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Research Article

Convergence Analysis Hilbert Space Approach for Fuzzy Integro-Differential Models

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In this paper, we present and demonstrate an innovative numerical method, which makes use of fuzzy numbers and fuzzy parameters that is effective in the solution of fuzzy type Volterra integro-differential equations, which was previously thought to be impossible using conventional methods. The first application of a technique for solving Volterra integro-differential equations of the fuzzy type, which was devised and tested in this paper, is shown here. This is the first time that this approach has been used. This system's overall quality may be improved as a consequence of the use of the Hilbert space replicating kernel idea, which is a possibility. Separate evaluations are made of the algorithms' correctness and sloppiness, as well as their foundations in the computationally effective kernel Hilbert space, which has been extensively researched in the past. Numerical examples are provided of the article to demonstrate how the technique outlined before may achieve convergence and accuracy. Here are a few illustrations to help understand that it is possible to deal with physical issues that require complicated geometric calculations with the assistance of the method explained in this article.

1. Introduction

It is possible to achieve better results using differential and probabilistic techniques rather than conventional processes. It is possible to uncover system features with more precision and less work using differential and probabilistic techniques rather than traditional processes, as opposed to traditional processes. In contrast to this, traditional procedures require the placement of more resources and time to be successful. In an attempt to attain this goal, one strategy that may be used is the application of fuzzy integro-differential equations which are more accurate than typical approaches [1]. A significant increase in the number of theoretical and computational computations using fuzzy Volterra integro-differential equations, as well as the number of publications referring to these equations, both in terms of quantity and quality, has been seen in recent years [2–5]. Both the number of products available and the quality of those products have increased as a result of this expansion (also known as fuzzy Volterra integro-differential equations, or FVIDEs). Only a few academics (including [6–9] and other published works) have looked at the consequences of fuzzy modelling in a quantum gravity context to the best of our knowledge at the time of writing. With the exception of these issues, other areas such as fragile biopolymers, quantum gravity, and quantum optics, among others, have gotten only a sliver of attention in population dynamics research despite the fact that they are important. This, on the other hand, has just recently become the case. Biswas and Roy developed a second-order fuzzy differential equations (FVDE) technique that is based on fuzzy differential equations in order to deal with fuzzy differential equations in practice [10, 11]. It may be used to solve fuzzy differential equations as well as other issues because it is based on the differentiability extension concept established by Seikkala. The concept of differentiability extension developed by Seikkala, which provided as inspiration for the method, laid the groundwork for its development. In fact, upon closer examination, it becomes clear that fuzzy integro-differential equation (as well as the theory that underpins them) is addressed more thoroughly in references [2, 3], [10], and [12] than fuzzy differential equations (2), (3), and (10) Using fuzzy integro-differential equations as a reference point, it is clear that fuzzy integro-differential equations (and the theory that underpins them) are treated more leniently in the first two papers. Recently, it is observed that the Xue et al. presented the comprehensive study of decline approximations for fuzzy viscoelastic integral model [12, 13] and compound learning control of ambiguous nonlinear fractional-order models with actuator liabilities grounded on command sifting and fuzzy estimation [14]. One can find comprehensive related literature in Refs. [12–16].

It is observed that the reproducing kernel philosophy has significant scientific applications in various fields like ordinary differential, numerical analysis, fractional differential, statistics, and probability models [17]. Ahmadian et al. recently developed some kind of reproducing kernel Hilbert space (RKHS) approaches to hand both ordinary and fractional-order fuzzy differential models [18, 19]. The author's show many advantages of the proposed scheme like to start the procedure and choose any point lies in the limits of integration, and it requires less effort to investigate the results. Later, various researchers used this strategy to explore the two-point fuzz BVPs model [20, 21], fuzzy differential model [22], periodic first-order BVPs of integro-differential model Fredholm type [23], systems of periodic second-order BVPs [24], Lane–Emden equation, and fractional-order model of Lane–Emden [25, 26].

On the basis of comprehensive literature review and author's best knowledge, it is observed that no one introduced the algorithm with less computational cost toproduce more accurate solutions, which motivates us to fill this gap and provide an efficient scheme for the problem given in Equation (1).

An in-depth explanation of the essay's organisational structure is provided in the next section, which also includes samples of the essay throughout the rest of the section. The next part will continue the topic that started in Part I of this chapter by addressing fuzzy integrals and fuzzy number theories. These two subjects were addressed in more detail in the part that came before it in the chapter's previous section. The following is a breakdown of how Part II of this chapter is organised. The use of methods such as erroneous integrals and fuzzy number theories, among other things, is required in order to create functions that take fuzzy numbers as both input and output values. In Section 2, we will discuss how to use failure integrals and fuzzy number theories in the construction of fuzzy-valued functions. Failure integrals and fuzzy number theories are two concepts that are used in the construction of fuzzy-valued functions. Consider the ideas of fuzzy integrals and fuzzy number theories, which were introduced in Section 3 of this chapter. It becomes clear that these concepts may be understood differently depending on the context in which they are used. In the third part of this chapter, we will go over in depth a kind of differential equation known as second-order integro-differential equation, also known as Volterra integro-differential equation, which is also known as Volterra integro-differential equation.

2. Important Concepts and Preliminaries

Many fundamental concepts and theorems will be reviewed in this part, and they will be used throughout the course. "Fuzzy numbers" (FN), "fuzzy functions" (FF), and derivative FF are only a few examples of what is available.

Definition 1 Reference [26]. A FN $U \longrightarrow [0,1]$ is a "fuzzy subset" (FS) of H with normal, convex, and superior "membership function" (MF) of "bounded support" (BS). Let $R_{\mathscr{F}}$ signify of FN. $0<\beta\leq 1$, set $[U]_\beta = \{S \in H | U(S) \geq \beta\}$ and $[U]_0 = \{S \in H | U(S) > 0\}$. Formerly, the β-level $[U]_\beta$ is a compact interlude $0 \leq \beta \leq 1$ and slightly $U \in R_{\mathscr{F}}$. The representation $[U]_\beta = [U_1(\beta), U_2(\beta)]$ signifies clearly the β-level set U. We remark to U_1 and U_2 inferior and superior divisions on U correspondingly.

Theorem 1 (see [27]). *Mapping $U \longrightarrow [0,1]$ is a FN with β-cut depiction $[U_1(\beta), U_2(\beta)]$ if and only if the succeeding circumstances are fulfilled:*

(i) U_1: $[0,1] \longrightarrow H$ is restricted nondeclining
(ii) U_2: $[0,1] \longrightarrow H$ is restricted nongrowing
(iii) $R \in (0,1]$, $\lim_{\beta \longrightarrow R^-} U_1(\beta) = U_1(R) \lim_{\beta \longrightarrow R^-} U_2(\beta) = U_2(R)$
(iv) $R \in (0,1]$, $\lim_{\beta \longrightarrow R^+} U_1(\beta) = U_1(R) \lim_{\beta \longrightarrow R^+} U_2(\beta) = U_2(R)$
(v) $U_1(\beta) \leq U_2(\beta) \beta \in [0,1]$

Definition 2 (see [26]). Suppose Y: $[a,c] \longrightarrow R_{\mathscr{F}}$ and $x_0 \in (a,c)$. We give or take Y is (1)-differentiable at x_0 and some element $Y'(x_0) \in R_{\mathscr{F}}$ and $j>0$ adequately near to 0, then there will be

$$Y(x_0+j) - Y(x_0), Y(x_0) - Y(x_0-j). \tag{1}$$

The limits

$$\lim_{j \longrightarrow 0^+} \frac{Y(x_0+j) - Y(x_0)}{j} = \lim_{j \longrightarrow 0^+} \frac{Y(x_0) - Y(x_0-j)}{j} = Y'(x_0). \tag{2}$$

In this circumstance, we signify $Y'(x_0)$ by $D_1^1 Y(x_0)$. Also, Y is (2)-differentiable, $j<0$ adequately near to 0, there be $Y(x_0+j) - Y(x_0), Y(x_0) - Y(x_0-j)$, and the limits

$$\lim_{j \longrightarrow 0^-} \frac{Y(x_0+j) - Y(x_0)}{j} = \lim_{j \longrightarrow 0^-} \frac{Y(x_0) - Y(x_0-j)}{j} = Y'(x_0). \tag{3}$$

In circumstance, this imitative is signified by $D_2^1 Y(x_0)$.

Theorem 2 (see [28]). *Let Y: $[a,c] \longrightarrow R_{\mathscr{F}}$ be a FF, then* $[Y(x)]_\beta = [Y_{1,\beta}(x), Y_{2,\beta}(x)], \beta \in [0,1]$.

(i) If Y (1)-differentiable, then $Y_{1,\beta}$ and $Y_{2,\beta}$ are DF and $[D_1^1 Y(x_0)]_\beta = [Y'_{1,\beta}(x), Y'_{2,\beta}(x)]$
(ii) If Y (2)-differentiable, then $Y_{1,\beta}$ and $Y_{2,\beta}$ are DF and $[D_2^1 Y(x_0)]_\beta = [Y'_{2,\beta}(x), Y'_{1,\beta}(x)]$

Theorem 3 (see [29]). *Let $D_1^1 Y: [a,c] \longrightarrow R_{\mathcal{F}}$ or $D_2^1 Y: [a,c] \longrightarrow R_{\mathcal{F}}$ be FF, where $[Y(x)]_\beta = [Y_{1,\beta}(x), Y_{2,\beta}(x)]$, $\beta \in [0,1]$.*

(i) If $D_1^1 Y$ is (1)-differentiable, then $Y_{1,\beta}'$ and $Y_{2,\beta}'$ are DF and $[Y''(x)]_\beta = [Y_{1,\beta}''(x), Y_{2,\beta}''(x)]$

(ii) If $D_1^1 Y$ is (2)-differentiable, then $Y_{1,\beta}'$ and $Y_{2,\beta}'$ are DF and $[Y''(x)]_\beta = [Y_{2,\beta}''(x), Y_{1,\beta}''(x)]$

(iii) If $D_2^1 Y$ is (1)-differentiable, then $Y_{1,\beta}'$ and $Y_{2,\beta}'$ are DF and$[Y''(x)]_\beta = [Y_{2,\beta}''(x), Y_{1,\beta}''(x)]$

(iv) If $D_2^1 Y$ is (2)-differentiable, then $Y_{1,\beta}'$ and $Y_{2,\beta}'$ are DF and $[Y''(x)]_\beta = [Y_{1,\beta}''(x), Y_{2,\beta}''(x)]$

Theorem 4 (see [30]). *Let $Y: [a,c] \longrightarrow R_{\mathcal{F}}$ be a continuous FF, where $[Y(x)]_\beta = [Y_{1,\beta}(x), Y_{2,\beta}(x)]$. If $Y_{1,\beta}(x)$ and $Y_{2,\beta}(x)$ are integrable functions (IF) over $[a,c]$, then $\int_a^c Y(x)dx \in R_{\mathcal{F}}$, then we get*

$$\left[\int_a^c Y(x)\mathrm{d}x\right]_\beta = \left[\int_a^c Y_{1,\beta}(x)\mathrm{d}x, \int_a^c Y_{2,\beta}(x)\mathrm{d}x\right]. \quad (4)$$

3. Modelling of Fuzzy Integral Equation

This section contains the modelling of the fuzzy integro-differential equation of Volterra type, where the fuzzy integro-differential equations of Volterra type are transformed into corresponding system of integro-differential equations [31]. It comprises the discovery of α-cut representation form of g. With the intention of develop the reproducing kernel Hilbert space algorithm to examine the accurate solutions of fuzzy integro-differential equations of Volterra type, first we assume

$$\begin{aligned} &x''(\eta) = f(\eta) + \int_a^\eta K(\eta,\xi) G(x(\xi)) d\xi, \quad a \le \eta, \\ &x(a) = b_1, x'(a) = b_2. \end{aligned} \quad (5)$$

According to section 2, the above second-order fuzzy Volterra equations is converted to the following system of equations as

$$\begin{aligned} x_{1,\alpha}''(\eta) &= f_{1,\alpha}(\eta) + \int_a^n \underline{V}(\eta,\alpha)\mathrm{d}s, \\ x_{2,\alpha}''(\eta) &= f_{2,\alpha}(\eta) + \int_\eta^\eta \overline{V}(\eta,\alpha)\mathrm{d}\xi, \\ x_{2,\alpha}''(\eta) &= f_{2,\alpha}(\eta) + \int_\eta^\eta \underline{V}(\eta,\alpha)\mathrm{d}s, \\ x_{1,\alpha}''(\eta) &= f_{1,\alpha}(\eta) + \int_\eta^\eta \overline{V}(\eta,\alpha)\mathrm{d}\xi. \end{aligned} \quad (6)$$

By means of the well-known Zadeh expansion principle given in reference [28] and if the function $G(x(\xi))$ present in Equation (1) is a function of strictly increasing, then

$$\begin{aligned} \overline{V}(\eta,\alpha) &= \begin{cases} K(\eta,\xi) G(x_{2,\alpha}(\xi)), & K(\eta,\xi) \ge 0, \\ K(\eta,\xi) G(x_{1,\alpha}(\xi)), & K(\eta,\xi) < 0, \end{cases} \\ \underline{V}(\eta,\alpha) &= \begin{cases} K(\eta,\xi) G(x_{1,\alpha}(\xi)), & K(\eta,\xi) \ge 0, \\ K(\eta,\xi) G(x_{2,\alpha}(\xi)), & K(\eta,\xi) < 0. \end{cases} \end{aligned} \quad (7)$$

Similarly, if the function $G(x(\xi))$ is strictly decreasing, then we obtain

$$\begin{aligned} \underline{V}(\eta,\alpha) &= \begin{cases} K(\eta,\xi) G(x_{2,\alpha}(\xi)), & K(\eta,\xi) \ge 0, \\ K(\eta,\xi) G(x_{1,\alpha}(\xi)), & K(\eta,\xi) < 0, \end{cases} \\ \overline{V}(\eta,\alpha) &= \begin{cases} K(\eta,\xi) G(x_{1,\alpha}(\xi)), & K(\eta,\xi) \ge 0, \\ K(\eta,\xi) G(x_{2,\alpha}(\xi)), & K(\eta,\xi) < 0. \end{cases} \end{aligned} \quad (8)$$

It is important to mention that the sufficient conditions for the existence and uniqueness of solution for the problem subject to Equation (1) are presented in reference [15].

4. Important Results and Convergence Analysis

This segment encloses the preliminaries, notation, development, and application of reproducing kernel Hilbert space scheme to seek the exact and numerical solutions of second kind Volterra integral equation. By means of Gram–Schmidt orthogonalization procedure, we build system of orthogonal function of $H_2^3[0,1] \oplus H_2^3[0,1]$.

Definition 3 Reference [32]. Let H and A be denoted by Hilbert space and abstract set, respectively, then a function $f_1: A \times A \longrightarrow \mathbb{R}$ is known as reproducing kernel of the Hilbert space H if it holds the following conditions:

$$\begin{aligned} &f_1(\cdot,t) \in H, \quad \forall t \in A, \\ &\phi, f_1(\cdot,t) = \phi(t), \quad \forall \phi \in H, t \in A. \end{aligned} \quad (9)$$

The second condition is also known as the reproducing property.

Definition 4 (see [32]). Suppose that $H_2^m[a,b]$ is an inner product space and defined as absolutely continuing, then $H_2^m[a,b]=\{x(\eta)|x(\eta),x'(\eta),\ldots,x^m(\eta)\text{areabsolutelycontinuous}, x^n(a)=0\text{for}n=0,1,2,\ldots,m-1\text{and}x^m(\eta)$ belongsto $L^2[a,b]\}$.

In the time being, the norm and inner product in $H_2^m[a,b]$ are defined as

$$\begin{aligned} &\sqrt{x(\eta), x(\eta)}_{H_2^m} = \|x(\eta)\|_{H_2^m}, \\ &\langle x(\eta), z(\eta)\rangle_{H_2^m} = \sum_{n=0}^{m-1} x^n(a) z^n(a) + \int_a^b x^m(v) z^m(v)\mathrm{d}v. \end{aligned} \quad (10)$$

In the above equation, the functions $x(\eta), z(\eta)$ are belongs to $H_2^m[a,b]$.

Definition 5 (see [32, 33]). Hilbert space $H_2^m[a,b] \oplus H_2^m[a,b]$ for $m = 1, 2, \ldots, n$, can be given as

$$H_2^m[a,b] \oplus H_2^m[a,b] = \left\{x = (x_1, x_2)^T | x_1, x_2 \in H_2^m[a,b]\right\}. \tag{11}$$

The inner product and norms of space $H_2^m[a,b] \oplus H_2^m[a,b]$ are given as

$$\begin{aligned} \sqrt{\sum_{i=1}^{2} \|x_i(\eta)\|_{H_2^m}^2} &= \|x(\eta)\|_{H_2^m \oplus H_2^m}, \\ \langle x(\eta), z(\eta)\rangle_{H_2^m \oplus H_2^m} &= \sum_{i=1}^{2} \langle x_i(\eta), z_i(\eta)\rangle_{H_2^m}. \end{aligned} \tag{12}$$

Definition 6 (see [32, 33]). Hilbert space $H_2^m[a,b]$ is called the reproducing kernel under the condition that $\forall \eta \in [a,b]$, $\exists R(\eta,\xi) \in H_2^m$ so that $x(\eta) = \langle x(\xi), R(\eta,\xi)\rangle_{H_2^1}$, $\forall x(\eta) \in H_2^1[a,b]$, $\xi \in [a,b]$.

Theorem 5 (see [32, 33]). *Let $H_2^m[a,b]$ be the Hilbert space which is also complete reproducing kernel space. Then, $R_\eta(\xi)$ reproducing kernel function is given as*

$$R_\eta(\xi) = \begin{cases} \sum_{i=1}^{2m-1} p_i(\eta)\xi^i, & \xi \le \eta, \\ \sum_{i=1}^{2m-1} q_i(\eta)\xi^i, & \xi > \eta. \end{cases} \tag{13}$$

The reproducing kernel function illustration $R_\eta(\xi)$ in the Hilbert space $H_2^3[0,1]$, by means of Maple 2015, is delivered by

$$R_\eta(\xi) = \begin{cases} 1 + \dfrac{1}{2! \cdot 3!}\eta^2\xi^2(\eta+3) + \eta\xi\left(1 - \dfrac{1}{4!}\eta^4\right) + \dfrac{1}{5!}\eta^5, & \xi \le \eta, \\ 1 + \dfrac{1}{2! \cdot 3!}\eta^2\xi^2(\eta+3) + \eta\xi\left(1 - \dfrac{1}{4!}\eta^4\right) + \dfrac{1}{5!}\eta^5, & \xi > \eta. \end{cases} \tag{14}$$

In order to apply the proposed scheme on the Hilbert space $H_2^3[a,b] \oplus H_2^3[a,b]$, first, we introduce a linear and invertible operator as

$$L: H_2^3[a,b] \oplus H_2^3[a,b] \longrightarrow H_2^1[a,b] \oplus H_2^1[a,b], \tag{15}$$

as $Lx(\eta) = x''(\eta)$ so that

$$L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}, \tag{16}$$

and $x(\eta) = (x_{1,\alpha}(\eta), x_{2,\alpha}(\eta))^T$. Therefore, the problem under study can be transformed as given below:

$$\begin{aligned} Lx(\eta) &= F(\eta, f(\eta), h(x(\eta))), \\ x(a) &= b_1, x'(a) = b_2. \end{aligned} \tag{17}$$

In the above equations, $h(x(\eta)) = \int_a^\eta K(\eta,\xi)G(x(\xi))\,d\xi$, $f(\eta) \in H_2^1[a,b] \oplus H_2^1[a,b]$, and $x(\eta) \in H_2^3[a,b] \oplus H_2^3[a,b]$. Also, $F(\eta, f(\eta), h(x(\eta)))$ is $F(\eta, f_{1,\alpha}(\eta), f_{2,\alpha}(\eta), h_{1,\alpha}(x(\eta)), h_{2,\alpha}(x(\eta)))$.

Theorem 6 (see [32, 33]). *The invertible and linear operator L defined as $L: H_2^3[a,b] \oplus H_2^3[a,b] \longrightarrow H_2^1[a,b] \oplus H_2^1[a,b]$ is bounded.*

Proof. It is very easy to prove that the operator L is linear from the space $H_2^3[a,b] \oplus H_2^3[a,b]$ to the space $H_2^1[a,b] \oplus H_2^1[a,b]$. Now, we have to prove that the operator L is bounded and bounded by some constant γ such that

$$\|Lx\|_{H_2^1 \oplus H_2^1} \le \gamma \|x\|_{H_2^3 \oplus H_2^3}, \quad \gamma > 0, \tag{18}$$

$\forall$ x belongs to $H_2^3[a,b] \oplus H_2^3[a,b]$, then we have

$$\begin{aligned} \|Lx(\eta)\|_{H_2^1 \oplus H_2^1}^2 &= \left\|L_1 x_{1,\alpha}(\eta)\right\|_{H_2^1}^2 + \left\|L_2 x_{2,\alpha}(\eta)\right\|_{H_2^1}^2 \\ &= \langle L_1 x_{1,\alpha}(\eta), L_1 x_{1,\alpha}(\eta)\rangle_{H_2^1} + \langle L_2 x_{2,\alpha}(\eta), L_2 x_{2,\alpha}(\eta)\rangle_{H_2^1} \\ &= \left(L_1 x_{1,\alpha}(\eta)\right)^2 + \left(L_2 x_{2,\alpha}(\eta)\right)^2 + \int_a^b \left[\left(L_1 x_{1,\alpha}(\eta)\right)_\eta\right]^2 d\eta + \int_a^b \left[\left(L_2 x_{2,\alpha}(\eta)\right)_\eta\right]^2 d\eta. \end{aligned} \tag{19}$$

By means of the property of reproducing kernel of $R_\eta(\xi)$, then

$$\begin{aligned} x_{1,\alpha}(\eta) &= \langle x_{1,\alpha}(\xi), R_\eta(\xi)\rangle_{H_2^3}, \\ x_{2,\alpha}(\eta) &= \langle x_{2,\alpha}(\xi), R_\eta(\xi)\rangle_{H_2^3}. \end{aligned} \tag{20}$$

Thus, we have

$$\begin{aligned} L_1 x_{1,\alpha}(\eta) &= \langle x_{1,\alpha}(\xi), L_1 R_\eta(\xi)\rangle_{H_2^3}, \\ L_1 R_\eta(\eta) &= \langle x_{2,\alpha}(\xi), L_2 R_\eta(\xi)\rangle_{H_2^3}, \\ \left(L_1 x_{1,\alpha}(\eta)\right)_\eta &= \langle x_{1,\alpha}(\xi), \left(L_1 R_\eta(\xi)\right)_\eta\rangle_{H_2^3}, \\ \left(L_1 R_\eta(\eta)\right) &= \langle x_{2,\alpha}(\xi), \left(L_2 R_\eta(\xi)_\eta\right)_\eta\rangle_{H_2^3}. \end{aligned} \tag{21}$$

By means of the continuity of the function $R_\eta(\xi)$ on the interval $[a,b]$, we obtain

$$\begin{aligned}
L_1 x_{1,\alpha}(\eta) &= \langle x_{1,\alpha}(\xi), L_1 R_\eta(\xi)\rangle_{H_2^3} \le \|x_{1,\alpha}\|_{H_2^3} \langle L_1 R_\eta(\xi)\rangle_{H_2^3} \le \gamma_1 \|x_{1,\alpha}\|_{H_2^3},\\
L_2 x_{2,\alpha}(\eta) &= \langle x_{2,\alpha}(\xi), L_2 R_\eta(\xi)\rangle_{H_2^3} \le \|x_{2,\alpha}\|_{H_2^3} \langle L_2 R_\eta(\xi)\rangle_{H_2^3} \le \gamma_1 \|x_{2,\alpha}\|_{H_2^3},\\
\left|\left(L_1 x_{1,\alpha}(\eta)\right)_\eta\right| &= \left|\langle x_{1,\alpha}(\xi), L_1 R_\eta(\xi)\rangle_{H_2^3}\right| \le \|x_{1,\alpha}\|_{H_2^3} \left\langle \left(L_1 R_\eta(\xi)\right)_\eta \right\rangle_{H_2^3} \le \gamma_3 \|x_{1,\alpha}\|_{H_2^3},\\
\left|\left(L_2 x_{2,\alpha}(\eta)\right)_\eta\right| &= \left|\langle x_{2,\alpha}(\xi), L_2 R_\eta(\xi)\rangle_{H_2^3}\right| \le \|x_{2,\alpha}\|_{H_2^3} \left\langle \left(L_2 R_\eta(\xi)\right)_\eta \right\rangle_{H_2^3} \le \gamma_4 \|x_{2,\alpha}\|_{H_2^3}.
\end{aligned} \tag{22}$$

Accordingly, we have

$$\begin{aligned}
\|Lx(\eta)\|^2_{H_2^1 \oplus H_2^1} &\le \gamma_1^2 \|x_{1,\alpha}\|^2_{H_2^3} + \gamma_2^2 \|x_{2,\alpha}\|^2_{H_2^3} + (b-a)\gamma_3^2 \|x_{1,\alpha}\|^2_{H_2^3} + (b-a)\gamma_4^2 \|x_{2,\alpha}\|^2_{H_2^3}\\
&= \left(\gamma_1^2 + (b-a)\gamma_3^2\right)\|x_{1,\alpha}\|^2_{H_2^3} + \left(\gamma_2^2 + (b-a)\gamma_4^2\right)\|x_{2,\alpha}\|^2_{H_2^3} \le \gamma^2\left(\|x_{1,\alpha}\|^2_{H_2^3} + \|x_{2,\alpha}\|^2_{H_2^3}\right)\\
&= \gamma^2 \sum_{i=1}^{2} \|x_{i,\alpha}\|^2_{H_2^3} = \gamma^2 \|x\|^2_{H_2^3 \oplus H_2^3}.
\end{aligned} \tag{23}$$

In the above equation, γ is maximum of $(\gamma_1^2 + (b-a)\gamma_3^2)$ and $(\gamma_2^2 + (b-a)\gamma_4^2)$.

To apply the illustration form of exact and numerical solutions of second kind Volterra integral equation, we next formulate the system of orthogonal functions $\{\chi_{ij}(\eta)\}$ for $i = 1,2,\ldots,\infty$, $j = 1,2$ of $H_2^3[a,b] \oplus H_2^3[a,b]$ space, and thus we assume

$$\chi_{ij}(\eta) = \begin{bmatrix} L_1^* & 0 \\ 0 & L_2^* \end{bmatrix} \Theta_{ij} = L^* \Theta_{ij}. \tag{24}$$

In the above equation, L^* is known as the adjoint operator of the operator L and $\Theta_{ij} = (\Theta_{i1}, \Theta_{i2})^T$. The system of orthogonal system $\{\overline{\chi}_{ij}(\eta)\}$ for $i = 1,2,\ldots,\infty$, $j = 1,2$ in $H_2^3[a,b] \oplus H_2^3[a,b]$ space can be computed by means of the Gram–Schmidt orthogonalization procedure of $\{\chi_{ij}(\eta)\}$ for $i = 1,2,\ldots,\infty$, $j = 1,2$ as given below:

$$\overline{\chi}_{lm}(\eta) = \sum_{i=1}^{l} \sum_{j=1}^{m} \beta_{ij}^{lm} \chi_{ij}(\eta). \tag{25}$$

The coefficients β_{ij}^{lm} can be obtained by means of the following relation

$$\begin{aligned}
\beta_{11}^{lm} &= \frac{1}{\|\chi_{11}\|}, \beta_{ij}^{lm} = \frac{1}{\sqrt{\|\chi_{ij}\|^2 - \sum_{q=1}^{i-1} \langle \chi_{ij}(\eta), \overline{\chi}_{iq}(\eta)\rangle^2}}, \quad \text{for } i = j \ne 1,\\
\beta_{ij}^{lm} &= -\frac{\sum_{q=j}^{i-1} \chi_{ij}(\eta), \overline{\chi}_{iq}(\eta) \beta_{qj}^{lm}}{\sqrt{\|\chi_{ij}\|^2 - \sum_{q=1}^{i-1} \langle \chi_{ij}(\eta), \overline{\chi}_{iq}(\eta)\rangle^2}}, \quad \text{for } i > j.
\end{aligned} \tag{26}$$

□

Theorem 7. *Let $\{\eta_i\}$ for $i = 1,2,\ldots,\infty$ is dense in the interval $[a,b]$, $x(\eta)$ is the solution of fuzzy integro-differential model given in Equation (1) and $x(\eta) \in H_2^3[a,b] \oplus H_2^3[a,b]$, then*

$$x(\eta) = \sum_{l=1}^{\infty} \sum_{m}^{2} \sum_{i=1}^{l} \sum_{j=1}^{m} \beta_{ij}^{lm} F_j(\eta_i, f(\eta_i), h(x(\eta_i))) \overline{\chi}_{lm}(\eta), \tag{27}$$

$x(\eta)$ is the convergent series in the logic of. $\|\cdot\|_{H_2^3 \oplus H_2^3}$.

Proof. In order to prove the required result, first we need to show that $\{\chi_{ij}(\eta)\}$ for $i = 1, 2, \ldots, \infty$, $j = 1, 2$ is a complete system and belongs to $H_2^3[a,b] \oplus H_2^3[a,b]$ as

$$\begin{aligned}\chi_{ij}(\eta) &= L^*\Theta_{ij}(\eta) = \langle L^*\Theta_{ij}(\xi), R_\eta(\xi)\rangle_{H_2^3 \oplus H_2^3} = \langle \Theta_{ij}(\xi), L_\xi R_\eta(\xi)\rangle_{H_2^1 \oplus {}_2^1} \\ &= L_\xi R_\eta(\xi)|_{\xi=\eta_i} \in H_2^3[a,b] \oplus H_2^3[a,b].\end{aligned} \tag{28}$$

In contrast, for each $x(\eta) \in H_2^3[a,b] \oplus H_2^3[a,b]$, assume $x(\eta), \chi_{ij}(\eta)_{H_2^3 \oplus H_2^3} = 0$, also

$$\begin{aligned}\langle x(\eta), \chi_{ij}(\eta)\rangle_{H_2^3 \oplus H_2^3} &= \langle x_{1,\alpha}(\eta), \chi_{i1}(\eta)\rangle_{H_2^3} + \langle x_{2,\alpha}(\eta), \chi_{i2}(\eta)\rangle_{H_2^3} \\ &= \langle x_{1,\alpha}(\eta), L_1^*\Theta_{i1}(\eta)\rangle_{H_2^3} + \langle x_{2,\alpha}(\eta), L_2^*\Theta_{i2}(\eta)\rangle_{H_2^3} \\ &= \langle L_1 x_{1,\alpha}(\eta), \Theta_{i1}(\eta)\rangle_{H_2^1} + \langle L_2 x_{2,\alpha}(\eta), \Theta_{i2}(\eta)\rangle_{H_2^1} = L_1 x_{1,\alpha}(\eta_i) + L_2 x_{2,\alpha}(\eta_i) = Lx(\eta_i).\end{aligned} \tag{29}$$

Since $\{\eta_i\}$ for $i = 1, 2, \ldots, \infty$ is dense, then $Lx(\eta) = 0$, and sin e L in invariable which implies that $x(\eta) = 0$. We know that the sequence $\{\chi_{ij}(\eta)\}$ for $i = 1, 2, \ldots, \infty$, $j = 1, 2$ is complete in the space $H_2^3[a,b] \oplus H_2^3[a,b]$ and

$$\overline{\chi}_{lm}(\eta) = \sum_{i=1}^{l}\sum_{j=1}^{m} \beta_{ij}^{lm}\chi_{ij}(\eta). \tag{30}$$

Then the system $\{\chi_{ij}(\eta)\}$ for $i = 1, 2, \ldots, \infty$, $j = 1, 2$ is complete orthonormal in the space $H_2^3[a,b] \oplus H_2^3[a,b]$. By means of the Fourier series expansion around $\{\chi_{ij}(\eta)\}$ for $i = 1, 2, \ldots, \infty$, $j = 1, 2$, we obtain

$$\begin{aligned}x(\eta) &= \sum_{l=1}^{\infty}\sum_{m=1}^{2} x(\eta), \overline{\chi}_{lm}(\eta)_{H_2^3 \oplus H_2^3}\overline{\chi}_{lm}(\eta) \\ &= \sum_{l=1}^{\infty}\sum_{m=1}^{2} x(\eta), \sum_{l=1}^{l}\sum_{m=1}^{m} \beta_{ij}^{lm}\chi_{ij}(\eta)_{H_2^3 \oplus H_2^3}\overline{\chi}_{lm}(\eta) \\ &= \sum_{l=1}^{\infty}\sum_{m=1}^{2}\sum_{i=1}^{l}\sum_{j=1}^{m} \beta_{ij}^{lm} x(\eta), \chi_{ij}(\eta)_{H_2^3 \oplus H_2^3}\overline{\chi}_{lm}(\eta) \\ &= \sum_{l=1}^{\infty}\sum_{m=1}^{2}\sum_{i=1}^{l}\sum_{j=1}^{m} \beta_{ij}^{lm} x(\eta), L^*\Theta_{ij}(\eta)_{H_2^3 \oplus H_2^3}\overline{\chi}_{lm}(\eta) \\ &= \sum_{l=1}^{\infty}\sum_{m=1}^{2}\sum_{i=1}^{l}\sum_{j=1}^{m} \beta_{ij}^{lm} Lx(\eta), \Theta_{ij}(\eta)_{H_2^1 \oplus H_2^1}\overline{\chi}_{lm}(\eta) \\ &= \sum_{l=1}^{\infty}\sum_{m=1}^{2}\sum_{i=1}^{l}\sum_{j=1}^{m} \beta_{ij}^{lm} F_j(\eta, f(\eta), h(x(\eta))), \Theta_{ij}(\eta)_{H_2^1 \oplus H_2^1}\overline{\chi}_{lm}(\eta) \\ &= \sum_{l=1}^{\infty}\sum_{m=1}^{2}\sum_{i=1}^{l}\sum_{j=1}^{m} \beta_{ij}^{lm} F_j(\eta_i, f(\eta_i), h(x(\eta_i)))\overline{\chi}_{lm}(\eta).\end{aligned} \tag{31}$$

This implies that the above series is nothing but Fourier series in the space $H_2^3[a,b] \oplus H_2^3[a,b]$. As $H_2^3[a,b] \oplus H_2^3[a,b]$ is the Hilbert space, the above series is convergent with the logic $\|\cdot\|_{H_2^3 \oplus H_2^3}$.

For numerical procedure, we place the initial function $x_0(\eta_i) = x(\eta_i)$ and the nth-term of the numerical solution of the problem under study is given as

$$x_n(\eta) = \sum_{l=1}^{\infty}\sum_{m=1}^{2}\sum_{i=1}^{l}\sum_{j=1}^{m}\beta_{ij}^{lm}F_j\left(\eta_i, f(\eta_i), h\left(x_{i-1}(\eta_i)\right)\right)\overline{\chi}_{lm}(\eta). \tag{32}$$

□

Theorem 8. *Let the exact solution of (11) be $x(\eta)$, and $x_n(\eta)$ denote its approximate solution, then*

(i) Suppose $\|x\|_{nH_2^3\oplus H_2^3}$ is the bounded sequence and the sequence $\{\eta_i\}_{i=1}^{\infty}$ dense on the interval $[a,b]$, then as $n\longrightarrow\infty$, $\|x_n - x\|_{H_2^3\oplus H_2^3}\longrightarrow 0$

(ii) As $n\longrightarrow\infty$ then $\|x_n - x\|_c\longrightarrow 0$

(iii) As $n\longrightarrow\infty$ then $(x_n)_{\eta\eta}\longrightarrow x_{\eta\eta}$ uniformly

(iv) If as $n\longrightarrow\infty\|x_{n-1}-x\|_{H_2^3\oplus H_2^3}\longrightarrow 0$, $\|x_{n-1}\|_{H_2^3\oplus H_2^3}$ is the bounded, as $n\longrightarrow\infty\eta_n\longrightarrow\tau$, then the function $F(\eta, f(\eta), h(x(\eta)))$ is continuous $\forall\eta\in[a,b]$ and the functions $h(\eta), f(\eta)$ are continuous, then

$$F\left(\eta_n, f(\eta_n), h\left(x_{n-1}(\eta_n)\right)\right)\longrightarrow F(\eta, f(\eta), h(x(\eta))). \tag{33}$$

Proof

(i) By means of orthonormality of $\left\{\chi_{ij}(\eta)\right\}_{(j,i)=(1,1)}^{(2,\infty)}$ and Equation (12), we have

$$\left\|x_{n+1}\right\|_{H_2^3\oplus H_2^3}^2 = \left\|x_0\right\|_{H_2^3\oplus H_2^3}^2 + \sum_{m=1}^{n+1}\sum_{l=1}^{2}A_{lm}^2, \tag{34}$$

Table 1: Comparison of exact $\mathbf{x}_{1,\alpha}, \mathbf{x}_{2,\alpha}$ and approximate $\widetilde{\mathbf{x}}_{1,\alpha}, \widetilde{\mathbf{x}}_{2,\alpha}$ solutions of problem (13) when $\eta = 0.5$ for different values ofα.

$\downarrow\alpha$	$x_{1,\alpha}$	$\widetilde{x}_{1,\alpha}$	$x_{2,\alpha}$	$x_{2,\alpha}$
Solution of problem (14)				
0.0	−0.89025	−0.88849	0.89025	0.88849
0.2	−0.71220	−0.71079	0.71220	0.71079
0.4	−0.5342	−0.53309	0.53415	0.53309
0.6	−0.35610	−0.35539	0.35610	0.35540
0.8	−0.17805	−0.17770	0.17805	0.17769
1.0	0	0.0		0.0
Solution of problem (15)				
0.0	−0.85445	−0.85217	0.85445	0.85217
0.2	−0.68356	−0.68174	0.68356	0.68174
0.4	−0.51267	−0.51130	0.51267	0.51130
0.6	−0.34178	−0.34087	0.34178	0.34087
0.8	−0.17089	−0.17043	0.17089	0.17043
1.0	0	0.0		0

where $A_{ij} = \sum_{i=1}^{l}\sum_{j=1}^{m}\beta_{ij}^{lm}F_j\left(\eta_i, f(\eta_i), h\left(x_{i-1}(\eta_i)\right)\right)\overline{\chi}_{lm}(\eta)$. We know that $\|x_n\|_{H_2^3\oplus H_2^3}$ is bounded and $\|x_n\|_{H_2^3\oplus H_2^3}\le\|x_{n+1}\|_{H_2^3\oplus H_2^3}$, then the sequence $\|x_n\|_{H_2^3\oplus H_2^3}$ is convergent and also $\sum_{m=1}^{2}A_{lm}^2\in l^2$ for $l = 1, 2, \ldots, \infty$. If $n < m$ as $m, n\longrightarrow\infty$, then

$$\left\|x_m\right\| - \left\|x_n\right\|_{H_2^3\oplus H_2^3}^2 = \sum_{l=n+1}^{m}\sum_{m=1}^{2}A_{lm}^2\longrightarrow 0. \tag{35}$$

Since the space $H_2^3[a,b]\oplus H_2^3[a,b]$ is Hilbert, then as $n\longrightarrow\infty$, $\|x_n\| - \|x\|_{H_2^3\oplus H_2^3}\longrightarrow 0$.

(ii) By means of the reproducing property of the function $R_\eta(\xi)$, then

$$\begin{aligned}\left|x_n(\eta) - x(\eta)\right| &= \left|\langle x_n(\xi) - x(\xi), R_\eta(\xi)\rangle_{H_2^3\oplus H_2^3}\right| \le \left\|x_n(\xi) - x(\xi)\right\|_{H_2^3\oplus H_2^3}\left\|R_\eta(\xi)\right\|_{H_2^3\oplus H_2^3}\\ &\le \gamma_5\left\|x_n - x\right\|_{H_2^3\oplus H_2^3}\longrightarrow 0,\quad \text{as}\, n\longrightarrow\infty.\end{aligned} \tag{36}$$

This implies that $x_n - x_C\longrightarrow 0$ as $n\longrightarrow\infty$.

(iii) We know that x_n uniformly converges to x, then

$$\begin{aligned}\left|x_n''(\eta) - x''(\eta)\right| &= \left|\frac{d^2}{d\eta^2}\left(x_n(\eta) - x(\eta)\right)\right| = \left|\frac{d^2}{d\eta^2}\langle x_n(\xi) - x(\xi), R_\eta(\xi)\rangle_{H_2^3\oplus H_2^3}\right|\\ &= \left|\langle x_n(\xi) - x(\xi), \frac{d^2}{d\eta^2}R_\eta(\xi)\rangle_{H_2^3\oplus H_2^3}\right| \le \left\|x_n(\xi) - x(\xi)\right\|_{H_2^3\oplus H_2^3}\left\|\frac{d^2}{d\eta^2}R_\eta(\xi)\right\|_{H_2^3\oplus H_2^3}\\ &\le \gamma_6\left\|x_n - x\right\|_{H_2^3\oplus H_2^3}\longrightarrow 0,\quad \text{as}\, n\longrightarrow\infty.\end{aligned} \tag{37}$$

This implies that $\|x_n'' - x_C''\|\longrightarrow 0$, as $n\longrightarrow\infty$.

(iv) First of all, we show that $x_{n-1}(\eta_n)\longrightarrow x(\tau)$ as $n\longrightarrow\infty$ as

$$\left|x_{n-1}(\eta_n)-x(\tau)\right| = \left|x_{n-1}(\eta_n)-x_{n-1}(\tau)+x_{n-1}(\tau)-x(\tau)\right| \cdot \left|x_{n-1}(\eta_n)-x_{n-1}(\tau)\right| + \left|x_{n-1}(\tau)-x(\tau)\right|. \quad (38)$$

We know that $n \longrightarrow \infty, \|x_{n-1}-x\|_{H_2^3\oplus H_2^3} \longrightarrow 0$, this implies that as $n \longrightarrow \infty, \|x_{n-1}(\tau)-x(\tau)\|_{H_2^3\oplus H_2^3} \longrightarrow 0$. Now

$$\begin{aligned}\left|x_{n-1}(\eta_n)-x_{n-1}(\tau)\right| &= \left|\langle x_{n-1}(\eta), R_{\eta_n}(\xi)\rangle - \langle x_{n-1}(\eta), R_\tau(\xi)\rangle\right| \\ &= \left|\langle x_{n-1}(\eta), R_{\eta_n}(\xi)-R_\tau(\xi)\rangle\right| \le \left\|x_{n-1}(\eta)\right\|_{H_2^3\oplus H_2^3}\left\|R_{\eta_n}(\xi)-R_\tau(\xi)\right\|_{H_2^3\oplus H_2^3}.\end{aligned} \quad (39)$$

As the function $R_\eta(\xi)$ is symmetric, as $n \longrightarrow \infty; \|R_{\eta_n}(\xi)-R_\tau(\xi)\|_{H_2^3\oplus H_2^3} \longrightarrow 0$, and this implies that $n \longrightarrow \infty, |x_{n-1}(\eta_n)-x_{n-1}(\tau)| \longrightarrow 0$. Therefore, $|x_{n-1}(\eta_n)-x(\tau)| \longrightarrow 0$, as $n \longrightarrow \infty$.

We know that the functions h and f are continuous and $\eta_n \longrightarrow \tau$, then as $n \longrightarrow \infty; f(\eta_n) \longrightarrow f(\tau)$ and $h(x_{n-1}(\eta_n)) \longrightarrow h(x(\tau))$ as $n \longrightarrow \infty$. By means of the continuity of the function F, we have $F(\eta_n, f(\eta_n), h(x_{n-1}(\eta_n))) \longrightarrow F(\tau, f(\tau), h(x(\tau)))$. □

5. Results and Discussion

A fuzzy integro-differential problem of the Volterra type is investigated in this part using the expected reproducing kernel Hilbert space method and the approximate solution of the integro-differential equation. It is important to note that the suggested technique is very simple to use, quick convergent, and trustworthy, and it may be used to find both approximate and precise solutions to the issue under consideration. The Maple code is being created for this mentioned issue utilising a replicating kernel Hilbert space method, and it is being used to produce fresh findings as well as to make comparisons between results [34–37].

Problem 1. Suppose the following fuzzy integro-differential model of Volterra type is given as

$$\begin{aligned}x_{\eta\eta} &= [-1+\alpha, 1-\alpha] + \int_0^\eta x(\xi)d\xi, \quad \alpha, t \in [0,1], \\ x(0) &= [-1+\alpha, 1-\alpha], \\ x_\eta(0) &= [0,0].\end{aligned} \quad (40)$$

Allowing to the anticipated reproducing kernel Hilbert space algorithm, the problem given in Equation (13) transformed into following system

$$\begin{aligned}x_{1,\alpha}'' &= -1+\alpha+\int_0^\eta x_{1,\alpha}(\xi)\mathrm{d}\xi, \\ x_{2,\alpha}'' &= 1-\alpha+\int_0^\eta x_{2,\alpha}(\xi)\mathrm{d}\xi, \quad 0\le\eta\le 1, \\ x_{1,\alpha}(0) &= -1+\alpha, x_{1,\alpha}'(0)=0, \\ x_{2,\alpha}(0) &= 1-\alpha, x_{2,\alpha}'(0)=0, \\ x_{1,\alpha}'' &= 1-\alpha+\int_0^\eta x_{2,\alpha}(\xi)\mathrm{d}\xi, \\ x_{2,\alpha}'' &= -1+\alpha+\int_0^\eta x_{1,\alpha}(\xi)\mathrm{d}\xi, \quad 0\le\eta\le 1, \\ x_{1,\alpha}(0) &= 1-\alpha, x_{1,\alpha}'(0)=0, \\ x_{2,\alpha}(0) &= -1+\alpha, x_{2,\alpha}'(0)=0.\end{aligned} \quad (41)$$

The exact solutions of the problems (14)-(15) are given as

$$\begin{aligned}x(\eta) &= \left[\left(\cos\frac{\eta}{\sqrt{2}}-\frac{1}{\sqrt{2}}\sin\frac{\eta}{\sqrt{2}}\right)(\alpha-1)e^{(1/2)\eta}, \left(\cos\frac{\eta}{\sqrt{2}}-\frac{1}{\sqrt{2}}\sin\frac{\eta}{\sqrt{2}}\right)(1-\alpha)e^{(1/2)\eta}\right], \\ x(\eta) &= \left[\left(\cos\frac{\sqrt{3}}{2}\eta-\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}\eta\right)(\alpha-1)e^{(1/2)\eta}, \left(\cos\frac{\sqrt{3}}{2}\eta-\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}\eta\right)(1-\alpha)e^{(1/2)\eta}\right].\end{aligned} \quad (42)$$

Numerous simulation have been implemented for various values of parameter α and η. The comparison between the exact and approximate solutions when $\eta = 0.5$ and numerous values of α for problem (13) are deliberated in Table 1. It can be observed that the proposed scheme demonstrates very accurate solution for solving the problem under study. Also, the accuracy does not depend on the selection of α, the analytic solution is obtained of the discussed problem using proposed scheme when $\eta = 1$.

Problem 2. Suppose the following fuzzy integro-differential model of Volterra type is given as

$$x_{\eta\eta} = \left[(-10 + 5\alpha)e^{1-\eta} - \left(1 + \frac{1}{2}\alpha\right)\eta, (-3 - 2\alpha)e^{1-\eta} - \left(0.6 + 0.1\alpha - 0.2\alpha^2\right)\eta\right]$$
$$+ \int_0^\eta [-0.2 + 0.1\alpha, -0.1]e^{\xi-1}x(\xi)d\xi \ , \alpha, t \in [0,1], \tag{43}$$
$$x(0) = [(-10 + 5\alpha)e, (-3 - 2\alpha)e], x_\eta(0) = [(3 + 2\alpha)e, (10 - 5\alpha)e].$$

Allowing the anticipated reproducing kernel Hilbert space algorithm, the problem given in Equation (13) transformed into following system

$$\begin{aligned}
x''_{1,\alpha} &= (-10 + 5\alpha)e^{1-\eta} - \left(1 + \frac{1}{2}\alpha\right)\eta + \int_0^\eta (-0.2 + 0.1\alpha)e^{\xi-1}x_{1,\alpha}(\xi)\mathrm{d}\xi,\\
x''_{2,\alpha} &= (-3 - 2\alpha)e^{1-\eta} - \left(0.6 + 0.1\alpha - 0.2\alpha^2\right)\eta - \int_0^\eta 0.1e^{\xi-1}x_{2,\alpha}(\xi)\mathrm{d}\xi,\\
x_{1,\alpha}(0) &= (-10 + 5\alpha)e, x'_{1,\alpha}(0) = (3 + 2\alpha)e,\\
x_{2,\alpha}(0) &= (-3 - 2\alpha)e, x'_{2,\alpha}(0) = (10 - 5\alpha)e,\\
x''_{1,\alpha} &= (-3 - 2\alpha)e^{1-\eta} - \left(0.6 + 0.1\alpha - 0.2\alpha^2\right)\eta - \int_0^\eta 0.1e^{\xi-1}x_{1,\alpha}(\xi)\mathrm{d}\xi,\\
x''_{2,\alpha} &= (-10 + 5\alpha)e^{1-\eta} - \left(1 + \frac{1}{2}\alpha\right)\eta + \int_0^\eta (-0.2 + 0.1\alpha)e^{\xi-1}x_{2,\alpha}(\xi)\mathrm{d}\xi,\\
x_{1,\alpha}(0) &= (-3 - 2\alpha)e, x'_{1,\alpha}(0) = (10 - 5\alpha)e,\\
x_{2,\alpha}(0) &= (-10 + 5\alpha)e, x'_{2,\alpha}(0) = (3 + 2\alpha)e.
\end{aligned} \tag{44}$$

The exact solutions of the problems (17)-(18) are given as

$$\begin{aligned}
x(\eta) &= \left[(-10 + 5\alpha)e^{1-\eta}(-3 - 2\alpha)e^{1-\eta}\right],\\
x(\eta) &= \left[(3 + 2\alpha)e^{1-\eta}, (10 - 5\alpha)e^{1-\eta}\right].
\end{aligned} \tag{45}$$

For $\eta \in [0, 1]$, reproducing the kernel Hilbert space algorithm applied for solving this problem when $\alpha = 0.5$ and numerous values of η. Figure 1 is plotted to show the comparison of approximate and exact solutions. It is significant to indicate that the estimated solution attained using the proposed scheme is well-matched with the exact solutions for every values of α, which is the beauty of the suggested algorithm [34].

Problem 3. Suppose the following fuzzy integro-differential model of Volterra type is given as

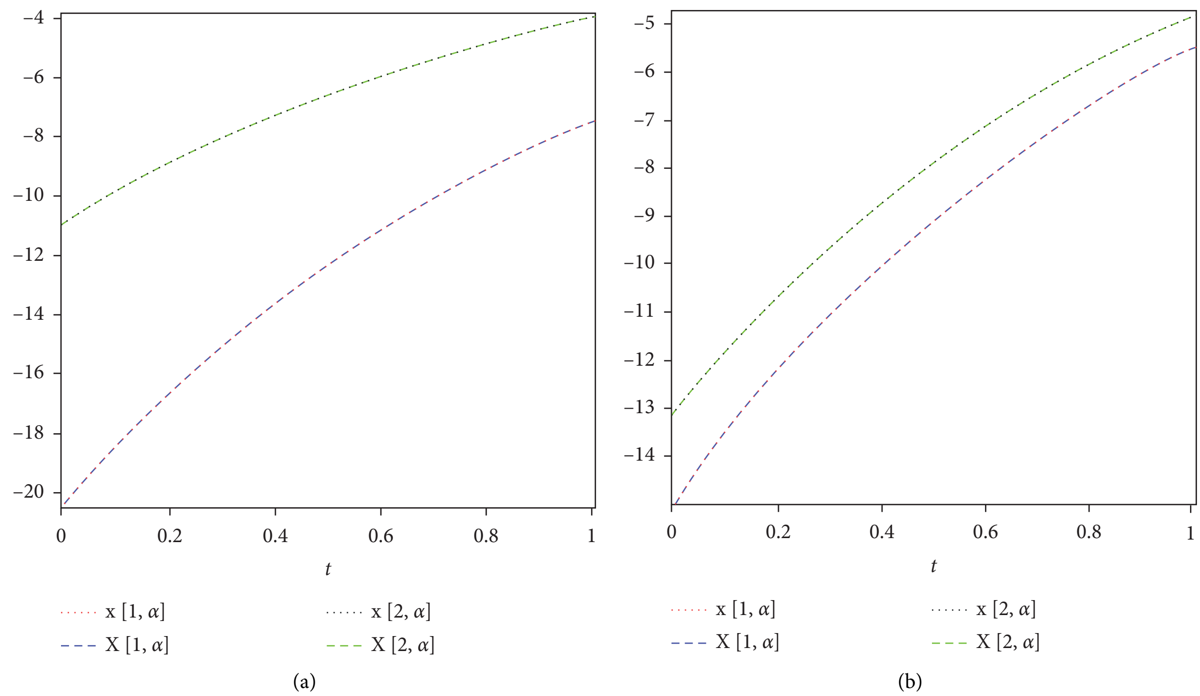

Figure 1: Comparison of the approximate $X_{1,\alpha}, X_{2,\alpha}$ and exact solutions $x_{1,\alpha}, x_{2,\alpha}$ for (a) $\alpha = 0.5$ and (b) $\alpha = 0.9$.

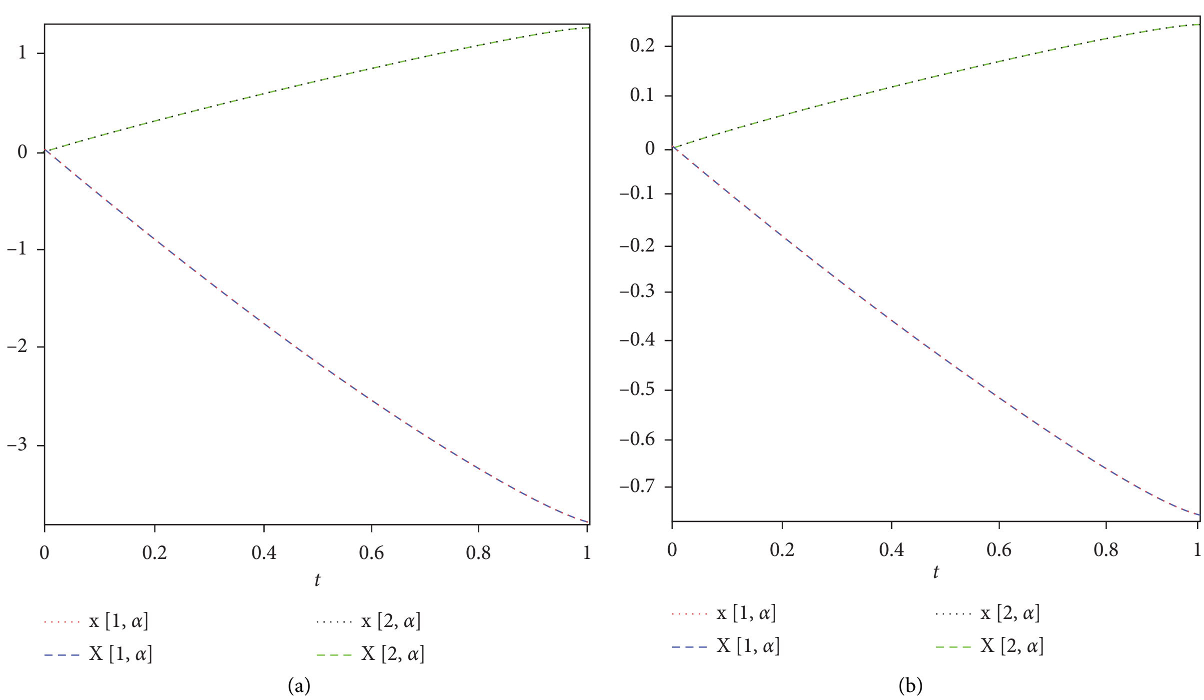

Figure 2: Comparison of the approximate $X_{1,\alpha}, X_{2,\alpha}$ and exact solutions $x_{1,\alpha}, x_{2,\alpha}$ for (a) $\alpha = 0.5$ and (b) $\alpha = 0.9$.

$$x_{\eta\eta} = \begin{bmatrix} (3+3\alpha)\sin\,\eta + \frac{3}{20}\left(\alpha^2 - 6\alpha + 5\right)\eta\,\cos^2\,\eta \\ \\ (9-9\alpha)\sin\,\eta - \frac{9}{20}\left(\alpha^2 - 6\alpha + 5\right)\eta\,\cos^2\,\eta \end{bmatrix} + \int_0^{\eta} [0.1 + 0.3\alpha, 0.5 - 0.1]\eta\,\cos\,\xi x(\xi)\mathrm{d}\xi, \quad \alpha, t \in [0,1],$$

$$x(0) = [(-10+5\alpha)e, (-3-2\alpha)e],$$

$$x_{\eta}(0) = [(3+2\alpha)e, (10-5\alpha)e]. \tag{46}$$

The exact solutions are given as

$$\begin{aligned} x(\eta) &= [(3+3\alpha)\sin\,\eta, (9-9\alpha)\sin\,\eta], \\ x(\eta) &= [(-9+9\alpha)\sin\,\eta, (3-3\alpha)\sin\,\eta]. \end{aligned} \tag{47}$$

For $\eta \in [0,1]$, reproducing the kernel Hilbert space algorithm applied for solving this problem when $\alpha = 0.5$ and numerous values of η. The comparison of the approximate and exact solutions is deliberated in Figure 2. It is important to mention that the approximate solution achieved by means of the proposed scheme is well-matched with the exact solutions for every values of α, which is the beauty of the suggested algorithm.

6. Conclusion

Using replicated kernel theory, we were able to create a whole new technique for solving complex second-order FVIDs, which we have just published. This method is explained in more depth farther down this page. The technique under consideration generates responses that are both broad in scope and particular in kind. Numerical simulations have been performed in order to demonstrate the robustness of the technique under discussion. In the future, we expect that our technique will be used to solve fuzzy ordinary differential algebraic equations, as well as fuzzy partial integro-differential equations and various types of fuzzy differential algebraic equations, among other things.

Data Availability

The datasets used and/or analysed during the current study are available from the corresponding author on reasonable request.

Conflicts of Interest

The authors declare that this article is free of conflicts of interest.

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