Hindawi
Journal of Mathematics
Volume 2022, Article ID 3836837, 11 pages
https://doi.org/10.1155/2022/3836837

Research Article

Some Inequalities for Operator (p, h)-Convex Function

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Received 25 October 2021; Revised 14 December 2021; Accepted 6 January 2022; Published 7 February 2022

Academic Editor: Ji Gao

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In this paper, we introduce operator (p, h)-convex functions and establish a Hermite–Hadamard inequality for these functions. As application, we obtain several trace and singular value inequalities of operators.

1. Introduction

In recent years, several extensions and generalizations have been considered for classical convexity and the theory of inequalities has made essential contributions to many areas of mathematics.

In 1973, Elliott Lieb published a ground-breaking article on operator inequalities [1]. This and a subsequent article by Lieb and Ruskai [2] have had a profound effect on quantum statistical mechanics and more recently on quantum information theory. Since then, a number of attempts have been made to elucidate and extend these results. Two elegant examples are those of Nielsen and Petz [3] and Ruskai [4], which use the analytic representations for operator convex functions. In addition, Hansen [5] has developed a powerful theory that utilizes geometric means of positive operators. The latter notion was formulated by Pusz and Woronowicz [6] and subsequently investigated by Ando [7] (see the discussion in Section 3) and by Kubo and Ando [8].

We recall some concepts of convexity that is well-known in the literature.

The following inequality holds for any convex function f defined on $\mathbb{R}$ and $a, b \in \mathbb{R}$, with $a < b$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

It was firstly discovered by Hermite in 1881 in the journal Mathematics (see [9]) and independetly proved in 1893 by Hadamard in [10]. The inequality (1) is known in the literature as the Hermite–Hadamard inequality.

The Hermite–Hadamard inequality has several applications in nonlinear analysis and the geometry of Banach space (see [11]).

The Hermite–Hadamard inequality has been the subject of intensive research; many applications, generalizations, and improvement of them can be found in the literature (see [12]).

In recent years, many scholars have been interested in modifying and extending the Hermite–Hadamard inequality.

In [13], Dragomir and Fitzpatrick proved the following version of Hermite–Hadamard inequality for s-convex functions in the second sense: let $f\colon [0,\infty) \longrightarrow [0,\infty)$ be an s-convex function, where $s \in (0,1)$ and $a, b \in [0,\infty)$, $a < b$. If $f \in L^1[a,b]$, then the following inequalities hold:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{s+1}. \tag{2}$$

The authors of [13] defined the operator s-convex and proved the following inequality for operator s-convex function. He proved that if $f\colon I \subseteq [0,\infty) \longrightarrow \mathbb{R}$ is an operator s-convex function, then the following inequalities hold:

$$\begin{aligned} 2^{s-1} f\left(\frac{A+B}{2}\right) &\leq \frac{1}{b-a}\int_a^b f(\lambda A + (1-\lambda)B)\,\mathrm{d}\lambda \\ &\leq \frac{f(A)+f(B)}{s+1}. \end{aligned} \tag{3}$$

The following inequalities due to the authors [14] give the Hermite–Hadamard inequalities for operator h-convex function.

Let f be an operator h-convex function. Then,

$$\frac{1}{2h(1/2)} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \leq (f(A) + f(B)) \int_0^1 h(t)\,\mathrm{d}t, \tag{4}$$

for any self-adjoint operators A and B with spectra in K.

Motivated by the above results, we investigate in this paper the operator version of the Hermite–Hadamard inequality for operator (α, h)-preinvex function.

Let $\mathscr{B}(\mathscr{H})$ stand for the C^*-algebra of all bounded linear operators on a complex separable Hilbert space $(\mathscr{H}, \langle, \rangle)$. An operator $A \in \mathscr{B}(\mathscr{H})$ is positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$ (we write $A \geq 0$). A positive invertible operator A is naturally denoted by $A > 0$. Let $\mathscr{B}(\mathscr{H})^+$ stand for all positive operators $\mathscr{B}(\mathscr{H})$. For self-adjoint operators $A, B \in \mathscr{B}(\mathscr{H})$, we write $B \geq A$ if $B - A \geq 0$. A linear map $\varphi\colon \mathscr{B}(\mathscr{H}) \longrightarrow \mathscr{B}(\mathscr{H})$ is positive if $\varphi(A) \geq 0$ whenever $A \geq 0$ and φ is said to be unital if $\varphi(I) = I$.

Let A be a self-adjoint operator in $\mathscr{B}(\mathscr{H})$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(sp(A))$ of all continuous functions defined on the spectrum of A denoting $sp(A)$ and C^*-algebra $C^*(A)$ generated by A and the identity operator $1_{\mathscr{B}(\mathscr{H})}$ on $\mathscr{H}$ as follows:

For any $f, g \in C(sp(A))$ and any $\alpha, \beta \in \mathbb{C}$, we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\overline{f}) = \Phi(f)^*$

(iii) $\|\Phi(f)\| = \|f\|\colon = \sup_{t\in sp(A)}|f(t)|$

(iv) $\Phi(f_0) = 1_{\mathscr{H}}$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for all $t \in sp(A)$

If f is a continuous complex valued function on $sp(A)$, the element $\Phi(f)$ of $C^*(A)$ is denoted by $f(A)$ and we call it the continuous functional calculus for a bounded self-adjoint operator A. If A is a bounded self-adjoint operator and f is a real-valued continuous function on $sp(A)$, then $f(t) \geq 0$ for any $t \in sp(A)$ implies that $f(A) \geq 0$, *i.e.*, $f(A)$ is a positive operator on $\mathscr{H}$. Moreover, if both f and g are real-valued functions on $sp(A)$ such that $f(t) \leq g(t)$ for any $t \in sp(A)$, then $f(A) \leq g(A)$ in the operator order in $\mathscr{B}(\mathscr{H})$.

An important and useful class of functions are called operator convex functions. A real-valued continuous function f on interval $K \subseteq [0, \infty)$ is said to be operator convex (operator concave) if

$$f(\lambda A + (1-\lambda)B) \leq (\geq) \lambda f(A) + (1-\lambda) f(B), \tag{5}$$

in the operator order in $\mathscr{B}(\mathscr{H})$, for all $\lambda \in [0, 1]$ and for every bounded self-adjoint operators A and B in $\mathscr{B}(\mathscr{H})$ whose spectra are contained in K.

$K \subseteq \mathbb{R}^+$ is p-convex set if

$$sp\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right) \subseteq K, \tag{6}$$

for every positive operators $A, B \in \mathscr{B}(\mathscr{H})^+$ with spectra in K, $\lambda \in (0, 1)$ and $p > 0$.

Let $J \subseteq \mathbb{R}^+$ such that $(0, 1) \subset J$. A function $h\colon J \longrightarrow \mathbb{R}^+$ is called super-multiplicative function if $h(xy) \geq h(x)h(y)$ for all $x, y \in J$.

In this paper, we assume that $K \subseteq \mathbb{R}^+$ is a p-convex set. We introduce operator (p, h)-convex function. We establish some properties of operator (p, h)-convex function. This paper is organized as follows: In Section 2, we will show some properties of operator (p, h)-convex functions. In Section 3, a new refinement of the Hermite–Hadamard type inequality is presented for operator (p, h)-convex functions. If f is an operator (p, h)-convex function, then

$$\frac{1}{2h(1/2)} f\left(\left(\frac{A^p + B^p}{2}\right)^{1/p}\right) \leq \int_0^1 f\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right)\mathrm{d}\lambda \leq (f(A) + f(B)) \int_0^1 h(\lambda)\,\mathrm{d}\lambda. \tag{7}$$

Our results enable us to obtain a new inequality for positive operators on $\mathscr{B}(\mathscr{H})$. For example, let $0 < s \leq p$ and φ be an unital linear operator positive, then

$$\frac{1}{2}\left(\varphi^p(A) + \varphi^p(B)\right)^{s/p} \leq \int_0^1 \left(\lambda\varphi^p(A) + (1-\lambda)\varphi^p(B)\right)^{s/p}\mathrm{d}\lambda \leq \frac{p}{p+s}\left(\varphi^s(A) + \varphi^s(B)\right), \tag{8}$$

for any positive operators A and B belonging to $M(\mathscr{H})$ with spectra in K.

In some special cases, we show that our result gives a generalized estimation for operator (p, h)-convex functions than the corresponding results obtained in [15].

In Section 4, we will show that if f is an operator (p, h)-convex function, then we have

$$\frac{1}{2h(1/2)} Tr\left(f\left(\left(\frac{A^p + B^p}{2}\right)^{1/p}\right)\right) \leq \int_0^1 Tr\left(f\left(\left(tA^p + (1-t)B^p\right)^{1/p}\right)\right)\mathrm{d}t \leq (Tr(f(A)) + Tr(f(B))) \int_0^1 h(t)\,\mathrm{d}t, \tag{9}$$

for any self-adjoint operators A and B. We establish several trace inequalities for positive operator on $\mathscr{B}(\mathscr{H})$.

2. Some Properties of Operator (p, h)-Convex Functions

In order to obtain the main result of this section, we need the following known definitions.

Let f be a real-valued continuous function defined on an interval $K \subseteq \mathbb{R}^+$. We say that f is of operator P-class function on K if

$$f(\lambda A + (1-\lambda)B) \leq f(A) + f(B), \quad (10)$$

for all self-adjoint operators A, B with spectra in K and all $\lambda \in [0,1]$.

For some properties of this class of operators, see [16].

Let f be a continuous real-valued function defined on an interval $K \subseteq \mathbb{R}^+$. We say that f is an operator Q-class function on K if

$$f(\lambda A + (1-\lambda)B) \leq \frac{f(A)}{\lambda} + \frac{f(B)}{1-\lambda}, \quad (11)$$

for all self-adjoint operators A, B with spectra in K and all $\lambda \in (0,1)$.

See [17] for some results and inequalities on operator Q-class function.

A continuous function $f\colon K \subseteq [0,\infty) \longrightarrow \mathbb{R}^+$ is said to be operator s-convex on K if

$$f(\lambda A + (1-\lambda)B) \leq \lambda^s f(A) + (1-\lambda)^s f(B), \quad (12)$$

for all $\lambda \in [0,1]$ and for every positive operators A and B in $\mathscr{B}(\mathscr{H})^+$, whose spectra are contained in K for some fixed $s \in (0,1]$.

For some properties about operator s-convex function, see [18].

Let $K, J \subseteq \mathbb{R}^+$, $(0,1) \subseteq J$, and $h\colon J \longrightarrow \mathbb{R}$ be a nonnegative function nonidentical to 0. We say that a continuous function $f\colon K \longrightarrow \mathbb{R}^+$ is said be a operator h-convex function on K if

$$f(\lambda A + (1-\lambda)B) \leq h(\lambda) f(A) + h(1-\lambda) f(B), \quad (13)$$

for every $A, B \in \mathscr{B}(\mathscr{H})^+$ whose spectra are contained in K for all $\lambda \in (0,1)$.

For some results on operator h-convex, see [14].

Lemma 1 (see [19]). *Let $A, B \in \mathscr{B}(\mathscr{H})^+$, then $AB + BA \geq 0$ if and only if $f(A+B) \leq f(A) + f(B)$ for all nonnegative operator monotone function f on $[0,\infty)$.*

Lemma 2 (see [20]). *Let φ be a unital positive linear map on $\mathscr{H}$ and f an operator monotone function on $[0,\infty)$, then for every $A \geq 0$,*

$$\varphi(f(A)) \leq f(\varphi(A)). \quad (14)$$

Lemma 3 (Davis-Choi-Jensen's inequality). *Let φ be a unital positive linear map on $\mathscr{H}$ and g an operator convex function on $[0,\infty]$, then for every $A \geq 0$,*

$$\varphi(g(A)) \geq g(\varphi(A)). \quad (15)$$

The following lemma is the special case of Lemmas 2 and 3.

Lemma 4 (see [20]). *Let φ be a unital positive linear map on $\mathscr{H}$, then*

(i) $\varphi(A^r) \leq (\varphi(A))^r$ for every $A \geq 0$ and $0 \leq r \leq 1$

(ii) $(\varphi(A))^r \leq \varphi(A^r)$ for every $A > 0$ and $1 \leq r \leq 2$ or $-1 \leq r \leq 0$

The following lemma is a consequence of theorem Hansen-Pedersen-Yensen's inequality.

Lemma 5 (see [21]). *Let f be a continuous function mapping the positive half-line $[0,\infty)$ into itself. Then, f is the operator monotone if and only if it is operator concave.*

Definition 1 (see [22]). Let $p > 0$ and $K, J \subseteq \mathbb{R}^+$, $(0,1) \subset J$, and $h\colon J \longrightarrow \mathbb{R}^+$ be a nonnegative function nonidentical to 0. A continuous function $f\colon K \longrightarrow \mathbb{R}^+$ is said to be operator (p, h)-convex (concave) if

$$f\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right) \leq (\geq) h(\lambda) f(A) + h(1-\lambda) f(B), \quad (16)$$

for any positive operators A, B with spectra in K and $\lambda \in (0,1)$.

This class contains several well-known classes of nonnegative operator convex function, operator P-class function, operator Q-class function, operator s-convex function, and operator h-convex functions on K. We can see some results operator (p, h)-convex functions by Dinh and Khue in [22].

Lemma 6 (see [22]). *Let φ be a unital positive linear map on $\mathscr{B}(\mathscr{H})$, A a positive operator in $\mathscr{H}$, and f an operator (p, h)-convex function on $\mathbb{R}^+$ such that $f(0) = 0$. Let h be a nonnegative and nonzero super-multiplicative function on J satisfying $2h(1/2) \leq \lambda^{-1} h(\lambda)$ $(\lambda \in (0,1))$. Then,*

$$f\left(\left(\varphi(A^p)\right)^{1/p}\right) \leq 2h\left(\frac{1}{2}\right) \varphi(f(A)). \quad (17)$$

Now, let us prove some properties of operator (p, h)-convex functions. In this paper, we suppose $(0,1) \subset J$ and $J, K \subseteq \mathbb{R}^+$.

Proposition 1. *Let $\lambda \in (0,1)$ and $h\colon J \longrightarrow \mathbb{R}^+$ be a nonzero function and $f\colon K \longrightarrow \mathbb{R}^+$ be an operator (p, h)-convex. Therefore, for positive operator A with spectra in K,*

(i If $h(\lambda) + h(1-\lambda) > 1$, then $f(A) \geq 0$

(ii) If $h(\lambda) + h(1-\lambda) < 1$, then $f(A) \leq 0$

Proof

(i) Suppose that A is a positive operator with spectra in K and $A = B$ in (16). Then, $f(A) \leq h(\lambda) f(A) + h(1-\lambda) f(A)$. Therefore,

$f(A) \le (h(\lambda) + h(1-\lambda)) f(A)$. Hence, $(h(\lambda) + h(1-\lambda) - 1) f(A) \ge 0$ and so $f(A) \ge 0$.

(ii) On account of the mentioned, (i). □

Proposition 2. *Let $h\colon J \longrightarrow \mathbb{R}^+$ be a nonzero function; therefore,*

(i) Let $f\colon K \longrightarrow \mathbb{R}^+$ be an operator (p,h)-concave. If $g\colon [0,\infty) \longrightarrow [0,\infty)$ is an operator monotone and $h(\lambda) + h(1-\lambda) = 1$, then $g^\circ f$ is an operator (p,h)-concave.

(ii) Let $f\colon K \longrightarrow \mathbb{R}^+$ be an operator (p,h)-convex and operator monotone. Then, for all $1 \le p \le 2$, f is the operator h-convex function.

Proof

(i) Let $f\colon K \longrightarrow \mathbb{R}^+$ be an operator (p,h)-concave and A and B are positive operators with spectra in K, then

$$f\left(\left(\lambda A^p + (1-\lambda) B^p\right)^{1/p} \right) \ge h(\lambda) f(A) + h(1-\lambda) f(B). \tag{18}$$

So,

$$\begin{aligned} g\left(f\left(\left(\lambda A^p + (1-\lambda)B^p\right)\right)\right) &\ge g\left(h(\lambda) f(A) + h(1-\lambda) f(B)\right) \\ &\ge h(\lambda) g(f(A)) + h(1-\lambda) g(f(B)). \end{aligned} \tag{19}$$

Therefore, $g^\circ f$ is the operator (p,h)-concave.

(ii) If $1 \le p \le 2$, then $r(x) = x^p$ is the operator convex. For positive operators A and B with spectra in K, we have $(\lambda A + (1-\lambda)B)^p \le \lambda A^p + (1-\lambda)B^p$. Since $1/2 \le 1/p \le 1$, then $k(x) = x^{1/p}$ is the operator monotone. Hence,

$$(\lambda A + (1-\lambda)B) \le \left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}. \tag{20}$$

We have

$$\begin{aligned} f(\lambda A + (1-\lambda)B) &\le f\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p} \right) \\ &\le h(\lambda) f(A) + h(1-\lambda) f(B). \end{aligned} \tag{21}$$

Then, f is the operator h-convex function. □

Proposition 3. *Let φ be a unital positive linear map on $\mathscr{B}(\mathscr{H})$, A and B be positive operators in $\mathscr{H}$ with spectra in K, and $f\colon K \longrightarrow \mathbb{R}^+$ be an operator (p,h)-convex function such that $f(0) = 0$. Let $h\colon J \longrightarrow (0,\infty)$ be a super-multiplicative function satisfying $2h(1/2) \le \lambda^{-1} h(\lambda)$ for $\lambda \in (0,1)$. We have*

$$f\left(\left(\varphi\left(\lambda A^p + (1-\lambda)B^p\right)\right)^{1/p} \right) \le 2h\left(\frac{1}{2}\right)\left(h(\lambda)\varphi(f(A)) + h(1-\lambda)\varphi(f(B))\right). \tag{22}$$

Proof. By Lemma 7, $f\left((\varphi(A^p))^{1/p}\right) \le 2h(1/2)\varphi(f(A))$. If $A = (\lambda A^p + (1-\lambda)B^p)^{1/p}$, then

$$\begin{aligned} f\left(\left(\varphi\left(\lambda A^p + (1-\lambda)B^p\right)\right)^{1/p} \right) &\le 2h\left(\frac{1}{2}\right)\varphi\left(f\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p} \right) \right) \\ &\le 2h\left(\frac{1}{2}\right)\varphi\left(h(\lambda) f(A) + h(1-\lambda) f(B)\right) \\ &\le 2h\left(\frac{1}{2}\right)\left(h(\lambda)\varphi(f(A)) + h(1-\lambda)\varphi(f(B))\right). \end{aligned} \tag{23}$$

□

Corollary 1. *With conditions, Proposition 3, we have*

$$f\left(\left(\varphi\left(\frac{A^p + B^p}{2} \right) \right)^{1/p} \right) \le 2h^2\left(\frac{1}{2}\right)\left(\varphi(f(A)) + \varphi(f(B))\right). \tag{24}$$

Proof. Put $\lambda = 1/2$ in inequality (22). □

Remark 1. For $\varphi(A) = A$, inequality (24) reduces to the inequality

$$f\left(\left(\frac{A^p + B^p}{2} \right)^{1/p} \right) \le 2h^2\left(\frac{1}{2}\right)\left(f(A) + f(B)\right). \tag{25}$$

Proposition 4. *Let $p > 0$ and $h\colon J \longrightarrow \mathbb{R}^+$ be a nonzero function, then the following statements are equivalent:*

(i) $f(x) = x^s$ is an operator (p,h)-convex function

(ii) $f(x) = x^{s/p}$ is an operator h-convex function

Proof

$(i) \Rightarrow (ii)$: suppose that $f(x) = x^s$ is an operator (p,h)-convex. Thus, $(\lambda A^p + (1-\lambda)B^p)^{s/p} \le h(\lambda) A^s + h(1-\lambda) B^s$ for each positive operator A and B with spectra in K and $\lambda \in (0,1)$. For $A = A^{1/p}$ and $B = B^{1/p}$, we have $(\lambda A + (1-\lambda)B)^{s/p} \le h(\lambda) A^{s/p} + h(1-\lambda) B^{s/p}$. Therefore, $f(x) = x^{s/p}$ is an operator h-convex.

$(ii) \Rightarrow (i)$: the proof is similarly $(i) \Rightarrow (ii)$. □

Remark 2. Let $f(x) = x^{s/p}$ for $0 < s \le p$ be a nonnegative operator monotone function on $(0, \infty)$ and $\mathcal{M}(\mathcal{H}) = \{(A, B) \in \mathcal{B}(\mathcal{H})^+ \times \mathcal{B}(\mathcal{H})^+ | AB + BA \ge 0\}$. By Lemma 1 for each $(A, B) \in \mathcal{M}(\mathcal{H})$, we have $f(A + B) \le f(A) + f(B)$. Put $A = \lambda A$ and $B = (1-\lambda)B$, $\lambda \in (0, 1)$. Hence, $f(\lambda A + (1-\lambda)B) \le f(\lambda A) + f((1-\lambda)B)$. Therefore,

$$(\lambda A + (1-\lambda)B)^{s/p} \le \lambda^{s/p} A^{s/p} + (1-\lambda)^{s/p} B^{s/p}. \tag{26}$$

Therefore, $f(x) = x^{s/p}$ for $h(\lambda) = \lambda^{s/p}$ is an operator h-convex for $0 < s \le p$.

The following example is a characteristic example of operator (p, h)-convex function.

Example 1. Let $0 < s \le p$. By Remark 2, $f(x) = x^{s/p}$ is an operator h-convex function for $h(\lambda) = \lambda^{s/p}$ on $M(\mathcal{H})$. By Proposition 4, $f(x) = x^s$ is an operator (p, h)-convex on $M(\mathcal{H})$.

In the following preposition, we eliminate condition $A, B \in M(\mathcal{H})$ and we get the new example of operator (p, h)-convex.

Proposition 5. *Let $p > 0$ and $f(x) = x^s$ and $h(\lambda) = \lambda$, then f is the operator (p, h)-convex if and only if $s \in [p, 2p]$.*

Proof. If f is an operator (p, h)-convex, for each positive operator A and B with spectra in K, $(\lambda A^p + (1-\lambda)B^p)^{s/p} \le h(\lambda)A^s + h(1-\lambda)B^s$. Hence, $(\lambda A + (1-\lambda)B)^{s/p} \le \lambda A^{s/p} + (1-\lambda)B^{s/p}$. The last inequality means that the function $g(t) = t^{s/p}$ is the operator convex, which is equivalent to the condition $s \in [p, 2p]$. □

Corollary 2. *Let $f(x) = x^s$, $h(\lambda) = \lambda$, and $p > 0$, then we have*

(i) *If $s \in [p, 2p] \cap [0, 1]$, then the function f is an operator (p, h)-convex and operator concave*

(ii) *If $s \in [p, 2p] \cap [1, 2]$, then the function f is an operator (p, h)-convex and operator convex*

Proof. Because $f(x) = x^s$ for $0 \le s \le 1$ is operator concave and $f(x) = x^s$ for $1 \le s \le 2$ is operator convex, then proof is trivial. □

Proposition 6. *Let $f\colon K \longrightarrow \mathbb{R}^+$ be an operator (p, h)-convex and $h\colon J \longrightarrow \mathbb{R}^+$ be a nonzero function and A, B are two positive operators with spectra in K, then $\varphi_{x,A,B}(t)\colon [0,1] \longrightarrow \mathbb{R}^+$ defined by $\varphi_{x,A,B}(t) = \langle f((tA^p + (1-t)B^p)^{1/p})x, x\rangle$ is a h-convex function on $[0, 1]$ for any $x \in \mathcal{H}$ with $\|x\| = 1$.*

Proof. For each $u, v \in [0, 1]$, we have

$$\begin{aligned}
\varphi_{x,A,B}(tu + (1-t)v) &= \langle f\Big((tu + (1-t)v\,)A^p + (1 - (tu + (1-t)v))B^p\Big)^{1/p} \Big)x, x\rangle \\
&= \langle f\Big(\big(t(uA^p + (1-u)B^p) + (1-t)(vA^p + (1-v)B^p)\big)^{1/p} \Big)x, x\rangle \\
&= \langle f\Big(\Big(t\Big[(uA^p + (1-u)B^p)^{1/p} \Big]^p + (1-t)\Big[(vA^p + (1-v)B^p)^{1/p} \Big]^p\Big)^{1/p} \Big)x, x\rangle \\
&\le h(t)\langle f\Big((uA^p + (1-u)B^p)^{1/p} \Big)x, x\rangle \\
&\quad + h(1-t)\langle f\Big((vA^p + (1-v)B^p)^{1/p} \Big)x, x\rangle \\
&= h(t)\varphi_{x,A,B}(u) + h(1-t)\varphi_{x,A,B}(v).
\end{aligned} \tag{27}$$

□

3. Hermite–Hadamard Inequality for Operator (p, h)-Convex Functions

Now, we are ready to state the main result of this paper.

Theorem 1. *Let f be a continuous operator (p, h)-convex function and $h\colon J \longrightarrow \mathbb{R}^+$ be a continuous nonzero function. Then, for any $A, B \in \mathcal{B}(\mathcal{H})^+$ with spectra in K, we have*

$$\frac{1}{2h(1/2)} f\left(\left(\frac{A^p + B^p}{2} \right)^{1/p} \right) \le \int_0^1 f\Big((\lambda A^p + (1-\lambda)B^p)^{1/p} \Big)\, \mathrm{d}\lambda \le (f(A) + f(B)) \int_0^1 h(\lambda)\, \mathrm{d}\lambda. \tag{28}$$

Proof. Let $\lambda = 1/2$ in Definition 1, then

$$f\left(\left(\frac{A^p+B^p}{2}\right)^{1/p}\right) \le h\left(\frac{1}{2}\right)(f(A)+f(B)). \tag{29}$$

Put $A_1 = (\lambda A^p + (1-\lambda)B^p)^{1/p}$ and $B_1 = ((1-\lambda)A^p + \lambda B^p)^{1/p}$. Then, we have $A_1^p + B_1^p = A^p + B^p$. So,

$$f\left(\left(\frac{A^p+B^p}{2}\right)^{1/p}\right) \le h\left(\frac{1}{2}\right)\left(f\left((\lambda A^p+(1-\lambda)B^p)^{1/p}\right) + f\left(((1-\lambda)A^p+\lambda B^p)^{1/p}\right)\right). \tag{30}$$

Therefore,

$$\begin{aligned} f\left(\left(\frac{A^p+B^p}{2}\right)^{1/p}\right) &\le h\left(\frac{1}{2}\right)\int_0^1 \left(f\left((\lambda A^p+(1-\lambda)B^p)^{1/p}\right) + f\left(((1-\lambda)A^p+\lambda B^p)^{1/p}\right)\right) d\lambda \\ &= 2h\left(\frac{1}{2}\right)\int_0^1 f\left((\lambda A^p+(1-\lambda)B^p)^{1/p}\right) d\lambda. \end{aligned} \tag{31}$$

As

$$\begin{aligned} 2h\left(\frac{1}{2}\right)\int_0^1 f\left((\lambda A^p+(1-\lambda)B^p)^{1/p}\right) d\lambda &\le 2h\left(\frac{1}{2}\right)\int_0^1 (h(\lambda)f(A)+h(1-\lambda)f(B))\, d\lambda \\ &\le 2h\left(\frac{1}{2}\right)\left(f(A)\int_0^1 h(\lambda)\, d\lambda + f(B)\int_0^1 h(1-\lambda)\, d\lambda\right) \\ &= 2h\left(\frac{1}{2}\right)(f(A)+f(B))\int_0^1 h(\lambda)\, d\lambda, \end{aligned} \tag{32}$$

thus (28) holds. □

Remark 3. If $p = 1$ in inequality (28), then Theorem 1 produces [14].

Corollary 3.3. *Let $p > 0$, $s \in [p, 2p]$, and φ be an unital linear positive map, then*

$$\begin{aligned} \left(\frac{\varphi^p(A)+\varphi^p(B)}{2}\right)^{s/p} &\le \int_0^1 (\lambda\varphi^p(A) + (1-\lambda)\varphi^p(B)^{(s/p)}\, d\lambda \\ &\le \frac{\varphi^s(A)+\varphi^s(B)}{2}, \end{aligned} \tag{33}$$

for any positive operators A and B with spectra in K.

Proof. By Corollary 2, $f(x) = x^s$ is the operator (p,h)-convex for $h(\lambda) = \lambda$ and $s \in [p, 2p]$. Put $f(x) = x^s$ in Theorem 1.

In particular, we have the following, □

Corollary 4. *Let $0 < s \le p$ and φ be an unital linear positive map, then*

$$\begin{aligned} \frac{1}{2}(\varphi^p(A)+\varphi^p(B))^{s/p} &\le \int_0^1 (\lambda\varphi^p(A) + (1-\lambda)\varphi^p(B)^{(s/p}\, d\lambda \\ &\le \frac{p}{p+s}(\varphi^s(A)+\varphi^s(B)), \end{aligned} \tag{34}$$

for any positive operators A and B belonging to $M(\mathscr{H})$ with spectra in K.

Proof. By Example 1, $f(x) = x^s$ is the operator (p,h)-convex for $h(\lambda) = \lambda^{s/p}$ and $0 < s \le p$. Therefore,

$$\begin{aligned} \frac{1}{2(1/2)^{s/p}}\left(\frac{A^p+B^p}{2}\right)^{s/p} &\le \int_0^1 (\lambda A^p+(1-\lambda)B^p)^{s/p}\, d\lambda \\ &\le (A^s+B^s)\int_0^1 \lambda^{s/p}\, d\lambda. \end{aligned} \tag{35}$$

So,

$$\begin{aligned} \frac{1}{2}(A^p+B^p)^{s/p} &\le \int_0^1 (\lambda A^p+(1-\lambda)B^p)^{s/p}\, d\lambda \\ &\le (A^s+B^s)\left(\frac{p}{s+p}\right). \end{aligned} \tag{36}$$

Put $A = \varphi(A)$ and $B = \varphi(B)$ in inequality (36). □

Corollary 5. *Let $0 < s \le p \le 1$ and φ be a positive operator, then*

$$\begin{aligned} \frac{1}{2}\varphi\left((A^p+B^p)^{s/p}\right) &\le \frac{1}{2}(\varphi^p(A)+\varphi^p(B))^{s/p} \\ &\le \left(\frac{p}{s+p}\right)(\varphi^s(A)+\varphi^s(B)). \end{aligned} \tag{37}$$

for each positive operators A and B in $M(\mathscr{H})$.

Proof. If $0 < s \leq p \leq 1$, then $\varphi(A^p + B^p) \leq \varphi^p(A) + \varphi^p(B)$ (by Lemma 4), we have

$$\begin{aligned}\frac{1}{2}\varphi\left(\left(A^p + B^p\right)^{s/p}\right) &\leq \frac{1}{2}\left(\varphi^p(A) + \varphi^p(B)\right)^{s/p}\\ &\leq \int_0^1 \left(\lambda\varphi^p(A) + (1-\lambda)\varphi^p(B)\right)^{s/p} \mathrm{d}\lambda\\ &\leq \left(\frac{p}{s+p}\right)\left(\varphi^s(A) + \varphi^s(B)\right).\end{aligned} \tag{38}$$

□

Definition 2. Let $f, g\colon K \longrightarrow \mathbb{R}^+$ be operator (p, h)-convex functions and A, B self-adjoint operators on Hilbert space $\mathscr{H}$ with spectra in K. We defined real functions $M(A, B)$ and $N(A, B)$ on Hilbert space $\mathscr{H}$ by

$$\begin{aligned}M(A, B)(x) = {}&\langle f(A)x, x\rangle\langle g(A)x, x\rangle\\ &+\langle f(B)x, x\rangle\langle g(B)x, x\rangle,\end{aligned} \tag{39}$$

and

$$\begin{aligned}N(A, B)(x) = {}&\langle f(A)x, x\rangle\langle g(B)x, x\rangle\\ &+\langle f(B)x, x\rangle\langle g(A)x, x\rangle,\end{aligned} \tag{40}$$

for any $x \in \mathscr{H}$.

Theorem 2. *Let $h_1, h_2\colon J \longrightarrow \mathbb{R}^+$ and $f, g\colon K \longrightarrow \mathbb{R}^+$ be continuous functions. If f is an operator (p, h_1)-convex and g is an operator (p, h_2)-convex function, then for any positive operators A and B with spectra in K and each $x \in \mathscr{H}$ with $\|x\| = 1$, the following inequality holds:*

$$\begin{aligned}&\int_1^0 \left\langle f\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right)x, x\right\rangle\left\langle g\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right)x, x\right\rangle \mathrm{d}\lambda\\ &\leq M(A, B)(x)\int_0^1 h_1(\lambda)h_2(\lambda)\,\mathrm{d}\lambda + N(A, B)(x)\int_0^1 h_1(\lambda)h_2(1-\lambda)\,\mathrm{d}\lambda.\end{aligned} \tag{41}$$

Proof. On account of the operator (p, h_1)-convexity of f and (p, h_2)-convexity of g, we give

$$\left\langle f\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right)x, x\right\rangle \leq \left\langle \left(h_1(\lambda)f(A) + h_1(1-\lambda)f(B)\right)x, x\right\rangle, \tag{42}$$

$$\left\langle g\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right)x, x\right\rangle \leq \left\langle \left(h_2(\lambda)g(A) + h_2(1-\lambda)g(B)\right)x, x\right\rangle, \tag{43}$$

for each $\lambda \in (0, 1)$ and $x \in \mathscr{H}$ with $\|x\| = 1$. We have

$$\begin{aligned}&\left\langle f\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right)x, x\right\rangle\left\langle g\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right)x, x\right\rangle\\ &\leq h_1(\lambda)h_2(\lambda)\langle f(A)x, x\rangle\langle g(A)x, x\rangle\\ &\quad + h_1(1-\lambda)h_2(\lambda)\langle f(B)x, x\rangle\langle g(A)x, x\rangle\\ &\quad + h_1(\lambda)h_2(1-\lambda)\langle f(A)x, x\rangle\langle g(B)x, x\rangle\\ &\quad + h_1(1-\lambda)h_2(1-\lambda)\langle f(B)x, x\rangle\langle g(B)x, x\rangle.\end{aligned} \tag{44}$$

Therefore,

$$\begin{aligned}&\int_0^1\langle f\Big(\big(\lambda A^p+(1-\lambda)B^p\big)^{1/p}\Big)x,x\rangle\langle g\Big(\big(\lambda A^p+(1-\lambda)B^p\big)^{1/p}\Big)x,x\rangle\,\mathrm{d}\lambda\\&\le\langle f(A)x,x\rangle\langle g(A)x,x\rangle\int_0^1 h_1(\lambda)h_2(\lambda)\,\mathrm{d}\lambda\\&\quad+\langle f(B)x,x\rangle\langle g(A)x,x\rangle\int_0^1 h_1(1-\lambda)h_2(\lambda)\,\mathrm{d}\lambda\\&\quad+\langle\langle f(A)x,x\rangle\langle g(B)x,x\rangle\int_0^1 h_1(\lambda)h_2(1-\lambda)\,\mathrm{d}\lambda\\&\quad+\langle f(B)x,x\rangle\langle g(B)x,x\rangle\int_0^1 h_1(1-\lambda)h_2(1-\lambda)\,\mathrm{d}\lambda.\end{aligned}\tag{45}$$

The proof is complete. □

Corollary 6. *In Theorem 2, if $h_1(\lambda)=h_2(\lambda)=h(\lambda)$, then*

$$\begin{aligned}&\int_0^1\langle f\Big(\big(\lambda A^p+(1-\lambda)B^p\big)^{1/p}\Big)x,x\rangle\langle g\Big(\big(\lambda A^p+(1-\lambda)B^p\big)^{1/p}\Big)x,x\rangle\,\mathrm{d}\lambda\\&\le M(A,B)(x)\int_0^1 h^2(t)\,\mathrm{d}t+N(A,B)(x)\int_0^1 h(t)h(1-t)\,\mathrm{d}t.\end{aligned}\tag{46}$$

Remark 4. If $h_1(\lambda)=h_2(\lambda)=\lambda$ in Theorem 2, then

$$\begin{aligned}&\int_0^1\langle f\Big(\big(\lambda A^p+(1-\lambda)B^p\big)^{1/p}\Big)x,x\rangle\langle g\Big(\big(\lambda A^p+(1-\lambda)B^p\big)^{1/p}\Big)x,x\rangle\,\mathrm{d}\lambda\\&\le\frac{1}{3}M(A,B)(x)+\frac{1}{6}N(A,B)(x).\end{aligned}\tag{47}$$

Hence, in this case, Theorem 2 produces [15], Theorem 3.

Theorem 3. *Let $h_1,h_2\colon J\longrightarrow\mathbb{R}^+$ and $f,g\colon K\longrightarrow\mathbb{R}^+$ be continuous functions. If f is an operator (p,h_1)-convex and g is an operator (p,h_2)-convex function, then for any positive operators A and B with spectra in K and each $x\in\mathscr{H}$ with $\|x\|=1$, the following inequality holds:*

$$\begin{aligned}&\frac{1}{2h_1(1/2)h_2(1/2)}\langle f\left(\left(\frac{A^p+B^p}{2}\right)^{1/p}\right)x,x\rangle\langle g\left(\left(\frac{A^p+B^p}{2}\right)^{1/p}\right)x,x\rangle\\&\le\int_0^1\langle f\Big(\big(\lambda A^p+(1-\lambda)B^p\big)^{1/p}\Big)x,x\rangle\langle g\Big(\big(\lambda A^p+(1-\lambda)B^p\big)^{1/p}\Big)x,x\rangle\,\mathrm{d}\lambda\\&\quad+M(A,B)(x)\int_0^1 h_1(\lambda)h_2(1-\lambda)\,\mathrm{d}\lambda+N(A,B)(x)\int_0^1 h_1(\lambda)h_2(\lambda)\,\mathrm{d}\lambda.\end{aligned}\tag{48}$$

Proof. Since $A^p + B^p/2 = \lambda A^p + (1-\lambda)B^p/2 + (1-\lambda)A^p + \lambda B^p/2$, for any $x \in \mathscr{H}$ with $\|x\| = 1$, we have

$$\begin{aligned}
&\langle f\left(\left(\frac{A^p + B^p}{2}\right)^{1/p}\right)x, x\rangle \langle g\left(\left(\frac{A^p + B^p}{2}\right)^{1/p}\right)x, x\rangle \\
&= \langle f\left(\frac{\lambda A^p + (1-\lambda)B^p}{2} + \frac{(1-\lambda)A^p + \lambda B^p}{2}\right)x, x\rangle \\
&\langle g\left(\frac{\lambda A^p + (1-\lambda)B^p}{2} + \frac{(1-\lambda)A^p + \lambda B^p}{2}\right)x, x\rangle \\
&\leq h_1\left(\frac{1}{2}\right)\left(\langle f\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right)x, x\rangle + \langle f\left(\left((1-\lambda)A^p + \lambda B^p\right)^{1/p}\right)x, x\rangle\right) \\
&h_2\left(\frac{1}{2}\right)\left(\langle g\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right)x, x\rangle + \langle g\left(\left((1-\lambda)A^p + \lambda B^p\right)^{1/p}\right)x, x\rangle\right) \\
&\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left(\langle f\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right)x, x\rangle \langle g\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right)x, x\rangle\right) \\
&\quad + h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left(\langle f\left(\left((1-\lambda)A^p + \lambda B^p\right)^{\frac{1}{p}}\right)x, x\rangle \langle g\left(\left((1-\lambda)A^p + \lambda B^p\right)^{\frac{1}{p}}\right)x, x\rangle\right) \\
&\quad + h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left(h_1(\lambda)\langle f(A)x, x\rangle + h_1(1-\lambda)\langle f(B)x, x\rangle\right) \\
&\quad \cdot \left(h_2(1-\lambda)\langle g(A)x, x\rangle + h_2(\lambda)\langle g(B)x, x\rangle\right) \\
&\quad + h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left(h_1(1-\lambda)\langle f(A)x, x\rangle + h_1(\lambda)\langle f(B)x, x\rangle\right) \\
&\quad \cdot \left(h_2(\lambda)\langle g(A)x, x\rangle + h_2(1-\lambda)\langle g(B)x, x\rangle\right) \\
&= h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left(\langle f\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right)x, x\rangle \langle g\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right)x, x\rangle\right) \\
&\quad + h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left(\langle f\left(\left((1-\lambda)A^p + \lambda B^p\right)^{1/p}\right)x, x\rangle \langle g\left(\left((1-\lambda)A^p + \lambda B^p\right)^{1/p}\right)x, x\rangle\right) \\
&\quad + h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left(\left(h_1(\lambda)h_2(1-\lambda) + h_1(1-\lambda)h_2(\lambda)\right)M(A, B)(x)\right) \\
&\quad + h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left(\left(h_1(\lambda)h_2(\lambda) + h_1(1-\lambda)h_2(1-\lambda)\right)N(A, B)(x)\right).
\end{aligned} \tag{49}$$

Integrating both sides of inequality over $[0, 1]$, we get the required inequality (48). □

Remark 5. In Theorem 3, if $h_1(\lambda) = h_2(\lambda) = \lambda$, then

$$\begin{aligned}
&\langle f\left(\left(\frac{A^p + B^p}{2}\right)^{1/p}\right)x, x\rangle \langle g\left(\left(\frac{A^p + B^p}{2}\right)^{1/p}\right)x, x\rangle \\
&\leq \int_0^1 \langle f\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right)x, x\rangle \langle g\left(\left(\lambda A^p + (1-\lambda)B^p\right)^{1/p}\right)x, x\rangle \, d\lambda \\
&\quad + \frac{1}{12}M(A, B)(x) + \frac{1}{6}N(A, B)(x).
\end{aligned} \tag{50}$$

Therefore, in this case, Theorem 3 produces [15], Theorem 4.

4. Some Trace Functional Inequalities for Operator

Let $K(\mathscr{H})$ be two sided ideal of compact operator $\mathscr{B}(\mathscr{H})$. For $A \in \mathscr{B}(\mathscr{H})$, let $\|A\| = \sup \{\|Ax\|: \|x\| = 1\}$ denote the usual operator norm of A and $|A| = (A^* A)^{1/2}$ be the absolute value of A. The direct sum $A \oplus B$ denotes the block diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ defined on $\mathscr{H} \oplus \mathscr{H}$ (see [20]). It is clear that $\|A \oplus B\| = \max (\|A\|, \|B\|)$.

For any operator A, the operator $A^* A$ is always positive and its unique positive square root is denoted by $|A|$.

We remind some basic properties of trace for operators. Let $\{e_i\}_{i\in I}$ be an orthonormal basis of $\mathscr{H}$; we say that $A \in \mathscr{B}(\mathscr{H})$ is the trace class if

$$\|A\|_1 = \sum_{i\in I} \langle |A| e_i, e_i \rangle < \infty. \tag{51}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote the set of trace class operators in $\mathscr{B}(\mathscr{H})$ by $\mathscr{B}_1(\mathscr{H})$. We define the trace of a trace class operator $A \in \mathscr{B}_1(\mathscr{H})$ to be

$$\operatorname{Tr}(A) := \sum_{i\in I} \langle A e_i, e_i \rangle, \tag{52}$$

where $\{e_i\}_{i\in I}$ is an orthonormal basis of $\mathscr{H}$. $\operatorname{Tr}(\cdot)$ is a bounded linear functional on $\mathscr{B}_1(\mathscr{H})$ with $\|\operatorname{Tr}\| = 1$.

Lemma 7. *Let $f: K \longrightarrow \mathbb{R}^+$ be an operator (p,h)-convex function and $h: J \longrightarrow \mathbb{R}^+$ be a nonzero function, then $g(t) = Tr(f((tA^p + (1-t)B^p)^{1/p}))$ is the h-convex on $(0,1)$ for self-adjoint operators A and B with spectra in K.*

Proof. Because trace functional is convexity and monotonicity, then for each $u, v \in (0,1)$ and $0 < \alpha < 1$, we obtain

$$\begin{aligned}
g(\alpha u + (1-\alpha)v) &= Tr\Big(f\big((\alpha u + (1-\alpha)v)A^p + (\alpha u + (1-\alpha)v)B^p\big)^{1/p} \Big) \\
&= Tr\Big(f\big(\alpha(uA^p + (1-u)B^p) + (1-\alpha)(vA^p + (1-v)B^p)\big)^{1/p} \Big) \\
&= Tr\bigg(f\bigg(\alpha\Big[\Big((uA^p + (1-u)B^p)^{1/p}\Big)^p\Big] + (1-\alpha)\Big[\Big((vA^p + (1-v)B^p)^{1/p}\Big)^p\Big]\bigg)^{1/p} \bigg) \\
&\leq Tr\Big(h(\alpha) f\big(uA^p + (1-u)B^p\big)^{1/p} + h(1-\alpha) f\Big(\big(vA^p + (1-v)B^p\big)^{1/p}\Big) \Big) \\
&= h(\alpha) Tr\Big(f\big(uA^p + (1-u)B^p\big)^{1/p} \Big) + h(1-\alpha) Tr\Big(f\Big(\big(vA^p + (1-v)B^p\big)^{1/p}\Big) \Big) \\
&= h(\alpha) g(u) + h(1-\alpha) g(v).
\end{aligned} \tag{53}$$

Therefore, g is h-convex. □

Theorem 4. *Let $f: K \longrightarrow \mathbb{R}^+$ be an operator (p,h)-convex function and $h: J \longrightarrow \mathbb{R}^+$ be a nonzero function, then we have*

$$\begin{aligned}
&\frac{1}{2h(1/2)} \operatorname{Tr}\left(f\left(\left(\frac{A^p + B^p}{2} \right)^{1/p} \right) \right) \\
&\leq \int_0^1 Tr\Big(f\Big(\big(tA^p + (1-t)B^p\big)^{1/p} \Big) \Big) \mathrm{d}t \\
&\leq \big(\operatorname{Tr}(f(A)) + \operatorname{Tr}(f(B))\big) \int_0^1 h(t)\,\mathrm{d}t,
\end{aligned} \tag{54}$$

for each self-adjoint operators A and B with spectra in K.

Proof. $g(t) = \operatorname{Tr}(f(tA^p + (1-t)B^p)^{1/p})$ is h-convex on $(0,1)$ by Lemma 4. Hence, by inequality (28),

$$\begin{aligned}
\frac{1}{2h(1/2)} g\left(\frac{0+1}{2}\right) &\leq \int_0^1 g(t)\,\mathrm{d}t \\
&\leq \left(\frac{g(0) + g(1)}{2} \right) \int_0^1 h(t)\,\mathrm{d}t.
\end{aligned} \tag{55}$$

Therefore, we have the desired result. □

Remark 6. We think that with the new definition operator (p,h)-convex function for the Jensen's inequality and operator convex inequalities in [23], generalization or new inequalities can be obtained.

Data Availability

This article is about pure mathematics and the authors of the article express their satisfaction that the use of its data is in accordance with the policies of the journal. The data are available from the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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