 &= (D_xD_y)(D_x+D_y)\left((t+8)x^4y^4+2tx^4y^5+2tx^5y^8+2x^8y^8+4(t-1)x^8y^9+(4t-7)x^9y^9\right)|_{x=y=1},\\ ND_3 &= (D_xD_y)\left[4(t+8)x^4y^4+4(t+8)x^4y^4+8tx^4y^5+10tx^4y^5+10tx^5y^8+16tx^5y^8+16x^8y^8+16x^8y^8\right.\\ &\quad \left.+32(t-1)x^8y^9+36(t-1)x^8y^9+9(4t-7)x^9y^9+9(4t-7)x^9y^9\right]|_{x=y=1},\\ ND_3 &= \left[128(t+8)x^4y^4+360tx^4y^5+1040tx^5y^8+2048x^8y^8+4896(t-1)x^8y^9+1458(4t-7)x^9y^9\right]|_{x=y=1},\\ ND_3 &= 12256t-12030. \end{aligned} \tag{160}$$

(7) ND_5 is defined as

$$ND_5 = (D_xS_y+S_xD_y)(NM(G))|_{x=y=1}. \tag{161}$$

Now, by using equation (148), we have

$$\begin{aligned} ND_5 &= (D_xS_y+S_xD_y)(f(x,y))|_{x=y=1},\\ ND_5 &= (D_xS_y+S_xD_y)\left((t+8)x^4y^4+2tx^4y^5+2tx^5y^8+2x^8y^8+4(t-1)x^8y^9+(4t-7)x^9y^9\right)|_{x=y=1},\\ ND_5 &= \left[(t+8)x^4y^4+(t+8)x^4y^4+\frac{8}{5}tx^4y^5+\frac{10}{4}tx^4y^5+\frac{10}{8}tx^5y^8+\frac{16}{5}tx^5y^8+2x^8y^8+2x^8y^8\right.\\ &\quad \left.+\frac{32}{9}(t-1)x^8y^9+\frac{36}{8}(t-1)x^8y^9+(4t-7)x^9y^9+(4t-7)x^9y^9\right]|_{x=y=1},\\ ND_5 &= 2(4t-7)+2(t+8)+\frac{145}{18}(t-1)+\frac{171}{20}t+4. \end{aligned} \tag{162}$$

(8) NH is defined as

$$NH = (2S_xJ)(NM(G))|_{x=y=1}. \tag{163}$$

Now, by using equation (148), we have

$$\begin{aligned} NH &= (2S_xJ)(f(x,y))|_{x=y=1},\\ NH &= (2S_xJ)\left((t+8)x^4y^4+2tx^4y^5+2tx^5y^8+2x^8y^8+4(t-1)x^8y^9+(4t-7)x^9y^9\right)|_{x=y=1},\\ NH &= \left[\frac{2}{8}(t+8)x^8+\frac{4}{9}tx^9+\frac{4}{13}tx^{13}+\frac{4}{16}x^{16}+\frac{8}{17}(t-1)x^{17}+\frac{2}{18}(4t-7)x^{18}\right]|_{x=1},\\ NH &= \frac{1}{9}(4t-7)+\frac{1}{4}(t+8)+\frac{8}{17}(t-1)+\frac{88}{117}t+\frac{1}{4}. \end{aligned} \tag{164}$$

(9) NI is defined as

$$NI = \left(S_x J D_x D_y\right)(NM(G))|_{x=y=1}. \tag{165}$$

Now, by using equation (148), we have

$$\begin{aligned}
NI &= \left(S_x J D_x D_y\right)(f(x,y))|_{x=y=1},\\
NI &= \left(S_x J D_x D_y\right)\left((t+8)x^4y^4 + 2tx^4y^5 + 2tx^5y^8 + 2x^8y^8 + 4(t-1)x^8y^9 + (4t-7)x^9y^9\right)|_{x=y=1},\\
NI &= \left(S_x J\right)\left[16(t+8)x^4y^4 + 40tx^4y^5 + 80tx^5y^8 + 128x^8y^8 + 288(t-1)x^8y^9 + 81(4t-7)x^9y^9\right]|_{x=y=1},\\
NI &= \left[\frac{16}{8}(t+8)x^8 + \frac{40}{9}tx^9 + \frac{80}{13}tx^{13} + \frac{128}{16}x^{16} + \frac{288}{17}(t-1)x^{17} + \frac{81}{18}(4t-7)x^{18}\right]|_{x=1},\\
NI &= \frac{9}{2}(4t-7) + 2(t+8) + \frac{288}{17}(t-1) + \frac{1240}{117}t + 8.
\end{aligned} \tag{166}$$

(10) S is defined as

$$S = \left(S_x^3 Q_{-2} J D_x^3 D_y^3\right)(NM(G))|_{x=y=1}. \tag{167}$$

Now, by using equation (148), we have

$$\begin{aligned}
S &= \left(S_x^3 Q_{-2} J D_x^3 D_y^3\right)(f(x,y))|_{x=y=1},\\
S &= \left(S_x^3 Q_{-2} J D_x^3 D_y^3\right)\left((t+8)x^4y^4 + 2tx^4y^5 + 2tx^5y^8 + 2x^8y^8 + 4(t-1)x^8y^9 + (4t-7)x^9y^9\right)|_{x=y=1},\\
S &= \left[\frac{4096}{216}(t+8)x^6 + \frac{16000}{343}tx^7 + \frac{128000}{1331}tx^{11} + \frac{524288}{2744}x^{14} + \frac{1492992}{3375}(t-1)x^{15} + \frac{531441}{4096}(4t-7)x^{16}\right]|_{x=1},\\
S &= 129.7463379(4t-7) + \frac{512}{27}(t+8) + 442.368(t-1) + 142.8155248t + 191.0670554.
\end{aligned} \tag{168}$$

(11) Fifth hyper M_1 Zagreb index is defined as

$$HM_1G_5(G) = \left(D_x^2 + D_y^2 + 2D_xD_y\right)(NM(G))|_{x=y=1}. \tag{169}$$

Now, by using equation (148), we have

$$\begin{aligned}
HM_1G_5\left(L\left(S\left(G_{t,s}\right)\right)\right) &= \left(D_x^2 + D_y^2 + 2D_xD_y\right)(f(x,y))|_{x=y=1},\\
HM_1G_5\left(L\left(S\left(G_{t,s}\right)\right)\right) &= \left(D_x^2 + D_y^2 + 2D_xD_y\right)\left((t+8)x^4y^4 + 2tx^4y^5 + 2tx^5y^8 + 2x^8y^8 + 4(t-1)x^8y^9 + (4t-7)x^9y^9\right)|_{x=y=1},\\
HM_1G_5\left(L\left(S\left(G_{t,s}\right)\right)\right) &= \Big[\big\{16(t+8)x^4y^4 + 16(t+8)x^4y^4 + 32tx^4y^5 + 50tx^4y^5 + 50tx^5y^8 + 128tx^5y^8\\
&\quad +128x^8y^8 + 128x^8y^8 + 256(t-1)x^8y^9 + 324(t-1)x^8y^9 + 81(4t-7)x^9y^9 + 81(4t-7)x^9y^9\big\}\\
&\quad +2\big\{16(t+8)x^4y^4 + 40tx^4y^5 + 80tx^5y^8 + 128x^8y^8 + 288(t-1)x^8y^9 + 81(4t-7)x^9y^9\big\}\Big]|_{x=y=1},\\
HM_1G_5\left(L\left(S\left(G_{t,s}\right)\right)\right) &= 3016t - 2400.
\end{aligned} \tag{170}$$

(12) Fifth hyper M_2 Zagreb index is defined as

$$HM_2G_5(G) = D_xD_y\left(D_xD_y\right)(NM(G))|_{x=y=1}. \quad (171)$$

Now, by using equation (148), we have

$$\begin{aligned}
HM_2G_5\left(L\left(S\left(G_{t,s}\right)\right)\right) &= D_xD_y\left(D_xD_y\right)(f(x,y))|_{x=y=1},\\
HM_2G_5\left(L\left(S\left(G_{t,s}\right)\right)\right) &= D_xD_y\left(D_xD_y\right)\left((t+8)x^4y^4 + 2tx^4y^5 + 2tx^5y^8 + 2x^8y^8 + 4(t-1)x^8y^9 + (4t-7)x^9y^9\right)|_{x=y=1},\\
HM_2G_5\left(L\left(S\left(G_{t,s}\right)\right)\right) &= \left(D_xD_y\right)\left[16(t+8)x^4y^4 + 40tx^4y^5 + 80tx^5y^8 + 128x^8y^8 + 288(t-1)x^8y^9 + 81(4t-7)x^9y^9\right]|_{x=y=1},\\
HM_2G_5\left(L\left(S\left(G_{t,s}\right)\right)\right) &= \left[256(t+8)x^4y^4 + 800tx^4y^5 + 3200tx^5y^8 + 8192x^8y^8 + 20736(t-1)x^8y^9 + 6561(4t-7)x^9y^9\right]|_{x=y=1},\\
HM_2G_5\left(L\left(S\left(G_{t,s}\right)\right)\right) &= 51236t - 56423.
\end{aligned} \quad (172)$$

(13) Fifth arithmetic-geometric index is defined as

$$AG_5(G) = \frac{1}{2}S_x^{1/2}S_y^{1/2}\left(D_x + D_y\right)(NM(G))|_{x=y=1}. \quad (173)$$

Now, by using equation (148), we have

$$\begin{aligned}
AG_5\left(L\left(S\left(G_{t,s}\right)\right)\right) &= \frac{1}{2}S_x^{1/2}S_y^{1/2}\left(D_x + D_y\right)(f(x,y))|_{x=y=1},\\
AG_5\left(L\left(S\left(G_{t,s}\right)\right)\right) &= \frac{1}{2}S_x^{1/2}S_y^{1/2}\left(D_x + D_y\right)\left((t+8)x^4y^4 + 2tx^4y^5 + 2tx^5y^8 + 2x^8y^8 + 4(t-1)x^8y^9 + (4t-7)x^9y^9\right)|_{x=y=1},\\
AG_5\left(L\left(S\left(G_{t,s}\right)\right)\right) &= \frac{1}{2}S_x^{1/2}S_y^{1/2}\left[4(t+8)x^4y^4 + 4(t+8)x^4y^4 + 8tx^4y^5 + 10tx^4y^5 + 10tx^5y^8 + 16tx^5y^8 + 16x^8y^8\right.\\
&\quad \left. +16x^8y^8 + 32(t-1)x^8y^9 + 36(t-1)x^8y^9 + 9(4t-7)x^9y^9 + 9(4t-7)x^9y^9\right]|_{x=y=1},\\
AG_5\left(L\left(S\left(G_{t,s}\right)\right)\right) &= \left[\frac{8}{2\sqrt{16}}(t+8)x^4y^4 + \frac{18}{2\sqrt{20}}tx^4y^5 + \frac{26}{2\sqrt{40}}tx^5y^8 + \frac{32}{2\sqrt{64}}x^8y^8 + \frac{68}{2\sqrt{72}}(t-1)x^8y^9 + \frac{18}{2\sqrt{81}}(4t-7)x^9y^9\right]|_{x=y=1},\\
AG_5\left(L\left(S\left(G_{t,s}\right)\right)\right) &= (4t-7) + (t+8) + \frac{17\sqrt{2}}{6}(t-1) + 4.067941659t + 2.
\end{aligned} \quad (174)$$

(14) Fifth geometric-arithmetic index is defined as

$$GA_5(G) = 2S_xJ\left(D_x^{1/2}D_y^{1/2}\right)(NM(G))|_{x=1}. \quad (175)$$

Now, by using Equation (16), we have

$$GA_5\big(L\big(S\big(G_{t,s}\big)\big)\big) = 2S_xJ\big(D_x^{1/2}D_y^{1/2}\big)(f(x,y))|_{x=1},$$

$$GA_5\big(L\big(S\big(G_{t,s}\big)\big)\big) = 2S_xJ\big(Dx^{1/2}D_y^{1/2}\big)\big((t+8)x^4y^4 + 2tx^4y^5 + 2tx^5y^8 + 2x^8y^8 + 4(t-1)x^8y^9 + (4t-7)x^9y^9\big)|_{x=1},$$

$$GA_5\big(L\big(S\big(G_{t,s}\big)\big)\big) = \left[\frac{2\sqrt{16}}{8}(t+8)x^8 + \frac{4\sqrt{20}}{9}tx^9 + \frac{4\sqrt{40}}{13}tx^{13} + \frac{4\sqrt{64}}{16}x^{16} + \frac{8\sqrt{72}}{17}(t-1)x^{17} + \frac{2\sqrt{81}}{18}(4t-7)x^{18}\right]|_{x=1},$$

$$GA_5\big(L\big(S\big(G_{t,s}\big)\big)\big) = (4t-7) + (t+8) + \frac{48\sqrt{2}}{17}(t-1) + 3.93363300t + 2. \tag{176}$$

□

Corollary 6. *Let $L(S(G_{t,s}))$ be the line graph of the subdivision graph of graphene network, with $t = 1$ and $s > 1$; then, its neighborhood degree-based graphical indices are given by*

(1) $M_1' = 190s - 94$
(2) $M_2^* = 704s - 517$
(3) $F_N^* = 1454s - 1070$
(4) $M_2^{nm} = (2/25)(s-2) + (2581/12960)(s-1) + (33/40)$
(5) $NR_\alpha = (16)^\alpha 10 + (20)^\alpha 4 + (25)^\alpha 2(s-2) + (40)^\alpha 4(s-1) + (64)^\alpha 2(s-1) + (72)^\alpha 4(s-1) + (81)^\alpha (s-1)$.
(6) $ND_3 = 10982s - 9482$
(7) $ND_5 = 4(s-2) + (1033/45)(s-1) + (141/5)$
(8) $NH = (2/5)(s-2) + 1.447083962(s-1) + (61/18)$
(9) $NI = 5(s-2) + (18453/442)(s-1) + (260/9)$
(10) $S = (15625/256)(s-2) + 955.5179823(s-1) + 282.9240903$
(11) $HM_1G_5(L(S(G_{t,s}))) = 2862s - 2104$
(12) $HM_2G_5(L(S(G_{t,s}))) = 43139s - 40229$
(13) $AG_5(L(S(G_{t,s}))) = 2(s-2) + 11.11789938\ (s-1) + 14.02492236$
(14) $GA_5(L(S(G_{t,s}))) = 2(s-2) + 10.88510763\ (s-1) + 13.97523196$

Proof. NM-Polynomial of $L(S(G_{t,s}))$, for $t = 1$ and $s > 1$, is given by

$$NM\big(L\big(S\big(G_{t,s}\big)\big)\big) = 10x^4y^4 + 4x^4y^5 + 2(s-2)x^5y^5 + 4(s-1)x^5y^8 + 2(s-1)x^8y^8 + 4(s-1)x^8y^9 + (s-1)x^9y^9. \tag{177}$$

Let

$$NM\big(L\big(S\big(G_{t,s}\big)\big)\big) = f(x,y), \tag{178}$$

and then, we have

$$f(x,y) = NM\big(L\big(S\big(G_{t,s}\big)\big)\big) = 10x^4y^4 + 4x^4y^5 + 2(s-2)x^5y^5 + 4(s-1)x^5y^8 + 2(s-1)x^8y^8 + 4(s-1)x^8y^9 + (s-1)x^9y^9. \tag{179}$$

(1) M_1' is defined as

$$M_1' = \big(D_x + D_y\big)(NM(G))|_{x=y=1}. \tag{180}$$

Now, by using equation (179), we have

$$M_1' = \left(D_x + D_y\right)(f(x,y))|_{x=y=1},$$
$$M_1' = \left(D_x + D_y\right)\left(10x^4y^4 + 4x^4y^5 + 2(s-2)x^5y^5 + 4(s-1)x^5y^8 + 2(s-1)x^8y^8 + 4(s-1)x^8y^9 + (s-1)x^9y^9\right)\Big|_{x=y=1},$$
$$M_1' = \left[40x^4y^4 + 40x^4y^4 + 16x^4y^5 + 20x^4y^5 + 10(s-2)x^5y^5 + 10(s-2)x^5y^5 + 20(s-1)x^5y^8 + 32(s-1)x^5y^8 + 16(s-1)x^8y^8 + 16(s-1)x^8y^8 + 32(s-1)x^8y^9 + 36(s-1)x^8y^9 + 9(s-1)x^9y^9 + 9(s-1)x^9y^9\right]\Big|_{x=y=1},$$
$$M_1' = 190s - 94. \tag{181}$$

(2) M_2^* is defined as

$$M_2^* = \left(D_xD_y\right)(NM(G))|_{x=y=1}. \tag{182}$$

Now, by using equation (179), we have

$$M_2^* = \left(D_xD_y\right)(f(x,y))\big|_{x=y=1},$$
$$M_2^* = \left(D_xD_y\right)\left(10x^4y^4 + 4x^4y^5 + 2(s-2)x^5y^5 + 4(s-1)x^5y^8 + 2(s-1)x^8y^8 + 4(s-1)x^8y^9 + (s-1)x^9y^9\right)\big|_{x=y=1},$$
$$M_2^* = \left[160x^4y^4 + 80x^4y^5 + 50(s-2)x^5y^5 + 160(s-1)x^5y^8 + 128(s-1)x^8y^8 + 288(s-1)x^8y^9 + 81(s-1)x^9y^9\right]|_{x=y=1},$$
$$M_2^* = 704s - 517. \tag{183}$$

(3) F_N^* is defined as

$$F_N^* = \left(D_x^2 + D_y^2\right)(NM(G))|_{x=y=1}. \tag{184}$$

Now, by using equation (179), we have

$$F_N^* = \left(D_x^2 + D_y^2\right)(f(x,y))|_{x=y=1},$$
$$F_N^* = \left(D_x^2 + D_y^2\right)\left(10x^4y^4 + 4x^4y^5 + 2(s-2)x^5y^5 + 4(s-1)x^5y^8 + 2(s-1)x^8y^8 + 4(s-1)x^8y^9 + (s-1)x^9y^9\right)|_{x=y=1},$$
$$F_N^* = \left[160x^4y^4 + 160x^4y^4 + 64x^4y^5 + 100x^4y^5 + 50(s-2)x^5y^5 + 50(s-2)x^5y^5 + 100(s-1)x^5y^8 + 256(s-1)x^5y^8 + 128(s-1)x^8y^8 + 128(s-1)x^8y^8 + 256(s-1)x^8y^9 + 324(s-1)x^8y^9 + 81(s-1)x^9y^9 + 81(s-1)x^9y^9\right]\Big|_{x=y=1},$$
$${F_N}^* = 1454s - 1070. \tag{185}$$

(4) M_2^{nm} is defined as

$$M_2^{nm} = \left(S_xS_y\right)(NM(G))\big|_{x=y=1}. \tag{186}$$

Now, by using equation (179), we have

$$M_2^{nm} = \left(S_x S_y\right)(f(x,y))|_{x=y=1},$$

$$M_2^{nm} = \left(S_x S_y\right)\left(10x^4y^4 + 4x^4y^5 + 2(s-2)x^5y^5 + 4(s-1)x^5y^8 + 2(s-1)x^8y^8 + 4(s-1)x^8y^9 + (s-1)x^9y^9\right)|_{x=y=1},$$

$$M_2^{nm} = \left[\frac{10}{16}x^4y^4 + \frac{4}{20}x^4y^5 + \frac{2}{25}(s-2)x^5y^5 + \frac{4}{40}(s-1)x^5y^8 + \frac{2}{64}(s-1)x^8y^8 + \frac{4}{72}(s-1)x^8y^9 + \frac{1}{81}(s-1)x^9y^9\right]|_{x=y=1},$$

$$M_2^{nm} = \frac{2}{25}(s-2) + \frac{2581}{12960}(s-1) + \frac{33}{40}. \tag{187}$$

(5) NR_α is defined as

$$NR_\alpha = \left(D_x^\alpha D_y^\alpha\right)(NM(G))|_{x=y=1}. \tag{188}$$

Now, by using equation (179), we have

$$NR_\alpha = \left(D_x^\alpha D_y^\alpha\right)(f(x,y))|_{x=y=1},$$
$$NR_\alpha = \left(D_x^\alpha D_y^\alpha\right)\left(10x^4y^4 + 4x^4y^5 + 2(s-2)x^5y^5 + 4(s-1)x^5y^8 + 2(s-1)x^8y^8 + 4(s-1)x^8y^9 + (s-1)x^9y^9\right)|_{x=y=1},$$
$$NR_\alpha = \left[(16)^\alpha 10x^4y^4 + (20)^\alpha 4x^4y^5 + (25)^\alpha 2(s-2)x^5y^5 + (40)^\alpha 4(s-1)x^5y^8 + (64)^\alpha 2(s-1)x^8y^8 + (72)^\alpha 4(s-1)x^8y^9 + (81)^\alpha (s-1)x^9y^9\right]|_{x=y=1},$$
$$NR_\alpha = (16)^\alpha 10 + (20)^\alpha 4 + (25)^\alpha 2(s-2) + (40)^\alpha 4(s-1) + (64)^\alpha 2(s-1) + (72)^\alpha 4(s-1) + (81)^\alpha (s-1). \tag{189}$$

(6) ND_3 is defined as

$$ND_3 = \left(D_x D_y\right)\left(D_x + D_y\right)(NM(G))|_{x=y=1}. \tag{190}$$

Now, by using equation (179), we have

$$ND_3 = \left(D_x D_y\right)\left(D_x + D_y\right)(f(x,y))|_{x=y=1},$$

$$ND_3 = \left(D_x D_y\right)\left(D_x + D_y\right)\left(10x^4y^4 + 4x^4y^5 + 2(s-2)x^5y^5 + 4(s-1)x^5y^8 + 2(s-1)x^8y^8 + 4(s-1)x^8y^9 + (s-1)x^9y^9\right)|_{x=y=1},$$

$$ND_3 = \left(D_x D_y\right)\left[40x^4y^4 + 40x^4y^4 + 16x^4y^5 + 20x^4y^5 + 10(s-2)x^5y^5 + 10(s-2)x^5y^5 + 20(s-1)x^5y^8 + 32(s-1)x^5y^8 + 16(s-1)x^8y^8 + 16(s-1)x^8y^8 + 32(s-1)x^8y^9 + 36(s-1)x^8y^9 + 9(s-1)x^9y^9 + 9(s-1)x^9y^9\right]|_{x=y=1},$$

$$ND_3 = \left[1280x^4y^4 + 720x^4y^5 + 500(s-2)x^5y^5 + 2080(s-1)x^5y^8 + 2048(s-1)x^8y^8 + 4896(s-1)x^8y^9 + 1458(s-1)x^9y^9\right]|_{x=y=1},$$

$$ND_3 = 10982s - 9482. \tag{191}$$

(7) ND_5 is defined as

$$ND_5 = \left(D_x S_y + S_x D_y\right)(NM(G))|_{x=y=1}. \tag{192}$$

Now, by using equation (179), we have

$$
\begin{aligned}
ND_5 &= \left(D_xS_y + S_xD_y\right)(f(x,y))|_{x=y=1},\\
ND_5 &= \left(D_xS_y + S_xD_y\right)\left(10x^4y^4 + 4x^4y^5 + 2(s-2)x^5y^5 + 4(s-1)x^5y^8 + 2(s-1)x^8y^8 + 4(s-1)x^8y^9 + (s-1)x^9y^9\right)|_{x=y=1},\\
ND_5 &= \left[10x^4y^4 + 10x^4y^4 + \frac{16}{5}x^4y^5 + \frac{20}{4}x^4y^5 + 2(s-2)x^5y^5 + 2(s-2)x^5y^5 + \frac{20}{8}(s-1)x^5y^8\right.\\
&\quad \left. + \frac{32}{5}(s-1)x^5y^8 + 2(s-1)x^8y^8 + 2(s-1)x^8y^8 + \frac{32}{9}(s-1)x^8y^9 + \frac{36}{8}(s-1)x^8y^9 + (s-1)x^9y^9 + (s-1)x^9y^9\right]|_{x=y=1},\\
ND_5 &= 4(s-2) + \frac{1033}{45}(s-1) + \frac{141}{5}.
\end{aligned}
\tag{193}
$$

(8) NH is defined as

$$NH = \left(2S_xJ\right)(NM(G))|_{x=y=1}. \tag{194}$$

Now, by using equation (179), we have

$$
\begin{aligned}
NH &= \left(2S_xJ\right)(f(x,y))|_{x=y=1},\\
NH &= \left(2S_xJ\right)\left(10x^4y^4 + 4x^4y^5 + 2(s-2)x^5y^5 + 4(s-1)x^5y^8 + 2(s-1)x^8y^8 + 4(s-1)x^8y^9 + (s-1)x^9y^9\right)|_{x=y=1},\\
NH &= \left[\frac{20}{8}x^8 + \frac{8}{9}x^9 + \frac{4}{10}(s-2)x^{10} + \frac{8}{13}(s-1)x^{13} + \frac{4}{16}(s-1)x^{16} + \frac{8}{17}(s-1)x^{17} + \frac{2}{18}(s-1)x^{18}\right]|_{x=1},\\
NH &= \frac{2}{5}(s-2) + 1.447083962(s-1) + \frac{61}{18}.
\end{aligned}
\tag{195}
$$

(9) NI is defined as

$$NI = \left(S_xJD_xD_y\right)(NM(G))|_{x=y=1}. \tag{196}$$

Now, by using equation (179), we have

$$
\begin{aligned}
NI &= \left(S_xJD_xD_y\right)(f(x,y))|_{x=y=1},\\
NI &= \left(S_xJD_xD_y\right)\left(10x^4y^4 + 4x^4y^5 + 2(s-2)x^5y^5 + 4(s-1)x^5y^8 + 2(s-1)x^8y^8 + 4(s-1)x^8y^9 + (s-1)x^9y^9\right)|_{x=y=1},\\
NI &= \left(S_xJ\right)\left[160x^4y^4 + 80x^4y^5 + 50(s-2)x^5y^5 + 160(s-1)x^5y^8 + 128(s-1)x^8y^8 + 288(s-1)x^8y^9 + 81(s-1)x^9y^9\right]|_{x=y=1},\\
NI &= \left[\frac{160}{8}x^8 + \frac{80}{9}x^9 + \frac{50}{10}(s-2)x^{10} + \frac{160}{13}(s-1)x^{13} + \frac{128}{16}(s-1)x^{16} + \frac{288}{17}(s-1)x^{17} + \frac{81}{18}(s-1)x^{18}\right]|_{x=1},\\
NI &= 5(s-2) + \frac{18453}{442}(s-1) + \frac{260}{9}.
\end{aligned}
\tag{197}
$$

(10) S is defined as

$$S = \left(S_x^3Q_{-2}JD_x^3D_y^3\right)(NM(G))|_{x=y=1}. \tag{198}$$

Now, by using equation (179), we have

$$S=\left(S_x^3Q_{-2}JD_x^3D_y^3\right)(f(x,y))|_{x=y=1},$$

$$S=\left(S_x^3Q_{-2}JD_x^3D_y^3\right)\left(10x^4y^4+4x^4y^5+2(s-2)x^5y^5+4(s-1)x^5y^8+2(s-1)x^8y^8+4(s-1)x^8y^9+(s-1)x^9y^9\right)|_{x=y=1},$$

$$S=\left[\frac{40960}{216}x^6+\frac{32000}{343}x^7+\frac{31250}{512}(s-2)x^8+\frac{256000}{1331}(s-1)x^{11}+\frac{524288}{2744}(s-1)x^{14}+\frac{1492992}{3375}(s-1)x^{15}+\frac{531441}{4096}(s-1)x^{16}\right]|_{x=1},$$

$$S=\frac{15625}{256}(s-2)+955.5179823(s-1)+282.9240903. \tag{199}$$

(11) Fifth hyper M_1 Zagreb index is defined as

$$HM_1G_5(G)=\left(D_x^2+D_y^2+2D_xD_y\right)(NM(G))|_{x=y=1}. \tag{200}$$

Now, by using equation (179), we have

$$HM_1G_5\left(L\left(S\left(G_{t,s}\right)\right)\right)=\left(D_x^2+D_y^2+2D_xD_y\right)(f(x,y))\big|_{x=y=1},$$

$$HM_1G_5\left(L\left(S\left(G_{t,s}\right)\right)\right)=\left(D_x^2+D_y^2+2D_xD_y\right)\left(10x^4y^4+4x^4y^5+2(s-2)x^5y^5+4(s-1)x^5y^8+2(s-1)x^8y^8\right.$$
$$\left.+4(s-1)x^8y^9+(s-1)x^9y^9\right)|_{x=y=1},$$

$$HM_1G_5\left(L\left(S\left(G_{t,s}\right)\right)\right)=\left[\left\{160x^4y^4+160x^4y^4+64x^4y^5+100x^4y^5+50(s-2)x^5y^5+50(s-2)x^5y^5\right.\right.$$
$$+100(s-1)x^5y^8+256(s-1)x^5y^8+128(s-1)x^8y^8+128(s-1)x^8y^8+256(s-1)x^8y^9$$
$$\left.+324(s-1)x^8y^9+81(s-1)x^9y^9+81(s-1)x^9y^9\right\}+2\left\{160x^4y^4+80x^4y^5+50(s-2)x^5y^5\right.$$
$$\left.\left.+160(s-1)x^5y^8+128(s-1)x^8y^8+288(s-1)x^8y^9+81(s-1)x^9y^9\right\}\right]\big|_{x=y=1},$$

$$HM_1G_5\left(L\left(S\left(G_{t,s}\right)\right)\right)=2862s-2104. \tag{201}$$

(12) Fifth hyper M_2 Zagreb index is defined as

$$HM_2G_5(G)=D_xD_y\left(D_xD_y\right)(NM(G))|_{x=y=1}. \tag{202}$$

Now, by using equation (179), we have

$$\begin{aligned}
HM_2G_5\big(L\big(S\big(G_{t,s}\big)\big)\big) &= D_xD_y\big(D_xD_y\big)(f(x,y))|_{x=y=1},\\
HM_2G_5\big(L\big(S\big(G_{t,s}\big)\big)\big) &= D_xD_y\big(D_xD_y\big)\big(10x^4y^4+4x^4y^5+2(s-2)x^5y^5+4(s-1)x^5y^8+2(s-1)x^8y^8\\
&\quad +4(s-1)x^8y^9+(s-1)x^9y^9\big)|_{x=y=1},\\
HM_2G_5\big(L\big(S\big(G_{t,s}\big)\big)\big) &= \big(D_xD_y\big)\big[160x^4y^4+80x^4y^5+50(s-2)x^5y^5+160(s-1)x^5y^8+128(s-1)x^8y^8\\
&\quad +288(s-1)x^8y^9+81(s-1)x^9y^9\big]|_{x=y=1},\\
HM_2G_5\big(L\big(S\big(G_{t,s}\big)\big)\big) &= \big[2560x^4y^4+1600x^4y^5+1250(s-2)x^5y^5+6400(s-1)x^5y^8+8192(s-1)x^8y^8\\
&\quad +20736(s-1)x^8y^9+6561(s-1)x^9y^9\big]|_{x=y=1},\\
HM_2G_5\big(L\big(S\big(G_{t,s}\big)\big)\big) &= 43139s-40229.
\end{aligned} \tag{203}$$

(13) Fifth arithmetic-geometric index is defined as

$$AG_5(G) = \frac{1}{2}S_x^{1/2}S_y^{1/2}\big(D_x+D_y\big)(NM(G))|_{x=y=1}. \tag{204}$$

Now, by using equation (179), we have

$$\begin{aligned}
AG_5\big(L\big(S\big(G_{t,s}\big)\big)\big) &= \frac{1}{2}S_x^{1/2}S_y^{1/2}\big(D_x+D_y\big)(f(x,y))|_{x=y=1},\\
AG_5\big(L\big(S\big(G_{t,s}\big)\big)\big) &= \frac{1}{2}S_x^{1/2}S_y^{1/2}\big(D_x+D_y\big)\big(10x^4y^4+4x^4y^5+2(s-2)x^5y^5+4(s-1)x^5y^8+2(s-1)x^8y^8\\
&\quad +4(s-1)x^8y^9+(s-1)x^9y^9\big)|_{x=y=1},\\
AG_5\big(L\big(S\big(G_{t,s}\big)\big)\big) &= \frac{1}{2}S_x^{1/2}S_y^{1/2}\big[40x^4y^4+40x^4y^4+16x^4y^5+20x^4y^5+10(s-2)x^5y^5+10(s-2)x^5y^5\\
&\quad +20(s-1)x^5y^8+32(s-1)x^5y^8+16(s-1)x^8y^8+16(s-1)x^8y^8+32(s-1)x^8y^9\\
&\quad +36(s-1)x^8y^9+9(s-1)x^9y^9+9(s-1)x^9y^9\big]|_{x=y=1},\\
AG_5\big(L\big(S\big(G_{t,s}\big)\big)\big) &= \Big[\frac{80}{2\sqrt{16}}x^4y^4+\frac{36}{2\sqrt{20}}x^4y^5+\frac{20}{2\sqrt{25}}(s-2)x^5y^5+\frac{52}{2\sqrt{40}}(s-1)x^5y^8+\frac{32}{2\sqrt{64}}(s-1)x^8y^8\\
&\quad +\frac{68}{\sqrt{72}}(s-1)x^8y^9+\frac{18}{2\sqrt{81}}(s-1)x^9y^9\Big]|_{x=y=1},\\
AG_5\big(L\big(S\big(G_{t,s}\big)\big)\big) &= 2(s-2)+11.11789938(s-1)+14.02492236.
\end{aligned} \tag{205}$$

(14) Fifth geometric-arithmetic index is defined as

$$GA_5(G) = 2S_xJ\big(D_x^{1/2}D_y^{1/2}\big)(NM(G))|_{x=1}. \tag{206}$$

Now, by using equation (179), we have

$$GA_5\left(L\left(S\left(G_{t,s}\right)\right)\right) = 2S_x J\left(D_x^{1/2} D_y^{1/2}\right)(f(x,y))\big|_{x=1},$$

$$GA_5\left(L\left(S\left(G_{t,s}\right)\right)\right) = 2S_x J\left(D_x^{1/2} D_y^{1/2}\right)\Big(10x^4y^4 + 4x^4y^5 + 2(s-2)x^5y^5 + 4(s-1)x^5y^8 + 2(s-1)x^8y^8 + 4(s-1)x^8y^9 + (s-1)x^9y^9\Big)\Big|_{x=1},$$

$$GA_5\left(L\left(S\left(G_{t,s}\right)\right)\right) = \left[\frac{20\sqrt{16}}{8}x^8 + \frac{8\sqrt{20}}{9}x^9 + \frac{4\sqrt{25}}{10}(s-2)x^{10} + \frac{8\sqrt{40}}{13}(s-1)x^{13} + \frac{4\sqrt{64}}{16}(s-1)x^{16} + \frac{8\sqrt{72}}{17}(s-1)x^{17} + \frac{2\sqrt{81}}{18}(s-1)x^{18}\right]\Bigg|_{x=1},$$

$$GA_5\left(L\left(S\left(G_{t,s}\right)\right)\right) = 2(s-2) + 10.88510763(s-1) + 13.97523196. \tag{207}$$

□

5. Conclusions

Graphene is an allotrope of carbon consisting of a single layer of atoms arranged in a two-dimensional honeycomb lattice nanostructure. It was an interesting problem to give an easy way to compute its topological indices. In this article, we established NM-polynomials for the graph of graphene and the line graph of the subdivision graph of graphene. By applying some basic roles of calculus, we computed serval of its neighborhood-based indices. The established NM-polynomials can also be used to compute the remaining neighborhood-based topological indices.

Data Availability

All data required for this research is including within this article.

Conflicts of Interest

The authors do not have any conflicts of interest.

Authors' Contributions

J.W. proposed the problem and supervised this work. S.N. proved the main results. I.A. analyzed the results and wrote the first version of the manuscript, and F.Y. wrote the final version of the manuscript and arranged the funding for this study.

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